

1 (10 points) Evaluate

$$\int_C x e^y ds,$$

along the curve  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$  from  $t = 0$  to  $t = \frac{\pi}{2}$ .

First things first.  $\int_C f ds = \int_C f(\vec{r}(t)) \|\vec{r}'(t)\| dt$ . So then,  $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ , and so  $\|\vec{r}'(t)\| = \sqrt{2}$ . The integral then becomes:

$$\begin{aligned} \int_C x e^y ds &= \int_0^{\frac{\pi}{2}} \cos t e^{\sin t} \sqrt{2} dt \\ & \quad u = \sin t \quad du = \cos t \\ &= \int_0^1 e^u \sqrt{2} du \\ &= \boxed{\sqrt{2}(e - 1)} \end{aligned}$$

2 (10 points) Evaluate

$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 x^2 - y dz dy dx.$$

The correction is made so that (in theory) this problem can be done in either cylindrical or spherical coordinates. However, most people had worked the problem before the correction was made, so let's do it that way so you can check your work:

$$\begin{aligned} & \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 x^2 - y dz dy dx \\ &= \int_0^\pi \int_0^2 \int_r^4 (r^2 \cos^2 \theta - r \sin \theta) r dz dr d\theta \\ &= \int_0^\pi \int_0^2 (r^2 \cos^2 \theta - r \sin \theta) r(4-r) dr d\theta \\ &= \int_0^\pi \int_0^2 (4r^3 - r^4) \cos^2 \theta - (4r^2 - r^3) \sin \theta dr d\theta \\ &= \int_0^\pi \left[ \left( r^4 - \frac{1}{5} r^5 \right) \cos^2 \theta - \left( \frac{4}{3} r^3 - \frac{1}{4} r^4 \right) \sin \theta \right]_{r=0}^{r=2} d\theta \\ &= \int_0^\pi \left( 16 - \frac{32}{5} \right) \cos^2 \theta - \left( \frac{32}{3} - 4 \right) \sin \theta d\theta \\ &= \int_0^\pi \left( 8 - \frac{16}{5} \right) (1 + \cos(2\theta)) - \left( \frac{32}{3} - 4 \right) \sin \theta d\theta \\ &= \left[ \left( 8 - \frac{16}{5} \right) \left( \theta + \frac{1}{2} \sin(2\theta) \right) + \left( \frac{32}{3} - 4 \right) \cos \theta \right]_0^\pi \\ &= \left( 8 - \frac{16}{5} \right) \pi - \frac{64}{3} + 8 \\ &= \boxed{\frac{24\pi}{5} - \frac{40}{3}} \\ &\approx 1.746 \end{aligned}$$

As for doing the problem after the correction, first see what it is with the upper  $z$  limit as 2:

$$\begin{aligned}
 & \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 x^2 - y \, dz \, dy \, dx \\
 &= \int_0^\pi \int_0^2 \int_r^2 (r^2 \cos^2 \theta - r \sin \theta) r \, dz \, dr \, d\theta \\
 &= \int_0^\pi \int_0^2 (r^2 \cos^2 \theta - r \sin \theta) r(2-r) \, dr \, d\theta \\
 &= \int_0^\pi \int_0^2 (2r^3 - r^4) \cos^2 \theta - (2r^2 - r^3) \sin \theta \, dr \, d\theta \\
 &= \int_0^\pi \left[ \left( \frac{1}{2} r^4 - \frac{1}{5} r^5 \right) \cos^2 \theta - \left( \frac{2}{3} r^3 - \frac{1}{4} r^4 \right) \sin \theta \right]_{r=0}^{r=2} d\theta \\
 &= \int_0^\pi \left( 8 - \frac{32}{5} \right) \cos^2 \theta - \left( \frac{16}{3} - 4 \right) \sin \theta \, d\theta \\
 &= \int_0^\pi \left( 4 - \frac{16}{5} \right) (1 + \cos(2\theta)) - \left( \frac{16}{3} - 4 \right) \sin \theta \, d\theta \\
 &= \left[ \left( 4 - \frac{16}{5} \right) \left( \theta + \frac{1}{2} \sin(2\theta) \right) + \left( \frac{16}{3} - 4 \right) \cos \theta \right]_0^\pi \\
 &= \left( 4 - \frac{16}{5} \right) \pi - \frac{32}{3} + 8 \\
 &= \boxed{\frac{4\pi}{5} - \frac{8}{3}} \\
 &\approx -.1534
 \end{aligned}$$

