Convergence of Composed Nonlinear Iterations

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SPPEXA Symposium 2016
Leibniz Supercomputing Centre
München, Freistaat Bayern    January 25, 2016
Nonlinear Preconditioning

Left Nonlinear Preconditioning


Right Nonlinear Preconditioning


- Nonlinearly preconditioned optimization on Grassman manifolds for computing approximate Tucker tensor decompositions, De Sterck, Howse, SISC, 2015.

Nonlinear Preconditioning

Algorithmic Formalism


<table>
<thead>
<tr>
<th>Type</th>
<th>Sym</th>
<th>Statement</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive</td>
<td>+</td>
<td>( \vec{x} + \alpha(M(F, \vec{x}, \vec{b}) - \vec{x}) ) ( + \beta(N(F, \vec{x}, \vec{b}) - \vec{x}) )</td>
<td>( M + N )</td>
</tr>
<tr>
<td>Multiplicative</td>
<td>*</td>
<td>( M(F, N(F, \vec{x}, \vec{b}), \vec{b}) )</td>
<td>( M \ast N )</td>
</tr>
<tr>
<td>Left Prec.</td>
<td>(-_L)</td>
<td>( M(\vec{x} - N(F, \vec{x}, \vec{b}), \vec{x}, \vec{b}) )</td>
<td>( M -_L N )</td>
</tr>
<tr>
<td>Right Prec.</td>
<td>(-_R)</td>
<td>( M(F(N(F, \vec{x}, \vec{b})), \vec{x}, \vec{b}) )</td>
<td>( M -_R N )</td>
</tr>
<tr>
<td>Inner Lin. Inv.</td>
<td>( \backslash )</td>
<td>( \vec{y} = \vec{J}(\vec{x})^{-1}\vec{r}(\vec{x}) = K(\vec{J}(\vec{x}), \vec{y}_0, \vec{b}) )</td>
<td>( N \backslash K )</td>
</tr>
</tbody>
</table>
Consider Linear Multigrid,

- **Local Fourier Analysis (LFA)**

- **Idealized Relaxation (IR)**
  - **Idealized Coarse-Grid Correction (ICG)**
  - On Quantitative Analysis Methods for Multigrid Solutions, Diskin, Thomas, Mineck, SISC, 2005.
How Helpful is Theory?

How about Nonlinear Multigrid?

- **Full Approximation Scheme (FAS)**
  - Analysis only for Picard

- **Overbroad conclusions based on experiments**

- **People feel helpless when it fails or stagnates**
How about Newton’s Method?

- We have an asymptotic theory

- We need a non-asymptotic theory

- People feel helpless when it fails or stagnates
How about Nonlinear Preconditioning?

- Some guidance
  - Nonlinear Preconditioning Techniques for Full-Space Lagrange-Newton Solution of PDE-Constrained Optimization Problems,

- Left preconditioning (Newton $-L$ NASM) handles local nonlinearities

- Right preconditioning (Nonlinear Elimination) handles nonlinear global coupling
Outline

1. Convergence Rates

2. Theory
What should be a Rate of Convergence? [Ptak, 1977]:

1. It should relate quantities which may be measured or estimated during the actual process.
2. It should describe accurately in particular the initial stage of the process, not only its asymptotic behavior.

\[ \| x_{n+1} - x^* \| \leq c \| x_n - x^* \|^q \]
What should be a Rate of Convergence? [Ptak, 1977]:

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What should be a Rate of Convergence? [Ptak, 1977]:

1. It should relate quantities which may be measured or estimated during the actual process.

2. It should describe accurately in particular the initial stage of the process, not only its asymptotic behavior . . .

\[ \|x_{n+1} - x_n\| \leq \omega(\|x_n - x_{n-1}\|) \]

where we have for all \( r \in (0, R] \)

\[ \sigma(r) = \sum_{n=0}^{\infty} \omega^{(n)}(r) < \infty \]
Define an approximate set $Z(r)$, where $x^* \in Z(0)$ implies $f(x^*) = 0$. 
Define an approximate set $Z(r)$, where $x^* \in Z(0)$ implies $f(x^*) = 0$.

For Newton’s method, we use

$$Z(r) = \left\{ x \left| \left\| f'(x)^{-1} f(x) \right\| \leq r, d(f'(x)) \geq h(r), \left\| x - x_0 \right\| \leq g(r) \right\},$$

where

$$d(A) = \inf_{\left\| x \right\| \geq 1} \left\| Ax \right\|,$$

and $h(r)$ and $g(r)$ are positive functions.
Define an approximate set $Z(r)$, where $x^* \in Z(0)$ implies $f(x^*) = 0$.

