Computational Magma Dynamics

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What am I going to talk about?

- Magma Dynamics Model
- FEM formulation and solver
- FEM+FVM formulation and solver
Why is this important?

- It is difficult to compare meshes, discretizations, and multilevel solvers

- **Comparison** is essential for making informed algorithmic choices

- Comparison in a **single code** seems necessary
Outline

1. Problem Definition
2. Solvers for FEM Formulation
3. Solvers for FEM+FVM Formulation
\[ \nabla p - \nabla \zeta_\phi (\nabla \cdot \vec{v}^S) - \nabla \cdot \left( 2\eta_\phi \dot{\varepsilon}^S \right) = 0 \]

\[ \nabla \cdot \left( -\frac{K_\phi}{\mu} \nabla p + \vec{v}^S \right) = 0 \]

\[ \frac{\partial \phi}{\partial t} - \nabla \cdot (1 - \phi) \vec{v}^S = 0 \]
Problem Definition

Closure Conditions

\[ K_\phi = K_0 \left( \frac{\phi}{\phi_0} \right)^n \]

\[ \eta_\phi = \eta_0 \exp \left( -\lambda (\phi - \phi_0) \right) \]

\[ \zeta_\phi = \zeta_0 \left( \frac{\phi}{\phi_0} \right)^{-m} \]
\[ \nabla p - \nabla \left( \left( \frac{\phi}{\phi_0} \right)^{-m} \nabla \cdot \vec{v}^S \right) - \nabla \cdot \left( 2e^{-\lambda(\phi-\phi_0)} \dot{\varepsilon}^S \right) = 0 \]

\[ \nabla \cdot \left( -\frac{R^2}{r_\zeta + 4/3} \left( \frac{\phi}{\phi_0} \right)^n \nabla p + \vec{v}^S \right) = 0 \]

\[ \frac{\partial \phi}{\partial t} - \nabla \cdot (1 - \phi) \vec{v}^S = 0 \]
Initial and Boundary conditions

Initially

\[ \phi = \phi_0 + A \cos(\vec{k} \cdot \vec{x}) \]

where

\[ A \ll \phi_0 \]

and on the top and bottom boundary

\[ K_{\phi} \nabla p \cdot \hat{n} = 0 \]

\[ \vec{v}^S = \pm \frac{\dot{\gamma}}{2} \hat{x} \]
Benchmark 0: $\lambda = 0$
There is no porosity feedback, and the initial pattern is stably advected:

$$\vec{k}(t) = \vec{k}_0 \left( \hat{x} \sin \theta_0 + \hat{y} (\cos \theta_0 - t \sin \theta_0) \right)$$

Benchmark 1: $\lambda > 0$
The porosity feedback causes localization, with initial compaction rate:

$$C = \nabla \cdot \vec{v}_S \bigg|_{t=0} = \frac{A \lambda \phi_0 \sin(2\theta_0)}{r_\zeta + 4/3} \cos(\vec{k} \cdot \vec{x})$$
Initial Porosity
Outline

1. Problem Definition
2. Solvers for FEM Formulation
3. Solvers for FEM+FVM Formulation
With PETSc’s DMMPlex, we can use

- Simplices,
- Hexes,
- 2D and 3D,

changing nothing but mesh creation.

I do this in SNES ex62
PETSc Viewers can output

- Meshes
- Solutions,
- Auxiliary and derived fields,

to HDF5/Xdmf using simple options.
PETSc Viewers can output

- `dm_view hdf5:sol_fv_1.h5`
- `magma_view_solution hdf5:sol_fv_1.h5::append`
- `compaction_vec_view hdf5:sol_fv_1.h5:HDF5_VIZ:append`

to HDF5/Xdmf using simple options.
Using continuous FE spaces,

which satisfy an inf-sup stability condition.
Using continuous FE spaces,

- $Q_2$ for velocity
- $Q_1$ for pressure
- $Q_1$ for porosity

which satisfy an inf-sup stability condition.
Using continuous FE spaces,

- `velocity_petscSpace_order 2`
  - `velocity_petscSpace_poly_tensor`
- `pressure_petscSpace_order 1`
  - `pressure_petscSpace_poly_tensor`
- `porosity_petscSpace_order 1`
  - `porosity_petscSpace_poly_tensor`

which satisfy an inf-sup stability condition.
Using continuous/discontinuous FE spaces, 

\[ Q_2 \] velocity \\
\[ P_{1_{\text{disc}}} \] pressure \\
\[ Q_1 \] porosity

which satisfy an inf-sup stability condition.
Using continuous/discontinuous FE spaces,