Now let's do this punk in spherical. It will be scary, so consider yourself warned. The important thing to understand is that since this is a cone with a flat top, the  $\rho$  limit is not a constant:

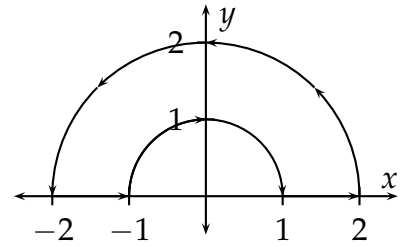
$$\begin{aligned}
 & \int_0^\pi \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos \phi}} (\rho^2 \sin^2 \phi \cos^2 \theta - \rho \sin \phi \sin \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos \phi}} \rho^4 \sin^3 \phi \cos^2 \theta - \rho^3 \sin^2 \phi \sin \theta \, d\rho \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^{\frac{\pi}{4}} \frac{32 \sin^3 \phi}{5 \cos^5 \phi} \cos^2 \theta - 4 \frac{\sin^2 \phi}{\cos^4 \phi} \sin \theta \, d\phi \, d\theta \\
 &= \int_0^\pi \int_0^{\frac{\pi}{4}} \frac{32}{5} \tan^3 \phi \sec^2 \phi \cos^2 \theta - 4 \tan^2 \phi \sec^2 \phi \sin \theta \, d\phi \, d\theta \\
 &\quad u = \tan \phi, \quad du = \sec^2 \phi \, d\phi \\
 &= \int_0^\pi \int_0^1 \frac{32}{5} u^3 \cos^2 \theta - 4u^2 \sin \theta \, du \, d\theta \\
 &= \int_0^\pi \frac{8}{5} \cos^2 \theta - \frac{4}{3} \sin \theta \, d\theta \\
 &= \int_0^\pi \frac{4}{5} (1 + \cos(2\theta)) - \frac{4}{3} \sin \theta \, d\theta \\
 &= \boxed{\frac{4\pi}{5} - \frac{8}{3}}
 \end{aligned}$$

3 (10 points)

Evaluate the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where  $\vec{F}(x, y) = \langle x^2y, x^3 + e^y \rangle$ , over the curve  $C$ , defined by going in a straight line from  $(1, 0)$  to  $(2, 0)$ , from here to  $(-2, 0)$  counterclockwise along the top of the circle  $x^2 + y^2 = 4$ , then in a straight line to  $(-1, 0)$ , then clockwise along the top of the circle  $x^2 + y^2 = 1$  back to  $(1, 0)$ . See the picture.



This is a Green's Theorem problem, since the line integral is over a closed curve. And the  $e^y$  in the second component is nasty.

$$\begin{aligned} & \int_C x^2y \, dx + (x^3 + e^y) \, dy \\ &= \iint_D 3x^2 - x^2 \, dA \\ &= \int_0^\pi \int_1^2 2r^2 \cos^2 \theta r \, dr \, d\theta \\ &= \int_1^2 r^3 \, dr \int_0^\pi 1 + \cos(2\theta) \, d\theta \\ &= \left[ \frac{1}{4}r^4 \right]_1^2 \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_0^\pi \\ &= \boxed{\frac{15\pi}{4}} \end{aligned}$$

- 4 (10 points) Find the volume between the planes  $z = y$ ,  $z = 2 - y$ , and the function  $z = x^2$ . Be careful setting up the integral.

