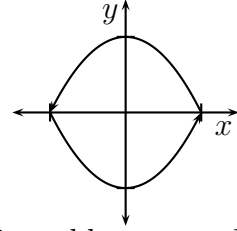


1 (10 points)

Evaluate

$$\int_C -xy \, dx + \frac{1}{2}x^2 \, dy,$$

where C is the curve given by $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$ for $-1 \leq t \leq 1$ and $\mathbf{r}(t) = \langle 2 - t, -t^2 + 4t - 3 \rangle$ for $1 \leq t \leq 3$. See the figure to the right.



The mistakes in the original version's parametric equations screwed this problem up, so I tried to be lenient on this problem. This is the way it was meant to go.

First notice that the first parametric equation is the equation $y = x^2 - 1$, tracing along the bottom from left to right. Then notice that the second equation has $x = 2 - t$, or $t = 2 - x$. Plugging this in to the second half gives $y = -(2-x)^2 + 4(2-x) - 3 = -4 + 4x - x^2 + 8 - 4x - 3 = 1 - x^2$, the parabola on the top, going right to left.

This is a closed loop, so Green's Theorem will make this easier.

$$\begin{aligned} \int_C -xy \, dx + \frac{1}{2}x^2 \, dy &= \iint_D \frac{\partial}{\partial x} \left(\frac{1}{2}x^2 \right) - \frac{\partial}{\partial y} (-xy) \, dA \\ &= \int_{-1}^1 \int_{x^2-1}^{1-x^2} 2x \, dy \, dx \\ &= \int_{-1}^1 2x(2 - 2x^2) \, dx \\ &= \int_{-1}^1 4x - 4x^3 \, dx \\ &= [2x^2 - x^4]_{-1}^1 \\ &= 2 - 1 - (2 - 1) = \boxed{0} \end{aligned}$$

2 (10 points)

Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

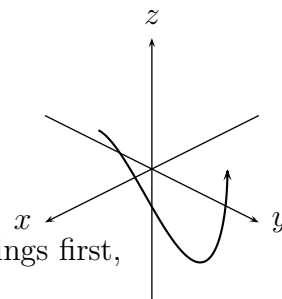
where $\mathbf{F}(x, y, z) = \langle -yz, xz, x^2 \rangle$ and C is the curve defined by $\mathbf{r}(\theta) = \langle \cos \theta, \sin \theta, \cos(2\theta) \rangle$ where $0 \leq \theta \leq \pi/4$.

This problem is just a problem in calculating a line integral. First things first,

$$\mathbf{r}'(\theta) = \langle -\sin \theta, \cos \theta, -2\sin(2\theta) \rangle.$$

With this, the integral can be found.

$$\begin{aligned} & \int_0^{\pi/4} \langle -\sin \theta \cos(2\theta), \cos \theta \cos(2\theta), \cos^2 \theta \rangle \cdot \langle -\sin \theta, \cos \theta, -2\sin(2\theta) \rangle d\theta \\ &= \int_0^{\pi/4} \cos(2\theta) - 2\cos^2 \theta \sin(2\theta) d\theta \\ &= \int_0^{\pi/4} \cos(2\theta) - \sin(2\theta) - \cos(2\theta) \sin(2\theta) d\theta \\ &= \left[\frac{1}{2} \sin(2\theta) + \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos^2(2\theta) \right]_0^{\pi/4} d\theta \\ &= \frac{1}{2} + 0 + 0 - \left(0 + \frac{1}{2} + \frac{1}{4} \right) = \boxed{-\frac{1}{4}} \end{aligned}$$

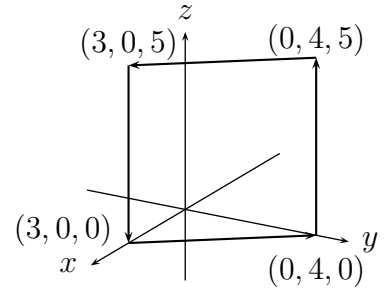


3 (10 points)

Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}(x, y, z) = \langle e^x + z, \cos y + xy, z^2 + y \rangle$ and C is the curve which is the union of four lines; starting at $(3, 0, 0)$, going to $(0, 4, 0)$, then to $(0, 4, 5)$, then to $(3, 0, 5)$, and back to $(3, 0, 0)$. Do this without calculating four line integrals.