-velocity_petscspace_order 2
-velocity_petscspacely_poly_tensor
-pressure_petscspacely_order 1
-pressure_petscspacely_poly_tensor
-pressure_petscdualspace_lagrange_continuity 0
-porosity_petscspacely_order 1
-porosity_petscspacely_poly_tensor

which satisfy an inf-sup stability condition.
We will use simple Backward Euler:

\[
\nabla p^{k+1} - \nabla \left( \left( \frac{\phi^{k+1}}{\phi_0} \right)^{-m} \nabla \cdot \vec{v}^{k+1} \right) - \nabla \cdot \left( 2e^{-\lambda(\phi^{k+1} - \phi_0)} \dot{\epsilon}^{k+1} \right) = 0
\]

\[
\nabla \cdot \left( - \frac{R^2}{r_\zeta + 4/3} \left( \frac{\phi^{k+1}}{\phi_0} \right)^n \nabla p^{k+1} + \vec{v}^{k+1} \right) = 0
\]

\[
\frac{\phi^{k+1} - \phi^k}{\Delta t} - \nabla \cdot (1 - \phi^{k+1}) \vec{v}^{k+1} = 0
\]
Begin with a Newton-Krylov solve with line search:

\[ \nabla K - L \text{NRICH} \]

Optimal linear preconditioner in Rhebergen, Wells, Wathen, and Katz, SISC.
We can separate the Stokes-like solve from the porosity advection:

\[
\begin{pmatrix}
A \oplus \text{Schur} & L \\
F & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 & I + G
\end{pmatrix}
\]
Solver Organization

Newton-Krylov

-pc_type fieldsplit
  -pc_fieldsplit_0_fields 0,1  -pc_fieldsplit_1_fields 2
  -pc_fieldsplit_type multiplicative
    -fieldsplit_0_pc_type fieldsplit
    -fieldsplit_0_pc_fieldsplit_type schur
    -fieldsplit_0_pc_fieldsplit_schur_precondition selfp
    -fieldsplit_0_pc_fieldsplit_schur_factorization_type full
    -fieldsplit_0_fieldsplit_velocity_pc_type lu
    -fieldsplit_0_fieldsplit_pressure_ksp_rtol 1.0e-9
    -fieldsplit_0_fieldsplit_pressure_pc_type gamg
      -fieldsplit_0_fieldsplit_pressure_ksp_monitor
      -fieldsplit_0_fieldsplit_pressure_kspgmres_restart 100
    -fieldsplit_fieldsplit_0_pressure_ksp_max_it 200
Or we can incorporate the porosity advection into the Stokes-like solve:

\[
\begin{bmatrix}
A & E \\
F & I+G
\end{bmatrix} \oplus \text{Schur } L
\]
Newton options
Newton-Krylov with Porosity

-snes_monitor -snes_converged_reason
-snes_type newtonls -snes_linesearch_type bt
-snes_fd_color -snes_fd_color_use_mat -mat_coloring_type greedy
-ksp_rtol 1.0e-10 -ksp_monitor -ksp_gmres_restart 200
-pc_type fieldsplit
   -pc_fieldsplit_0_fields 0,2 -pc_fieldsplit_1_fields 1
   -pc_fieldsplit_type schur -pc_fieldsplit_schur_precondition selfp
      -pc_fieldsplit_schur_factorization_type full
         -fieldsplit_0_pc_type lu
   -fieldsplit_pressure_ksp_rtol 1.0e-9 -fieldsplit_pressure_pc_type gamg
      -fieldsplit_pressure_ksp_monitor
   -fieldsplit_pressure_ksp_gmres_restart 100
   -fieldsplit_pressure_ksp_max_it 200
0 TS dt 0.01 time 0
0 SNES Function norm 5.292194079127e-03
  Linear pressure_ solve converged due to CONVERGED_RTOL its 10
0 KSP Residual norm 4.618093146920e+00
  Linear pressure_ solve converged due to CONVERGED_RTOL its 10
1 KSP Residual norm 3.01815330707e-03
  Linear pressure_ solve converged due to CONVERGED_RTOL its 11
2 KSP Residual norm 4.274869628519e-13
  Linear solve converged due to CONVERGED_RTOL its 2
1 SNES Function norm 2.766906985362e-06
  Linear pressure_ solve converged due to CONVERGED_RTOL its 8
0 KSP Residual norm 2.55890235972e-02
  Linear pressure_ solve converged due to CONVERGED_RTOL its 8
1 KSP Residual norm 1.638293944976e-07
  Linear pressure_ solve converged due to CONVERGED_RTOL its 8
2 KSP Residual norm 1.771928779400e-14
  Linear solve converged due to CONVERGED_RTOL its 2
2 SNES Function norm 1.188754322734e-11
  Nonlinear solve converged due to CONVERGED_FNORM_RELATIVE its 2
1 TS dt 0.01 time 0.01
Later Newton convergence

