1 (10 points)

(a) (6 points) Solve the following differential equation:

$$ty' + 2y = \cos t + \frac{\sin t}{t},\ y(\pi) = y_0$$

First divide everything by $t$ to get $y' + \frac{2y}{t} = \frac{cost}{t} + \frac{sint}{t^2}$. Next find the integrating factor and multiply the differential equation by it: $e^{\int \frac{2}{t}} = e^{2\ln t} = e^{\ln t^2} = t^2$. Now we have

$$(t^2 y)' = t\cos t + \sin t.$$

Integrate both sides to get

$$t^2 y = t\sin t + C.$$

At this point it's easy to find $C$. Plug in $y(\pi) = y_0$ to get $\pi^2 y_0 = 0 + C = C$ Finally, solve for $y$ to get

$$y(t) = \frac{\sin t}{t} + \frac{y_0 \pi^2}{t^2}.$$

(b) (2 points) What is the behavior of $y(t)$ as $t \to \infty$?
Well, as $t \to \infty$, no matter what $y_0$ is, $y(t) \to 0$.

(c) (2 points) Find the value of $y_0$ which makes $y(t)$ able to be continuous for all $t$.
If $y_0 = 0$, then $y(t) = \frac{sint}{t}$, which is continuous everywhere if defined to be 1 at $t = 0$.

2 (10 points)

(a) (6 points) Solve the following differential equation for $y(x)$:

$$\frac{dy}{dx} = 2(1 + x)y,\ y(0) = 1$$

Separate variables to get

$$\frac{dy}{y} = 2(1 + x)dx,$$

then integrate both sides to get

$$\ln|y| = 2x + x^2 + C.$$

Since $y(0) = 1$, $C = 0$ (or if you exponentiated before finding the constant, you get the $C = 1$), and when you solve for $y$, you get, $y(x) = e^{2x+x^2}$.

(b) (4 points) There is one critical point for $y(x)$. Find it, and classify it as either a maximum or a minimum.
Looking at the differential equation, you see that a critical point must occur at either $x = -1$ or $y = 0$. However, since the answer here is an exponential, $y$ will never be zero, so the only critical point is -1. Furthermore, if you differentiate the equation again, you obtain

$$y'' = 2y + 2(1 + x)y'$$

. But $y' = 0$ at a critical point, and $y$ is an exponential, so it's always positive, so $y'' > 0$, and the critical point is a minimum.

3 (10 points) Consider the differential equation:

$$(e^{-x} + \frac{1}{x+y} + axy)dx + (\frac{1}{x+y} + x^2 + 1)dy = 0.$$

Find the value of a that makes this equation exact, and then using that value of $a$, solve the differential equation.

$M(x,y) = e^{-x} + \frac{1}{x+y} + axy$, and $N(x,y) = \frac{1}{x+y} + x^2 + 1$, and to be exact, we need $M_y = N_x$, which means

$$M_y = 0 + \frac{-1}{(x+y)^2} + ax = \frac{-1}{(x+y)^2} + 2x = N_x.$$

This happens when $a = 2$. Integrating $M(x,y)$ with respect to $x$ then gives

$$\Psi(x,y) = -e^{-x} + \ln|x+y| + x^2y + f(y).$$

Differentiating this with respect to $y$ gives

$$\Psi_y(x,y) = \frac{1}{x+y} + x^2 + f'(y) = \frac{1}{x+y} + x^2 + 1.$$

So $f'(y) = 1$, or $f(y) = y + C$, and the solution is:

$$\Psi(x,y) = -e^{-x} + \ln|x+y| + x^2y + y + C.$$

4 (8 points) Consider the differential equation:

$$\frac{dy}{dt} = y(e^y - 1)(y^2 - a), \text{ where } a > 0.$$

Identify and classify (as stable, unstable, or semistable) the equilibrium solutions to this equation.

The zeros of this equation are the equilibrium solutions. These occur at the points $-\sqrt{a}, 0,$ and $\sqrt{a}$. Off towards infinity $f(y)$ is positive. It has a single root at $\pm\sqrt{a}$ and a double root at zero. So when it passes through $\sqrt{a}$ it changes sign, negative on the right, positive on the left, making $\sqrt{a}$ unstable. Zero is a double root, so the equation doesn't change sign as $y$ passes through zero, so it is semistable. It changes sign though going through $-\sqrt{a}$, making it negative on the right and positive on the left, a stable point.

5 (12 points) Fred is sick of mixing salt, but doesn't have a whole lot of imagination, so he decides to mix sugar in water. He's got a tank of 100 gallons of pure water hanging around, and it has a capacity of 400 gallons. So he starts pumping in 3 gallons of water per minute of a solution containing 1 pound of sugar per gallon. He also pumps water out of the tank, only at 1 gallon of water per minute.

(a) (7 points) Give a function describing how much sugar is in the tank at any time $t$.

We've set up differential equations like this before. The setup is just like problem #4 in the homework, and you have

$$\frac{dQ}{dt} = 3 - \frac{Q}{100 + 2t}.$$

The coefficient on $t$ is two because you have a net gain of 2 gallons of water per minute. This is a linear equation, so put it in that form and find the integrating factor:

$$\frac{dQ}{dt} + \frac{1}{100 + 2t} Q = 3,$$

$$\mu(t) = e^{\int \frac{1}{100+2t}} = e^{\frac{1}{2} \ln 100+2t} = e^{\ln (100+2t)^{\frac{1}{2}}} = \sqrt{100 + 2t}$$

So multiplying in the integrating factor gives

$$(\sqrt{100 + 2t} Q)' = 3\sqrt{100 + 2t}$$

and integrating both sides gives

$$\sqrt{100 + 2t} Q = (100 + 2t)^{\frac{3}{2}} + C$$

$Q(0) = 0$, so $C = -1000$. This all gives us that

$$Q(t) = 100 + 2t - \frac{1000}{\sqrt{100 + 2t}}.$$

(b) (3 points) Find the amount of sugar in the tank when it reaches capacity. How does this amount compare to the total amount of sugar that has been put into the tank?

The tank reaches capacity when the amount of water in it is 400 gallons. This happens when $t = 150$. Simply plug in $Q(150) = 400 - \frac{1000}{\sqrt{400}} = 350$. Compare this to how much sugar has been put in the tank: $3 * 150 = 450$.

(c) (2 points) If the tank had infinite capacity, find the limiting concentration of sugar in the tank.

Look at $\frac{Q(t)}{100 + 2t}$ (quantity of sugar divided by total amount of water at the time) $= 1 - \frac{1000}{(100 + 2t)^{\frac{3}{2}}}$. This clearly goes to 1 as $t$ goes to infinity.