

1 (10 points) Find the general solution of the following differential equation:

$$y'' + y' = 2t - e^{-t} - 2 \sin t$$

First find the homogeneous solution. The characteristic equation is  $\alpha^2 + \alpha = 0$ , which has roots  $\alpha = 0, \alpha = -1$ . So then  $y_h(t) = c_1 + c_2 e^{-t}$ .

Now make the initial guess for the inhomogeneous part of the solution:

$$Y(t) = At + B + Ce^{-t} + D \cos t + E \sin t$$

The polynomial term and the exponential term have redundancies with the homogeneous solution, so each must be multiplied by  $t$ :

$$Y(t) = At^2 + Bt + Cte^{-t} + D \cos t + E \sin t$$

$$Y'(t) = 2At + B + Ce^{-t} - Cte^{-t} - D \sin t + E \cos t$$

$$Y''(t) = 2A - 2Ce^{-t} + Cte^{-t} - D \cos t - E \sin t$$

Now plug this in to the differential equation:

$$2A - 2Ce^{-t} + Cte^{-t} - D \cos t - E \sin t + 2At + B + Ce^{-t} - Cte^{-t} - D \sin t + E \cos t = 2t - e^{-t} - 2 \sin t$$

Looking at this you immediately get that  $C = 1, A = 1$ , and so  $B = -2$ . We also have that  $-D + E = 0$  and  $-E - D = -2$ , so  $D = E = 1$ . So this gives our solution,

$$y(t) = c_1 + c_2 e^{-t} + t^2 - 2t + e^{-t} \cos t + \sin t$$

2 (10 points) Solve the initial value problem

$$y'' + 6y' + 9y = te^{-3t}, \quad y(0) = 1/3, \quad y'(0) = 0$$

using variation of parameters. Recall that

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)} dt,$$

where  $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ .

The homogeneous solution is  $y_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}$ . The Wronskian of this solution is:  $e^{-3t}(e^{-3t} - 3te^{-3t}) - te^{-3t}(-3)e^{-3t} = e^{-6t}$ .

$$Y(t) = -e^{-3t} \int \frac{te^{-3t}te^{-3t}}{e^{-6t}} dt + te^{-3t} \int \frac{e^{-3t}te^{-3t}}{e^{-6t}} dt$$

So  $Y(t) = -e^{-3t} \int t^2 dt + te^{-3t} \int t dt = -\frac{1}{3}t^3 e^{-3t} + \frac{1}{2}t^3 e^{-3t} = \frac{1}{6}t^3 e^{-3t}$

This gives that  $y(t) = c_1 e^{-3t} + c_2 t e^{-3t} + \frac{1}{6}t^3 e^{-3t}$ .  $y(0) = 1/3$ , so this immediately gives that  $c_1 = 1/3$ .  $y'(0) = 0$ , and  $y'(t) = -3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t} + \frac{1}{2}t^2 e^{-3t} - \frac{1}{2}t^3 e^{-3t}$ , so then  $0 = -1 + c_2$ , or  $c_2 = 1$ . This gives us the answer:

$$y(t) = \frac{1}{3}e^{-3t} + te^{-3t} + \frac{1}{6}t^3 e^{-3t}$$

- 3 (10 points) Find the general solution of the following differential equation:

$$y^v + y^{iv} - y''' - 3y'' + 2y = 0$$

Okay, this has characteristic equation  $\alpha^5 + \alpha^4 - \alpha^3 - 3\alpha^2 + 2 = 0$ . There are two main ways to do this one way is to see that  $\alpha = 1$  and  $\alpha = -1$  both work, and differentiating you get that  $\alpha = 1$  is in fact a double root. If you like, you could then multiply all those terms together, getting  $(\alpha - 1)^2(\alpha + 1) = (\alpha - 1)(\alpha^2 - 1) = \alpha^3 - \alpha^2 - \alpha + 1$ , then use long division to just end up with a quadratic that can be handled with the quadratic equation.

The other way to do this would be to shed off each root as you find them. The easiest roots to find are always  $\alpha = 1$ , so divide off that root:  $(\alpha - 1)(\alpha^4 + 2\alpha^3 + \alpha^2 - 2\alpha - 2)$ . This new polynomial also has 1 as a root, so divide it off as well:  $(\alpha - 1)^2(\alpha^3 + 3\alpha^2 + 4\alpha + 2)$ . This cubic has a root of -1, so divide that off:  $(\alpha - 1)^2(\alpha + 1)(\alpha^2 + 2\alpha + 2)$ . This final polynomial is a quadratic that can't be factored, so using the quadratic formula you get:

$$\alpha = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$$

This give that the general solution of this equation is:

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t} + e^{-t}[c_4 \cos t + c_5 \sin t]$$

- 4 (10 points) Consider the following differential equation:

$$x^2 y'' + (2a + 1)xy' + (a^2 - a)y = 0$$

Find all values of  $a$  such that the solution to this differential equation goes to zero as  $x \rightarrow \infty$ . For these values of  $a$ , solve the differential equation.