0 TS dt 0.01 time 0.63
  0 SNES Function norm 9.366565251786e-03
    Linear pressure_ solve converged due to CONVERGED_RTOL its 16
    Linear pressure_ solve converged due to CONVERGED_RTOL its 16
    Linear pressure_ solve converged due to CONVERGED_RTOL its 16
    Linear solve converged due to CONVERGED_RTOL its 2
  1 SNES Function norm 4.492625910272e-03
    Linear solve converged due to CONVERGED_RTOL its 2
  2 SNES Function norm 3.66181450068e-03
    Linear solve converged due to CONVERGED_RTOL its 2
  3 SNES Function norm 2.523116582272e-03
    Linear solve converged due to CONVERGED_RTOL its 2
  4 SNES Function norm 3.022638159491e-04
    Linear solve converged due to CONVERGED_RTOL its 2
  5 SNES Function norm 9.761317324448e-06
    Linear solve converged due to CONVERGED_RTOL its 2
  6 SNES Function norm 1.14794474432e-08
    Linear solve converged due to CONVERGED_RTOL its 2
  7 SNES Function norm 8.729160299009e-14
    Nonlinear solve converged due to CONVERGED_FNORM_RELATIVE its 7
1 TS dt 0.01 time 0.64
Solvers for FEM Formulation

Newton failure

0 TS dt 0.01 time 0.64
Time 0.64 $L_2$ Error: 0.494811 [0.0413666, 0.491642, 0.0376071]

0 SNES Function norm 9.68233054059e-03
Linear solve converged due to CONVERGED_RTOL iterations 2

1 SNES Function norm 6.841434267123e-03
Linear solve converged due to CONVERGED_RTOL iterations 3

2 SNES Function norm 4.412420553822e-03
Linear solve converged due to CONVERGED_RTOL iterations 5

3 SNES Function norm 3.309326919835e-03
Linear solve converged due to CONVERGED_RTOL iterations 6

4 SNES Function norm 3.022494350289e-03
Linear solve converged due to CONVERGED_RTOL iterations 7

5 SNES Function norm 2.941050948582e-03
Linear solve converged due to CONVERGED_RTOL iterations 7

\vdots

9 SNES Function norm 2.631941422878e-03
Linear solve converged due to CONVERGED_RTOL iterations 7

10 SNES Function norm 2.631897334054e-03
Linear solve converged due to CONVERGED_RTOL iterations 10

11 SNES Function norm 2.631451174722e-03
Linear solve converged due to CONVERGED_RTOL iterations 15

\vdots
We can combine Newton-Krylov with Nonlinear CG:

\[(\text{NCG} -_L \text{NRICH}) \star (\bigwedge K -_L \text{NRICH})\]
NCG*Newton options

-snes_monitor -snes_converged_reason
-snes_type composite -snes_composite_type multiplicative
-snes_composite_sneses ncg,newtonls
  -sub_0_snes_monitor -sub_1_snes_monitor
  -sub_0_snes_type ncg -sub_0_snes_linesearch_type cp
    -sub_0_snes_max_it 5
  -sub_1_snes_linesearch_type bt -sub_1_snes_fd_color
    -sub_1_snes_fd_color_use_mat -mat_coloring_type greedy
  -sub_1_ksp rtol 1.0e-10 -sub_1_ksp_monitor -sub_1_ksp_gmres_restart 200
  -sub_1_pc_type fieldsplit -sub_1_pc_fieldsplit_0_fields 0,2
    -sub_1_pc_fieldsplit_1_fields 1
  -sub_1_pc_fieldsplit_type schur
    -sub_1_pc_fieldsplit_schur_precondition selfp
    -sub_1_pc_fieldsplit_schur_factorization_type full
      -sub_1_fieldsplit_0_pc_type lu
  -sub_1_fieldsplit_pressure_ksp rtol 1.0e-9
    -sub_1_fieldsplit_pressure_pc_type gamg
  -sub_1_fieldsplit_pressure_ksp_gmres_restart 100
  -sub_1_fieldsplit_pressure_ksp_max_it 200
Solvers for FEM Formulation