An important piece of this problem is nowhere does it say that the shape is bounded by any of the coordinate axes. The shape is below the two planes, but above the parabola. The easiest way to do this is to integrate in  $y$ , then  $z$ , then  $x$ :

$$\begin{aligned}
 \int_{-1}^1 \int_{x^2}^1 \int_z^{2-z} dy dz dx &= \int_{-1}^1 \int_{x^2}^1 2 - 2z dz dx \\
 &= \int_{-1}^1 \left[ 2x - z^2 \right]_{z=x^2}^{z=1} dx \\
 &= \int_{-1}^1 1 - 2x^2 + x^4 dx \\
 &= \left[ x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^1 \\
 &= \boxed{\frac{16}{15}}
 \end{aligned}$$

Many people tried to integrate in  $z$  first, but in order to do that, you *must* split the integral up into two pieces. Some people did this, but failed to realize that the second limits are also a little weird:

$$\begin{aligned}
 \int_{-1}^1 \int_{x^2}^1 \int_{x^2}^y dz dy dx + \int_{-1}^1 \int_1^{2-x^2} \int_{x^2}^{2-y} dz dy dx \\
 &= \int_{-1}^1 \int_{x^2}^1 y - x^2 dy dx + \int_{-1}^1 \int_1^{2-x^2} 2 - y - x^2 dy dx \\
 &= \int_{-1}^1 \left[ \frac{1}{2}y^2 - x^2y \right]_{x^2}^1 dx + \int_{-1}^1 \left[ 2y - \frac{1}{2}y^2 - x^2y \right]_1^{2-x^2} dx \\
 &= \int_{-1}^1 1 - 2x^2 + x^4 dx \\
 &= \boxed{\frac{16}{15}}
 \end{aligned}$$

Well, if we've done these, might as well integrate in  $x$  first. Some people tried that one:

$$\begin{aligned}
 \int_0^1 \int_z^{2-z} \int_{-\sqrt{z}}^{\sqrt{z}} dx dy dz &= \int_0^1 \int_z^{2-z} 2\sqrt{z} dy dz \\
 &= \int_0^1 2\sqrt{z}(2 - 2z) dz \\
 &= \int_0^1 4z^{1/2} - 4z^{3/2} dz \\
 &= \frac{8}{3} - \frac{8}{5} \\
 &= \boxed{\frac{16}{15}}
 \end{aligned}$$

5 (10 points) Evaluate the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

where  $\vec{F}(x, y, z) = \left\langle y \cos(xy) + \frac{1}{z}, x \cos(xy) + 2yz, -\frac{x}{z^2} + y^2 \right\rangle$  and  $\vec{r}(t) = \langle t^2, \cos^2(\pi t), 2t \rangle$ , and  $t$  goes from 0 to  $\frac{1}{2}$ .

Fundamental theorem for line integrals. No one who tried it the straightforward way got beyond setting up the integral. So, we want a function  $f$  where  $\nabla f = \vec{F}$ . Taking antiderivatives however you like to do so, and checking to make sure that it works all the way across gives that

$$f(x, y, z) = \sin(xy) + \frac{x}{z} + y^2z.$$

If you did this correctly, that was pretty much the whole problem. At this point, in theory,

$$\int_C \vec{F} \cdot d\vec{r} = f(1/4, 0, 1) - f(0, 1, 0),$$

but plugging in the second point has you dividing by zero. This is complicated by the fact that if you plug it in instead as  $f(\vec{r}(1/2)) - f(\vec{r}(0))$ , the part where you divide by zero cancels, and everything looks okay. So it was changed to evaluating from  $1/2$  to  $1$ , which gives:

$$f(1, 1, 2) - f(1/4, 0, 1) = \boxed{\sin(1) + \frac{9}{4}}$$