For $r \in (0, R]$, 

\[ Z(r) \subset U(Z(\omega(r)), r) \]

implies 

\[ Z(r) \subset U(Z(0), \sigma(r)). \]
Nondiscrete Induction

For the fixed point iteration

\[ x_{n+1} = Gx_n, \]

if I have

\[ x_0 \in Z(r_0) \]

and for \( x \in Z(r) \),

\[ \| Gx - x \| \leq r \]

then

\[ Gx \in Z(\omega(r)) \]
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then

\[ x^* \in Z(0) \]

\[ x_n \in Z(\omega^{(n)}(r_0)) \]
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and for \( x \in Z(r) \),

\[ \| Gx - x \| \leq r \]
\[ Gx \in Z(\omega(r)) \]

then

\[ \| x_{n+1} - x_n \| \leq \omega^{(n)}(r_0) \]
\[ \| x_n - x^* \| \leq \sigma(\omega^{(n)}(r_0)) \]
Nondiscrete Induction

For the fixed point iteration

\[ x_{n+1} = Gx_n, \]

if I have

\[ x_0 \in Z(r_0) \]

and for \( x \in Z(r) \),

\[ \| Gx - x \| \leq r \]

\[ Gx \in Z(\omega(r)) \]

then

\[ \| x_n - x^* \| \leq \sigma(\omega(\| x_n - x_{n-1} \|)) \]

\[ = \sigma(\| x_n - x_{n-1} \|) - \| x_n - x_{n-1} \| \]
Newton's Method

\[ \omega_N(r) = cr^2 \]
Newton’s Method

\[ \omega_N(r) = \frac{r^2}{2\sqrt{r^2 + a^2}} \]

\[ \sigma_N(r) = r + \sqrt{r^2 + a^2} - a \]

where

\[ a = \frac{1}{k_0} \sqrt{1 - 2k_0r_0}, \]

\( k_0 \) is the (scaled) Lipschitz constant for \( f' \), and \( r_0 \) is the (scaled) initial residual.
Newton’s Method

\[ \omega_N(r) = \frac{r^2}{2\sqrt{r^2 + a^2}} \]

\[ \sigma_N(r) = r + \sqrt{r^2 + a^2} - a \]

This estimate is *tight* in that the bounds hold with equality for some function \( f \),

\[ f(x) = x^2 - a^2 \]

using initial guess

\[ x_0 = \frac{1}{k_0}. \]

Also, if equality is attained for some \( n_0 \), this holds for all \( n \geq n_0 \).
Newton’s Method

\[ \omega_N(r) = \frac{r^2}{2\sqrt{r^2 + a^2}} \]

\[ \sigma_N(r) = r + \sqrt{r^2 + a^2} - a \]

If \( r \gg a \), meaning we have an inaccurate guess,

\[ \omega_N(r) \approx \frac{1}{2}r, \]

whereas if \( r \ll a \), meaning we are close to the solution,

\[ \omega_N(r) \approx \frac{1}{2a}r^2. \]
Left vs. Right

Left:

\[ \mathcal{F}(x) \implies x - \mathcal{N}(\mathcal{F}, x, b) \]

Right:

\[ x \implies y = \mathcal{N}(\mathcal{F}, x, b) \]

Heisenberg vs. Schrödinger Picture
Left vs. Right

Left:

\[ \mathcal{F}(x) \implies x - \mathcal{N}(\mathcal{F}, x, b) \]

Right:

\[ x \implies y = \mathcal{N}(\mathcal{F}, x, b) \]

Heisenberg vs. Schrödinger Picture
We start with $x \in Z(r)$, apply $\mathcal{N}$ so that

$$y \in Z(\omega_\mathcal{N}(r)),$$

and then apply $\mathcal{M}$ so that

$$x' \in Z(\omega_\mathcal{M}(\omega_\mathcal{N}(r))).$$

Thus we have

$$\omega_{\mathcal{M} - R \mathcal{N}} = \omega_\mathcal{M} \circ \omega_\mathcal{N}$$
\( \mathcal{N} - R \) NRICH

\[
\omega_\mathcal{N} \circ \omega_{\text{NRICH}} = \frac{1}{2} \frac{r^2}{\sqrt{r^2 + a^2}} \circ cr,
\]

\[
= \frac{1}{2} \frac{c^2 r^2}{\sqrt{c^2 r^2 + a^2}},
\]

\[
= \frac{1}{2} \frac{cr^2}{\sqrt{r^2 + (a/c)^2}},
\]

\[
= \frac{1}{2} \frac{r^2}{c \sqrt{r^2 + \tilde{a}^2}},
\]
Convergence Rates

Non-Abelian

\[ \mathcal{N} - R \text{ NRICH: } \frac{1}{2} c \frac{r^2}{\sqrt{r^2 + \tilde{a}^2}} \]

\[ \text{NRICH} - R \mathcal{N} \]

\[ \omega_{\text{NRICH}} \circ \omega_{\mathcal{N}} = cr \circ \frac{1}{2} \frac{r^2}{\sqrt{r^2 + a^2}}, \]

\[ = \frac{1}{2} c \frac{r^2}{\sqrt{r^2 + a^2}}, \]

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The first method also changes the onset of second order convergence.
Outline