Closed loop? Yeah, this is a Stokes's Theorem problem. It doesn't take much to notice the shape here is a square since each line segment is five units long. Since we're changing to a surface integral, we need to parameterize this surface. Using the plane parameterization from class, I use my reference point $(3, 0, 0)$, and take the two vectors from there to adjacent points:

$$\mathbf{r}(u, v) = \langle 3, 0, 0 \rangle + u\langle -3, 4, 0 \rangle + v\langle 0, 0, 5 \rangle.$$

Note that u and v in this parameterization go from 0 to 1. This is not the only parameterization, but this is how I did it. Having done this, the normal vector is needed:

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 0 \\ 0 & 0 & 5 \end{vmatrix} \\ &= \langle 20, 15, 0 \rangle \end{aligned}$$

Observe that this vector does point in the correct direction. It is pointing in the positive x and y direction, which fits with the orientation provided by the closed loop. Now we also need the curl of \mathbf{F} .

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ e^x + z & \cos y + xy & z^2 + y \end{vmatrix} \\ &= \langle 1, 1, y \rangle \end{aligned}$$

At this point all the hard stuff is done, it's just a matter of calculating the integral using Stokes's Theorem.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \int_0^1 35 \, du \, dv \\ &= \boxed{35} \end{aligned}$$

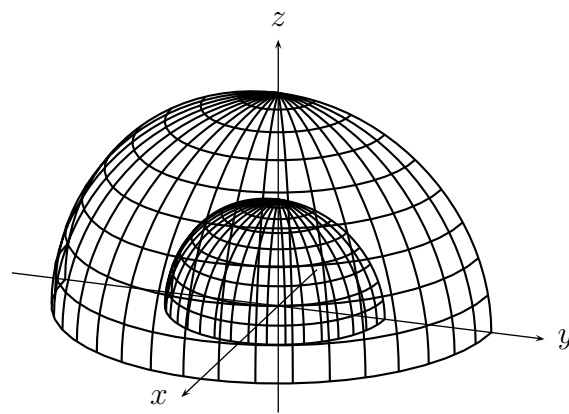
4 (10 points)

Evaluate

$$\iiint_D \sqrt{x^2 + y^2} \, dV,$$

where D is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$; and where $z \geq 0$ and $x \geq 0$.

Since the last test had a cylindrical coordinates problem, I figured a spherical problem was in order. So that's what this is all about.



Drawing a picture makes it easier to see what the limits are in each direction.

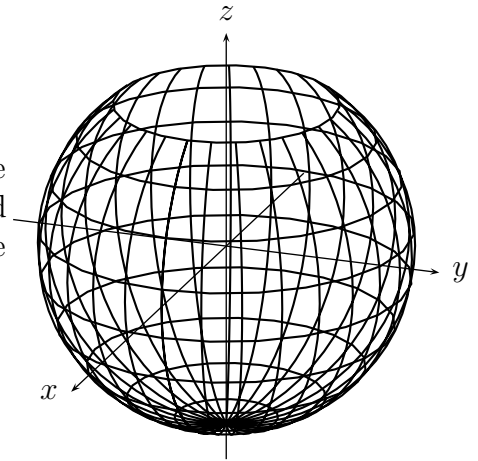
$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho^3 \sin^2 \phi \, d\rho \, d\phi \, d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} \rho^4 \right]_1^2 \sin^2 \phi \, d\phi \, d\theta \\ &= \frac{15\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{1}{2} \cos(2\phi) \, d\phi \\ &= \frac{15\pi}{4} \left[\frac{\phi}{2} - \frac{\sin(2\phi)}{4} \right]_0^{\frac{\pi}{2}} \\ &= \boxed{\frac{15\pi}{16}} \end{aligned}$$

5 (10 points)

Evaluate

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle zy, 2zx, xy \rangle$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 4$. The surface is oriented inward towards the origin. Do this without integrating over the surface of the sphere.



Seeing the curl of a function with a surface integral is a dead giveaway that this is another Stokes's Theorem problem. In theory you could change this to the line integral and calculate that, but I find it much easier to calculate this integral across the disk at the top of the partial sphere.