The characteristic equation for this differential equation is  $r(r-1) + (2a+1)r + a^2 - a = 0$ , or  $r^2 + 2ar + a^2 - a = 0$ . Plug this in to the quadratic equation to get  $r = \frac{-2a \pm \sqrt{(2a)^2 - 4(a^2 - a)}}{2} = -a \pm \sqrt{a}$ . So if you want a solution that decays as  $x \rightarrow \infty$ , you need both of these roots to be negative. The original intent was for this to split in cases of  $a > 0$ ,  $a < 0$ , and  $a = 0$ . However, for  $0 < a \leq 1$ , there is one positive root and one negative root, which decays at neither 0 nor  $\infty$ . According to the original intent, the case  $a > 0$  is the one that decays as  $x \rightarrow \infty$ , since it has to negative roots. It then has solution

$$y(x) = c_1 x^{-a+\sqrt{a}} + c_2 x^{-a-\sqrt{a}}$$

. If  $a < 0$ , then the solution decays at zero since  $-a$  is then positive and  $\sqrt{a}$  is imaginary. The solution is then:

$$y(x) = x^{-a}[c_1 \cos(\sqrt{-a} \ln x) + c_2 \sin(\sqrt{-a} \ln x)]$$

If  $a = 0$ , we have a double root of zero, so the solution is

$$y(x) = c_1 + c_2 \ln x$$

- 5 (10 points) You are given a weight. You have a spring handy, and when you attach weight to the spring, it stretches the spring 15cm.

- (a) (2 points) Find the undamped frequency of the spring. Your answer should not involve the mass. It will help if you write the numbers that appear in fraction form.

$$\omega_0 = \sqrt{\frac{k}{m}}, \text{ and } mg = kL, \text{ so } \omega_0 = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8}{.15}} = \frac{14}{\sqrt{3}}.$$

- (b) (4 points) You also happen to have a dashpot handy (an object that imposes a damping force), and it gives a resistive force of 7 Newtons when presented with a velocity of 1m/s. When the dashpot is attached to the spring with the weight on the end, the spring has quasi-frequency that is half the undamped frequency of the spring. Find the mass of the object on the end of the spring. It will be a very simple number, and it may help if you don't plug in any numbers until you have a formula solved in terms of the mass.

Pretty clearly  $\gamma = 7$ . Okay, so the general differential equation here is  $mu'' + \gamma u' + ku = 0$ , but here we have that  $k = \frac{mg}{L}$ , so plugging that in and dividing everything by  $m$  gives

$$u'' + \frac{\gamma}{m}u' + \frac{g}{L}u = 0 \text{ So the characteristic equation has roots } \alpha = \frac{-\gamma}{2m} \pm \frac{\sqrt{\left(\frac{\gamma}{m}\right)^2 - \frac{4g}{L}}}{2}$$

However, since we've got a quasi-frequency, that means that the discriminant is negative,

so if we actually want to find the frequency, it is  $\mu = \frac{\sqrt{\frac{4g}{L} - \left(\frac{\gamma}{m}\right)^2}}{2}$ . This is supposed to be one-half the undamped frequency, or  $\mu = \frac{1}{2}\omega_0$ . This gives you:

$$\frac{\sqrt{\frac{4g}{L} - \left(\frac{\gamma}{m}\right)^2}}{2} = \frac{1}{2}\sqrt{\frac{g}{L}}$$

$$\frac{4g}{L} - \left(\frac{\gamma}{m}\right)^2 = \frac{g}{L}$$

$$\frac{3g}{L} = \left(\frac{\gamma}{m}\right)^2$$

$$m^2 = \frac{L\gamma^2}{3g} = \frac{(.15)(49)}{3(9.8)} = \frac{(15)(49)(10)}{3(100)(98)} = \frac{(15)(10)}{3(100)(2)} = \frac{1}{4}$$

So  $m = \frac{1}{2}kg$ .

- (c) (4 points) Now that you have all the information about the spring, give the function of the displacement of the mass from equilibrium if it is started from equilibrium with a downward velocity of 1m/s.

All the stuff we've gotten so far gives us that the solution of the differential equation must be  $u(t) = e^{-\gamma t} \left[ c_1 \cos \frac{7}{\sqrt{3}}t + c_2 \sin \frac{7}{\sqrt{3}}t \right]$ . Keep in mind that all of this can be done (except for the power of the exponential) *without* ever finding  $m$ . After all, the quasi-frequency is half the undamped frequency, which you found in part (a). Now, the initial conditions are  $u(0) = 0$ ,  $u'(0) = 1$ . The first condition gives that  $c_1 = 0$ . The second condition gives that  $c_2 = \frac{\sqrt{3}}{7}$ , and the solution is:

$$u(t) = \frac{\sqrt{3}}{7}e^{-\gamma t} \sin \frac{7}{\sqrt{3}}t$$