NCG*Newton convergence

0 TS dt 0.01 time 0.64
  0 SNES Function norm 9.682733054059e-03
  0 SNES Function norm 9.682733054059e-03
  1 SNES Function norm 3.705698943518e-02
  2 SNES Function norm 4.981898384331e-02
  3 SNES Function norm 5.710183285964e-02
  4 SNES Function norm 5.476973798534e-02
  5 SNES Function norm 6.464724668855e-02
  0 SNES Function norm 6.464724668855e-02
    0 KSP Residual norm 1.021155502263e+00
    1 KSP Residual norm 9.145207488003e-05
    2 KSP Residual norm 3.899752904206e-09
    3 KSP Residual norm 1.001750831581e-12
  1 SNES Function norm 8.940296814443e-03
    1 SNES Function norm 8.940296814443e-03
    2 SNES Function norm 4.290429277269e-02
    3 SNES Function norm 1.154466745956e-02
    4 SNES Function norm 2.938816182982e-03
    5 SNES Function norm 4.148507767082e-04
    6 SNES Function norm 1.892807106900e-05
    7 SNES Function norm 4.912625424547e-08
    8 SNES Function norm 3.851626525260e-13
  1 TS dt 0.01 time 0.65
We can use Newton-Krylov as a level solver for FAS:

\[ \text{FAS}(\mathcal{N} \setminus \mathcal{K}, \mathcal{N} \setminus \mathcal{K}) \]
FAS-Newton options

Top level

-\texttt{snes\_monitor} -\texttt{snes\_converged\_reason}
-\texttt{snes\_type fas} -\texttt{snes\_fas\_type full} -\texttt{snes\_fas\_levels 4}
-\texttt{fas\_levels\_3\_snes\_monitor} -\texttt{fas\_levels\_3\_snes\_converged\_reason}
-\texttt{fas\_levels\_3\_snes\_atol 1.0e-9} -\texttt{fas\_levels\_3\_snes\_max\_it 2}
-\texttt{fas\_levels\_3\_snes\_type newtonls} -\texttt{fas\_levels\_3\_snes\_linesearch\_type bt}
-\texttt{fas\_levels\_3\_snes\_fd\_color} -\texttt{fas\_levels\_3\_snes\_fd\_color\_use\_mat}
-\texttt{fas\_levels\_3\_ksp\_rtol 1.0e-10} -\texttt{mat\_coloring\_type greedy}
-\texttt{fas\_levels\_3\_ksp\_gmres\_restart 50} -\texttt{fas\_levels\_3\_ksp\_max\_it 200}
-\texttt{fas\_levels\_3\_pc\_type fieldsplit}
-\texttt{fas\_levels\_3\_pc\_fieldsplit\_0\_fields 0,2}
-\texttt{fas\_levels\_3\_pc\_fieldsplit\_1\_fields 1}
-\texttt{fas\_levels\_3\_pc\_fieldsplit\_type schur}
-\texttt{fas\_levels\_3\_pc\_fieldsplit\_schur\_precondition selfp}
-\texttt{fas\_levels\_3\_pc\_fieldsplit\_schur\_factorization\_type full}
-\texttt{fas\_levels\_3\_fieldsplit\_0\_pc\_type lu}
-\texttt{fas\_levels\_3\_fieldsplit\_pressure\_ksp\_rtol 1.0e-9}
-\texttt{fas\_levels\_3\_fieldsplit\_pressure\_pc\_type gamg}
-\texttt{fas\_levels\_3\_fieldsplit\_pressure\_ksp\_gmres\_restart 100}
-\texttt{fas\_levels\_3\_fieldsplit\_pressure\_ksp\_max\_it 200}
FAS-Newton options

2nd level

- fas_levels_2_snnes_monitor -fas_levels_2_snnes_converged_reason
- fas_levels_2_snnes_atol 1.0e-9 -fas_levels_2_snnes_max_it 2
- fas_levels_2_snnes_type newtonls -fas_levels_2_snnes_linesearch_type bt
- fas_levels_2_snnes_fd_color -fas_levels_2_snnes_fd_color_use_mat
- fas_levels_2_ksp_rtol 1.0e-10 -fas_levels_2_ksp_gmres_restart 50
- fas_levels_2_pc_type fieldsplit
  - fas_levels_2_pc_fieldsplit_0_fields 0,2
  - fas_levels_2_pc_fieldsplit_1_fields 1
- fas_levels_2_pc_fieldsplit_type schur
  - fas_levels_2_pc_fieldsplit_schur_precondition selfp
  - fas_levels_2_pc_fieldsplit_schur_factorization_type full
    - fas_levels_2_fieldsplit_0_pc_type lu
  - fas_levels_2_fieldsplit_pressure_ksp_rtol 1.0e-9
    - fas_levels_2_fieldsplit_pressure_pc_type gamg
  - fas_levels_2_fieldsplit_pressure_ksp_gmres_restart 100
  - fas_levels_2_fieldsplit_pressure_ksp_max_it 200
FAS-Newton options