Having the surface oriented towards the origin seems weird at first. But if you imagine deforming the sphere until it becomes the disk on top, you see that that is just the orientation required to have the normal vector to the lid facing upwards. The parameterization of the lid is just $\mathbf{r}(\rho, \theta) = \langle \rho \cos \theta, \rho \sin \theta, 4 \rangle$ with ρ going from 0 to 3 and θ going from 0 to 2π . The normal vector is easy to calculate, it's just $\langle 0, 0, \rho \rangle$. It just remains to find the curl of \mathbf{F} .

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ zy & 2zx & xy \end{vmatrix} \\ &= \langle -x, 0, z \rangle \end{aligned}$$

Putting this together into the surface integral gives

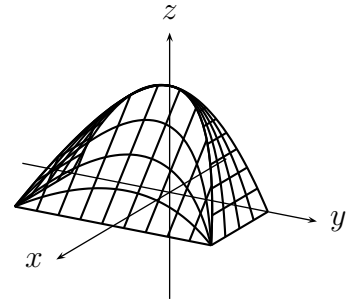
$$\begin{aligned} \int_0^{2\pi} \int_0^3 4\rho \, d\rho \, d\theta &= \int_0^{2\pi} 18 \, d\theta \\ &= \boxed{36\pi} \end{aligned}$$

6 (10 points)

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle x + \cos(y), e^z - y, zy^2 \rangle$ and S is the surface of the region D bounded by $z = 1 - y^2$, $x + z = 1$, $x = 0$, and $z = 0$. See the figure to the right.



You knew there was going to be a Divergence Theorem problem on the final, and this is it. The picture is here to help you decide the order of integration and to make it clearer how the boundaries fit together. Despite this a couple people tried to integrate in z first. Thankfully one of those people did think to split it into two integrals. The easiest thing to do is integrate in x , then z , then y . I don't know why so many tried to integrate in y earlier, because then you're dealing with square roots.

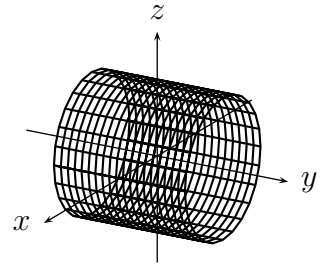
$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_D \nabla \cdot \mathbf{F} \, dV \\ &= \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} y^2 \, dx \, dz \, dy \\ &= \int_{-1}^1 \int_0^{1-y^2} y^2(1-z) \, dz \, dy \\ &= \int_{-1}^1 \left[y^2 \left(z - \frac{1}{2}z^2 \right) \right]_{z=0}^{z=1-y^2} dy \\ &= \int_{-1}^1 y^2 \left(1 - y^2 - \frac{1}{2}(1 - y^2)^2 \right) dy \\ &= \int_{-1}^1 \frac{1}{2}y^2 - \frac{1}{2}y^6 \, dy \\ &= \left(\frac{1}{3} - \frac{1}{7} \right) = \boxed{\frac{4}{21}} \end{aligned}$$

7 (10 points)

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle x, x^2z, zy \rangle$, and S is the surface of the cylinder $x^2 + z^2 = 4$ between $y = -2$ and $y = 2$ oriented away from the y -axis (outward). This *does not* include the "lids" of the cylinder.



This problem is just seeing if you can set up and evaluate a surface integral since the others done on the test were over planes. First you must parameterize the surface. $r(y, \theta) = \langle 2 \cos \theta, y, 2 \sin \theta \rangle$ with y going from -2 to 2 and θ going from 0 to 2π is the easiest way to do this. Next, the normal vector:

$$\begin{aligned} \mathbf{r}_y \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -2 \sin \theta & 0 & 2 \cos \theta \end{vmatrix} \\ &= \langle 2 \cos \theta, 0, 2 \sin \theta \rangle \end{aligned}$$

This normal vector does have the correct orientation, so we can proceed with it.

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_{-2}^2 \langle 2 \cos \theta, 8 \cos^2 \theta \sin \theta, 2y \sin \theta \rangle \cdot \langle 2 \cos \theta, 0, 2 \sin \theta \rangle dy d\theta \\ &= \int_0^{2\pi} \int_{-2}^2 4 \cos^2 \theta + 4y \sin^2 \theta dy d\theta \\ &= \int_0^{2\pi} 16 \cos^2 \theta d\theta \\ &= \int_0^{2\pi} 8 + 8 \cos(2\theta) d\theta \\ &= \boxed{16\pi} \end{aligned}$$