

1 (10 points) Consider the function

$$f(x, y) = x^4 - 2x^2 + y^2 + 2$$

(a) Find the critical points of this function and classify them.

First find the first derivatives and set them equal to zero.  $f_x(x, y) = 4x^3 - 4x = 4x(x - 1)(x + 1)$ .  $f_y(x, y) = 2y$ . So the critical points are when these are zero at the same time. In other words, they are the points  $(0, 0)$ ,  $(1, 0)$ , and  $(-1, 0)$ . To classify them, the second derivatives of  $f$  are needed.  $f_{xx}(x, y) = 12x^2 - 4$ ,  $f_{yy}(x, y) = 2$ , and  $f_{xy}(x, y) = 0$ . Plugging these in to the classification formula  $D = f_{xx}f_{yy} - [f_{xy}]^2$ , we see that  $D(0, 0) < 0$ , so this is a saddle point. For  $(1, 0)$  and  $(-1, 0)$ ,  $D > 0$  and  $f_{xx} > 0$ , so they are both local minima.

(b) Consider the function  $\vec{r} = \langle \sqrt{t}, t \rangle$ , defined for  $t > 0$ . Using the chain rule, find the single critical point of  $f$  along  $\vec{r}$ .

As for this problem, recall the chain rule formula:  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ . In part (a) the partials of  $f$  with respect to  $x$  and  $y$  have already been found, so most of what is needed is the derivatives of  $x$  and  $y$  with respect to  $t$ . This is obtained by calculating  $\vec{r}'(t)$ .  $\vec{r}'(t) = \langle x'(t), y'(t) \rangle = \langle \frac{1}{2\sqrt{t}}, 1 \rangle$ . Put this together, making sure to evaluate the partials of  $f$  at  $t$

$$\begin{aligned} \frac{df}{dt} &= \left( 4t^{3/2} - 4t^{1/2} \right) \cdot \frac{1}{2t^{1/2}} + 2t \cdot 1 \\ &= 4t - 2 \end{aligned}$$

This is zero when  $t = \frac{1}{2}$ , or at the point  $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ . The  $f$  value here is 2.

2 (10 points) Find the minimum of  $f(x, y, z) = x^2 + y^2 + z^2$  restricted to the surfaces  $x + y = 4$  and  $y + z = 6$ .

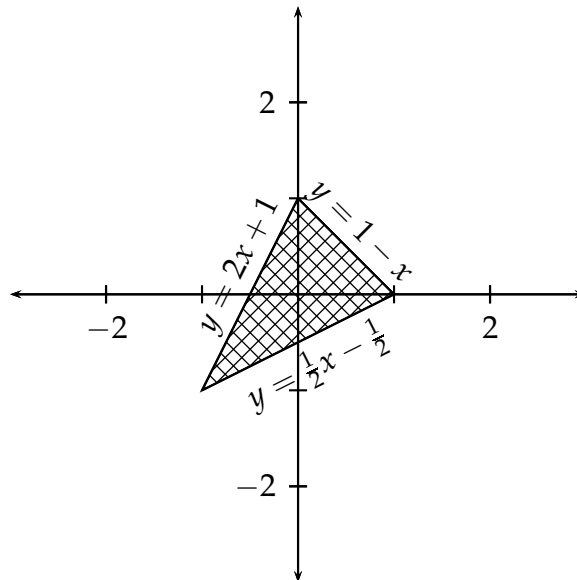
This is all about Lagrange multipliers, though it is technically possible to do without them. Say  $4 = g(x, y, z) = x + y$  and  $6 = h(x, y, z) = y + z$ . Take the partial derivatives and put them together in the form  $\nabla f = \lambda \nabla g + \mu \nabla h$ . This gives the set of equations:

$$2x = \lambda, \quad 2y = \lambda + \mu, \quad 2z = \mu, \quad x + y = 4, \quad y + z = 6$$

This quickly gives that  $x + z = y$ , and using this the equations are solved, giving that the only critical point is  $(\frac{2}{3}, \frac{10}{3}, \frac{8}{3})$   $f$  at this point is  $\frac{56}{3}$ .