1st level

-fas_levels_1_snes_monitor -fas_levels_1_snes_converged_reason
-fas_levels_1_snes_atol 1.0e-9
-fas_levels_1_snes_type newtonls -fas_levels_1_snes_linesearch_type bt
-fas_levels_1_snes_fd_color -fas_levels_1_snes_fd_color_use_mat
-fas_levels_1_ksp_rtol 1.0e-10 -fas_levels_1_ksp_gmres_restart 50
-fas_levels_1_pc_type fieldsplit
   -fas_levels_1_pc_fieldsplit_0_fields 0,2
   -fas_levels_1_pc_fieldsplit_1_fields 1
   -fas_levels_1_pc_fieldsplit_type schur
      -fas_levels_1_pc_fieldsplit_schur_precondition selfp
      -fas_levels_1_pc_fieldsplit_schur_factorization_type full
         -fas_levels_1_fieldsplit_0_pc_type lu
         -fas_levels_1_fieldsplit_pressure_ksp_rtol 1.0e-9
         -fas_levels_1_fieldsplit_pressure_pc_type gamg
FAS-Newton options

Coarse level

- fas_coarse_snes_monitor -fas_coarse_snes_converged_reason
- fas_coarse_snes_atol 1.0e-9
- fas_coarse_snes_type newtonls -fas_coarse_snes_linesearch_type bt
- fas_coarse_snes_fd_color -fas_coarse_snes_fd_color_use_mat
- fas_coarse_ksp_rtol 1.0e-10 -fas_coarse_ksp_gmres_restart 50
- fas_coarse_pc_type fieldsplit
  - fas_coarse_pc_fieldsplit_0_fields 0,2
  - fas_coarse_pc_fieldsplit_1_fields 1
  - fas_coarse_pc_fieldsplit_type schur
    - fas_coarse_pc_fieldsplit_schur_precondition selfp
    - fas_coarse_pc_fieldsplit_schur_factorization_type full
      - fas_coarse_fieldsplit_0_pc_type lu
    - fas_coarse_fieldsplit_pressure_ksp_rtol 1.0e-9
    - fas_coarse_fieldsplit_pressure_pc_type gamg
FAS-Newton convergence

0 TS dt 0.01 time 0.64
  0 SNES Function norm 9.682733054059e-03
  2 SNES Function norm 4.412420553822e-03
    2 SNES Function norm 8.022096211721e-15
  1 SNES Function norm 2.773743832538e-04
    1 SNES Function norm 5.627093528843e-11
    1 SNES Function norm 4.405884464849e-10
  2 SNES Function norm 8.985059910030e-08
  1 SNES Function norm 4.672651281994e-15
    0 SNES Function norm 3.160322858961e-15
    0 SNES Function norm 4.672651281994e-15
  1 SNES Function norm 1.046571008046e-14
  2 SNES Function norm 1.804845173803e-02
    2 SNES Function norm 2.776600115290e-12
    0 SNES Function norm 1.354009326059e-12
      0 SNES Function norm 3.160322858961e-15
      0 SNES Function norm 5.881604627760e-13
      0 SNES Function norm 1.354011456281e-12
      0 SNES Function norm 2.776600115290e-12
    2 SNES Function norm 9.640723411562e-05
  1 SNES Function norm 9.640723411562e-05
  2 SNES Function norm 1.057876040732e-08
  3 SNES Function norm 5.623618219189e-11
1 TS dt 0.01 time 0.65
On fine levels, we can replace Newton-Krylov with Nonlinear Gauss-Siedel:

\[ \text{FAS(NGS, } \mathcal{N} \backslash K) \]
Top level

-snes_monitor -snes_converged_reason
-snes_type fas -snes_fas_type full -snes_fas_levels 4
-fas_levels_3_snes_monitor -fas_levels_3_snes_converged_reason
-fas_levels_3_snes_atol 1.0e-9 -fas_levels_3_snes_max_it 10
-fas_levels_3_snes_type ngs -fas_levels_3_snes_linesearch_type nleqerr
FAS-NGS convergence

0 TS dt 0.01 time 0.64
Time 0.64 L_2 Error: 0.494811 [0.0413666, 0.491642, 0.0376071]
 0 SNES Function norm 9.68e-03 [1.96e-03, 1.71e-14, 9.65e-03]
   0 SNES Function norm 9.68273054059e-03
   3 SNES Function norm 9.06994580453e-01
      3 SNES Function norm 3.790367845975e-11
   0 SNES Function norm 1.884126634610e+00
   1 SNES Function norm 6.752057466899e-02
      2 SNES Function norm 3.799909413083e-11
   0 SNES Function norm 1.450032375835e-01
   1 SNES Function norm 2.567674743706e-04
  0 SNES Function norm 1.027806561203e+00
  3 SNES Function norm 1.582489644172e+00
     1 SNES Function norm 4.847533456932e-01
        3 SNES Function norm 7.36666076108e-15
     1 SNES Function norm 1.744390611632e-02
   3 SNES Function norm 1.473321454964e+00
      1 SNES Function norm 1.47e+00 [1.44e+00, 2.92e-01, 8.82e-04]
  0 SNES Function norm 9.962396109825e+03
  1 SNES Function norm 3.189537494940e+86
Nonlinear fas_levels_2_ solve did not converge, DIVERGED_FNORM_NAN
Nonlinear solve did not converge due to DIVERGED_INNER
Outline