- 3 (10 points) Find the integral of the function  $f(x, y) = 12x$  on the triangle bounded by the points  $(0, 1)$ ,  $(1, 0)$ , and  $(-1, -1)$ .

First things first, draw the picture.



With this picture in hand, it is clearer that the integral must be split into two pieces.

$$\begin{aligned}
 & \int_{-1}^0 \int_{\frac{1}{2}x - \frac{1}{2}}^{2x+1} 12x \, dy \, dx + \int_{-1}^0 \int_{\frac{1}{2}x - \frac{1}{2}}^{1-x} 12x \, dy \, dx \\
 &= \int_{-1}^0 12x \left( 2x + 1 - \frac{1}{2}x + \frac{1}{2} \right) dx + \int_0^1 12x \left( 1 - x - \frac{1}{2}x + \frac{1}{2} \right) dx \\
 &= \int_{-1}^0 18x^2 + 18x \, dx + \int_0^1 -18x^2 + 18x \, dx \\
 &= \left[ 6x^3 + 9x^2 \right]_{-1}^0 + \left[ -6x^3 + 9x^2 \right]_0^1 \\
 &= 6 - 9 - 6 + 9 \\
 &= 0
 \end{aligned}$$

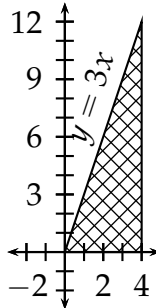
4 (10 points) Evaluate

$$\int_0^{12} \int_{y/3}^4 \frac{1}{\sqrt{x^2+9}} dx dy$$

This integral in  $x$  is possible (though difficult). However,

$$\int \frac{1}{\sqrt{x^2+9}} dx = \ln \left| \sqrt{x^2+9} + x \right| + C,$$

is difficult to integrate again. So it is best to switch the limits of integration. To do that, first draw the picture.



This allows you to see the new limits of integration.

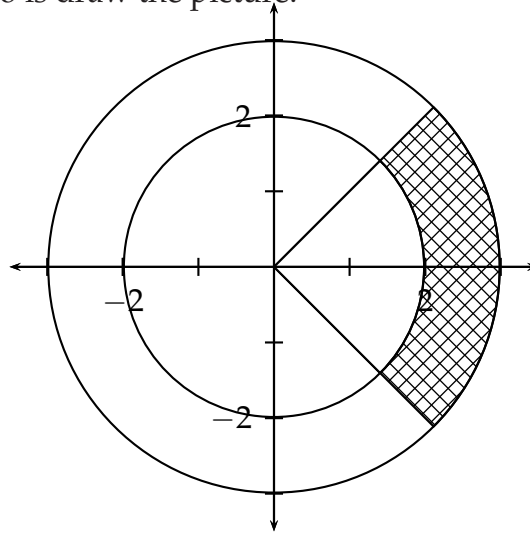
$$\begin{aligned} \int_0^{12} \int_{y/3}^4 \frac{1}{\sqrt{x^2+9}} dx dy &= \int_0^4 \int_0^{3x} \frac{1}{\sqrt{x^2+9}} dx dy \\ &= \int_0^4 \frac{3x}{\sqrt{x^2+9}} dx \\ &= \left[ 3\sqrt{x^2+9} \right]_0^4 \\ &= 15 - 9 \\ &= 6. \end{aligned}$$

5 (10 points) Evaluate

$$\iint_D \frac{x}{x^2 + y^2} dA$$

where  $4 \leq x^2 + y^2 \leq 9$  and  $-x \leq y \leq x$ .

Of course the first thing to do is draw the picture.



With this done it becomes clear what the limits must be in polar coordinates. So make the switch.

$$\begin{aligned} \iint_D \frac{x}{x^2 + y^2} dA &= \int_{-\pi/4}^{\pi/4} \int_2^3 \frac{r \cos \theta}{r^2} r dr d\theta \\ &= \int_{-\pi/4}^{\pi/4} \int_2^3 \cos \theta dr d\theta \\ &= \int_{-\pi/4}^{\pi/4} \cos \theta d\theta \\ &= [\sin \theta]_{-\pi/4}^{\pi/4} \\ &= \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \\ &= \sqrt{2} \end{aligned}$$