1. Problem Definition
2. Solvers for FEM Formulation
3. Solvers for FEM+FVM Formulation
Discretization

Using continuous FE spaces and simple FV,

\[ Q_2 \] velocity
\[ Q_1 \] pressure
\[ FV \] porosity

which we connect by cell/face interpolants.

```c
/* Set discretization object */
if (user->useFV)
    PetscDSSetDiscretization(prob, 2, fv);
else {
    PetscDSSetDiscretization(prob, 2, fe[2]);
}
/* Set pointwise residual functions */
PetscDSSetResidual(prob, 2, f0_advection, f1_scalar_zero);
PetscDSSetRiemannSolver(prob, 2, riemann_coupled_advection);
```
Discretization

Using continuous FE spaces and simple FV,

-velocity_petscspacet_order 2
  -velocity_petscspacet_poly_tensor
-pressure_petscspacet_order 1
  -pressure_petscspacet_poly_tensor
-use_fv

which we connect by cell/face interpolants.

/* Set discretization object */
if (user -> useFV) {
    PetscDSSetDiscretization(prob, 2, fv);
} else {
    PetscDSSetDiscretization(prob, 2, fe[2]);
}
/* Set pointwise residual functions */
PetscDSSetResidual(prob, 2, f0_advection, f1_scalar_zero);
PetscDSSetRiemannSolver(prob, 2, riemann_coupled_advection);
Using continuous FE spaces and simple FV, 

$Q_2$ velocity 
$Q_1$ pressure 
$FV$ porosity 

which we connect by cell/face interpolants.

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/* Set discretization object */
if (user->useFV) {
    PetscDSSetDiscretization(prob, 2, fv);
} else {
    PetscDSSetDiscretization(prob, 2, fe[2]);
}

/* Set pointwise residual functions */
PetscDSSetResidual(prob, 2, f0_advection, f1_scalar_zero);
PetscDSSetRiemannSolver(prob, 2, riemann_coupled_advection);
```
We can use a simple split scheme:

\[
\nabla p^{k+1} - \nabla \left( \left( \frac{\phi^k}{\phi_0} \right)^{-m} \nabla \cdot \vec{v}^{k+1} \right) - \nabla \cdot \left( 2e^{-\lambda(\phi^k - \phi_0)} \dot{\epsilon}^{k+1} \right) = 0
\]

\[
\nabla \cdot \left( -\frac{R^2}{r_\zeta + 4/3} \left( \frac{\phi^k}{\phi_0} \right)^n \nabla p^{k+1} + \vec{v}^{k+1} \right) = 0
\]

\[
\frac{\phi^{k+1} - \phi^k}{\Delta t} - \nabla \cdot (1 - \phi^k) \vec{v}^{k+1} = 0
\]
Or one that couples the algebraic and evolution equations:

\[ \nabla p^{k+1} - \nabla \left( \left( \frac{\phi^{k+1}}{\phi_0} \right)^{-m} \nabla \cdot \vec{v}^{k+1} \right) - \nabla \cdot \left( 2e^{-\lambda(\phi^{k+1} - \phi_0)}\dot{\varepsilon}^{k+1} \right) = 0 \]

\[ \nabla \cdot \left( - \frac{R^2}{r_\varsigma + 4/3} \left( \frac{\phi^{k+1}}{\phi_0} \right)^n \nabla p^{k+1} + \vec{v}^{k+1} \right) = 0 \]

\[ \frac{\phi^{k+1} - \phi^k}{\Delta t} - \nabla \cdot (1 - \phi^k)\vec{v}^{k+1} = 0 \]
Newton options

-snes_atol 1.0e-10 -snes_monitor_field -snes_converged_reason
-snes_linesearch_type basic -snes_fd_color -snes_fd_color_use_mat
-mat_coloring_type greedy -mat_coloring_greedy_symmetric 0
-ksp_rtol 1.0e-10 -ksp_monitor -ksp_gmres_restart 200
-pc_type fieldsplit
-pc_fieldsplit_0_fields 0,2 -pc_fieldsplit_1_fields 1
-pc_fieldsplit_type schur -pc_fieldsplit_schur_precondition selfp
-pc_fieldsplit_schur_factorization_type full
-fieldsplit_0_ksp_rtol 1.0e-8 -fieldsplit_0_pc_type lu
-fieldsplit_pressure_ksp_rtol 1.0e-9 -fieldsplit_pressure_pc_type svd
Early Newton convergence

5 TS dt 0.005 time 0.025
  0 SNES Function norm 6.52e-02 [1.46e-14, 4.91e-16, 6.52e-02]
  0 KSP Residual norm 4.26e-04
  1 KSP Residual norm 1.78e-17
  1 SNES Function norm 2.19e-03 [2.96e-08, 3.91e-09, 2.19e-03]
  2 SNES Function norm 7.51e-05 [3.40e-11, 4.55e-12, 7.51e-05]
  3 SNES Function norm 2.58e-06 [2.46e-13, 1.28e-14, 2.58e-06]
  4 SNES Function norm 8.86e-08 [1.39e-14, 6.64e-16, 8.86e-08]

6 TS dt 0.005 time 0.03
Late Newton convergence

0 TS dt 0.005 time 0.825
  0 SNES Function norm 2.14e+00 [ 1.40e-14, 3.67e-16, 2.14e+00]
  0 KSP Residual norm 3.53e-01
  1 KSP Residual norm 1.03e-10
  2 KSP Residual norm 2.82e-16
  1 SNES Function norm 5.13e-02 [ 2.01e-04, 1.47e-04, 5.13e-02]
  2 SNES Function norm 2.47e-02 [ 9.20e-06, 7.73e-06, 2.47e-02]
  3 SNES Function norm 7.81e-03 [ 2.13e-06, 1.67e-06, 7.81e-03]
  4 SNES Function norm 2.12e-03 [ 1.81e-07, 1.41e-07, 2.12e-03]
  5 SNES Function norm 4.72e-04 [ 1.08e-08, 8.28e-09, 4.72e-04]
  6 SNES Function norm 1.12e-04 [ 5.76e-10, 4.41e-10, 1.12e-04]
  7 SNES Function norm 2.63e-05 [ 3.21e-11, 2.50e-11, 2.63e-05]
  8 SNES Function norm 6.17e-06 [ 1.77e-12, 1.26e-12, 6.17e-06]
  9 SNES Function norm 1.45e-06 [ 1.07e-13, 9.84e-14, 1.45e-06]
 10 SNES Function norm 3.40e-07 [ 1.78e-14, 4.74e-15, 3.40e-07]
 11 SNES Function norm 7.99e-08 [ 1.36e-14, 1.88e-15, 7.99e-08]
 12 SNES Function norm 1.88e-08 [ 1.34e-14, 5.72e-16, 1.88e-08]

1 TS dt 0.005 time 0.83
FAS-Newton options

Top level

-snes_atol 1.0e-9  -snes_monitor_field  -snes_converged_reason
-snes_type fas  -snes_fas_type full  -snes_fas_levels 3
-fas_levels_2_snes_monitor  -fas_levels_2_snes_converged_reason
-fas_levels_2_snes_atol 1.0e-9  -fas_levels_2_snes_max_it 2
-fas_levels_2_snes_type newtonls
-fas_levels_2_snes_linesearch_type basic
-fas_levels_2_snes_fd_color  -fas_levels_2_snes_fd_color_use_mat
-fas_levels_2_ksp_rtol 1.0e-10  -fas_levels_2_ksp_gmres_restart 50
-fas_levels_2_pc_type fieldsplit
-fas_levels_2_pc_fieldsplit_0_fields 0,2
-fas_levels_2_pc_fieldsplit_1_fields 1
-fas_levels_2_pc_fieldsplit_type schur
-fas_levels_2_pc_fieldsplit_schur_precondition selfp
-fas_levels_2_pc_fieldsplit_schur_factorization_type full
-fas_levels_2_fieldsplit_0_pc_type lu
-fas_levels_2_fieldsplit_pressure_ksp_rtol 1.0e-9
-fas_levels_2_fieldsplit_pressure_pc_type svd
-fas_levels_2_fieldsplit_pressure_ksp_gmres_restart 100
-fas_levels_2_fieldsplit_pressure_ksp_max_it 200
FAS-Newton options

Coarse level

-fas_coarse_snes_max_it 10 -fas_coarse_snes_max_linear_solve_fail 10
-fas_coarse_snes_atol 1.0e-9
-fas_coarse_snes_monitor -fas_coarse_snes_converged_reason
-fas_coarse_snes_type newtonls -fas_coarse_snes_linesearch_type bt
-fas_coarse_snes_fd_color -fas_coarse_snes_fd_color_use_mat
-fas_coarse_ksp_rtol 1.0e-10 -fas_coarse_ksp_gmres_restart 50
-fas_coarse_pc_type fieldsplit
-fas_coarse_pc_fieldsplit_0_fields 0,2
-fas_coarse_pc_fieldsplit_1_fields 1
-fas_coarse_pc_fieldsplit_type schur
-fas_coarse_pc_fieldsplit_schur_precondition selfp
-fas_coarse_pc_fieldsplit_schur_factorization_type full
-fas_coarse_fieldsplit_0_pc_type lu
-fas_coarse_fieldsplit_pressure_ksp_rtol 1.0e-9
-fas_coarse_fieldsplit_pressure_pc_type svd
FAS-Newton convergence

0 TS dt 0.005 time 0.825
  0 SNES Function norm 2.14e+00 [1.40e-14, 3.67e-16, 2.14e+00]
  0 SNES Function norm 2.136811983007e+00
  2 SNES Function norm 2.467490038458e-02
    0 SNES Function norm 2.892788645925e-02
    5 SNES Function norm 6.686368379854e-11
  0 SNES Function norm 5.034219273717e-02
  1 SNES Function norm 1.054842559307e-03
    0 SNES Function norm 1.663080254945e-03
    4 SNES Function norm 2.126370356882e-10
  0 SNES Function norm 2.599480303180e-03
  1 SNES Function norm 9.990047497418e-05
  0 SNES Function norm 4.798584798600e-02
  2 SNES Function norm 1.288870672992e-03
    1 SNES Function norm 3.770621359658e-05
    2 SNES Function norm 1.008431552413e-06
  0 SNES Function norm 2.502531975042e-03
  2 SNES Function norm 4.730240156687e-05
    1 SNES Function norm 4.73e-05 [1.04e-10, 7.85e-11, 4.73e-05]
  2 SNES Function norm 3.98e-09 [1.38e-14, 4.18e-16, 3.98e-09]
1 TS dt 0.005 time 0.83
FAS-NGS options

Top level

-snes_atol 1.0e-9 -snes_monitor_field -snes_converged_reason
-snes_type fas -snes_fas_type full -snes_fas_levels 3
   -fas_levels_2_snes_monitor -fas_levels_2_snes_converged_reason
   -fas_levels_2_snes_atol 1.0e-9 -fas_levels_2_snes_max_it 10
   -fas_levels_2_snes_type ngs -fas_levels_2_snes_linesearch_type bt
Solvers for FEM+FVM Formulation

FAS-NGS convergence

<table>
<thead>
<tr>
<th>Step</th>
<th>SNES Function norm</th>
<th>[Min, Avg, Max]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.14e+00</td>
<td>[1.39e-14, 3.66e-16, 2.13e+00]</td>
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<tr>
<td>1</td>
<td>2.87e-04</td>
<td>[2.08e-04, 4.80e-06, 1.98e-04]</td>
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<td>3.15e-05</td>
<td>[2.30e-05, 9.56e-07, 2.19e-05]</td>
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<td>1.65e-05</td>
<td>[1.14e-05, 5.44e-07, 1.21e-05]</td>
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<tr>
<td>4</td>
<td>1.07e-05</td>
<td>[7.38e-06, 3.48e-07, 7.89e-06]</td>
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<tr>
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<td>[4.85e-06, 2.26e-07, 5.20e-06]</td>
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<td>[3.20e-06, 1.48e-07, 3.44e-06]</td>
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<tr>
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<td>3.09e-06</td>
<td>[2.12e-06, 9.82e-08, 2.27e-06]</td>
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<td>[1.40e-06, 6.52e-08, 1.50e-06]</td>
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<td>[9.35e-07, 4.34e-08, 1.00e-06]</td>
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<td>[6.21e-07, 2.89e-08, 6.64e-07]</td>
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<td>[4.13e-07, 1.94e-08, 4.41e-07]</td>
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<td>[2.75e-07, 1.30e-08, 2.94e-07]</td>
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<td>[1.23e-07, 6.01e-09, 1.31e-07]</td>
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<tr>
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<td>1.20e-07</td>
<td>[8.31e-08, 4.12e-09, 8.80e-08]</td>
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<td>8.12e-08</td>
<td>[5.64e-08, 2.85e-09, 5.94e-08]</td>
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</tr>
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<td>[1.43e-08, 7.48e-10, 1.44e-08]</td>
</tr>
</tbody>
</table>

1 TS dt 0.005 time 0.83
What does this mean?

- PETSc allows comparison between Meshes, Discretizations, and Solvers
- Can allow more robust, scalable solves
- Can allow better physical fidelity

http://www.mcs.anl.gov/petsc
What does this mean?

There are a bunch of unanswered analytical questions:

- Is the hybrid method stable? Working on a proof.
- Can we use the Implicit-Input/Explicit-Output scheme to increase the timestep?
- What is the accuracy/degree of freedom?

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