1. Convergence Rates

2. Theory
\[ \mathcal{N} - R \text{ NRICH} \]

\[
\omega_{\mathcal{N}} \circ \omega_{\text{NRICH}} = \frac{1}{2} \left( \frac{r^2}{\sqrt{r^2 + a^2}} \right) \circ cr, \\
= \frac{1}{2} \left( \frac{c^2 r^2}{\sqrt{c^2 r^2 + a^2}} \right), \\
= \frac{1}{2} \left( \frac{cr^2}{\sqrt{r^2 + (a/c)^2}} \right), \\
= \frac{1}{2} \left( \frac{r^2}{c \sqrt{r^2 + \tilde{a}^2}} \right),
\]
\[ \mathcal{N} - R \ NRICH: \frac{1}{2} c \frac{r^2}{\sqrt{r^2 + a^2}} \]

NRICH $- R \mathcal{N}$

\[
\omega_{\text{NRICH}} \circ \omega_{\mathcal{N}} = cr \circ \frac{1}{2} \frac{r^2}{\sqrt{r^2 + a^2}}, \\
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\[ \text{NRICH} - R \ \mathcal{N}: \ \frac{1}{2} c \frac{r^2}{\sqrt{r^2 + a^2}} \]

The first method also changes the onset of second order convergence.
Composed Rates of Convergence

**Theorem**

If $\omega_1$ and $\omega_2$ are convex rates of convergence, then $\omega = \omega_1 \circ \omega_2$ is a rate of convergence.
If \( \omega_1 \) and \( \omega_2 \) are convex rates of convergence, then \( \omega = \omega_1 \circ \omega_2 \) is a rate of convergence.

First we show that

\[
\omega(s) \leq \frac{s}{r} \omega(r),
\]

which means that convex rates of convergence are non-decreasing.

This implies that compositions of convex rates of convergence are also convex and non-decreasing.
Theorem:

If $\omega_1$ and $\omega_2$ are convex rates of convergence, then $\omega = \omega_1 \circ \omega_2$ is a rate of convergence.

Then we show that

$\omega(r) < r \quad \forall r \in (0, R)$

by contradiction.
Theorem

If $\omega_1$ and $\omega_2$ are convex rates of convergence, then $\omega = \omega_1 \circ \omega_2$ is a rate of convergence.

This is enough to show that

$$\omega_1(\omega_2(r)) < \omega_1(r),$$

and in fact

$$\omega_1 \circ \omega_2)^{(n)}(r) < \omega_1^{(n)}(r).$$
Let

- $p$ (1 for our case) and $m$ (2 for our case) be two positive integers,
- $X$ be a complete metric space and $D \subset X^p$,
- $G : D \rightarrow X^p$ and $F : D \rightarrow X^{p+1}$ be defined by $F_u = (u, Gu)$,
- $F_k = P_k F$, $-p + 1 \leq k \leq m$, the components of $F$,
- $P = P_m$,
- $Z(r) \subset D$ for each $r \in T^p$,
- $\omega$ be a rate of convergence of type $(p, m)$ on $T$,
- $u_0 \in D$ and $r_0 \in T^p$.
Multidimensional Induction Theorem

If the following conditions hold

\[ u_0 \in Z(r_0), \]
\[ PFZ(r) \subset Z(\tilde{\omega}(r)), \]
\[ \| F_k u - F_{k+1} u \| \leq \omega_k(r), \]

for all \( r \in T^p, \ u \in Z(r), \) and \( k = 0, \ldots, m - 1, \) then

1. \( u_0 \) is admissible, and \( \exists x^* \in X \) such that \((P_k u_n)_{n \geq 0} \to x^*,\)
2. and the following relations hold for \( n > 1, \)

\[ Pu_n \in Z(\tilde{\omega}(r_0)), \]
\[ \| P_k u_n - P_{k+1} u_n \| \leq \omega_k^{(n)}(r_0), \quad 0 \leq k \leq m - 1, \]
\[ \| P_k u_n - x^* \| \leq \sigma_k(\tilde{\omega}(r_0)), \quad 0 \leq k \leq m; \]
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for all \( r \in T^p, \; u \in Z(r), \) and \( k = 0, \ldots, m - 1, \) then

1. \( u_0 \) is admissible, and \( \exists x^* \in X \) such that \( (P_k u_n)_{n \geq 0} \to x^*, \)
2. and the following relations hold for \( n > 1, \)

\[ \| P_k u_n - x^* \| \leq \sigma_k(r_n), \quad 0 \leq k \leq m. \]

where \( r_n \in T^p \) and \( P u_{n-1} \in Z(r_n). \)
Theorem

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Multidimensional Induction Theorem

**Theorem**

*If the following conditions hold*

\[ u_0 \in Z(r_0), \]
\[ PFZ(r) \subset Z(\omega \circ \psi(r)), \]
\[ \| F_0 u - F_1 u \| \leq r, \]
\[ \| F_1 u - F_2 u \| \leq \psi(r), \]

*for all* \( r \in T^p, \ u \in Z(r), \) and \( k = 0, \ldots, m - 1, \) *then*

1. \( u_0 \) *is admissible, and* \( \exists x^* \in X \) *such that* \( (P_k u_n)_{n \geq 0} \to x^*, \)
2. *and the following relations hold for* \( n > 1, \)

\[ Pu_n \in Z(\tilde{\omega}(r_0)), \]
\[ \| P_k u_n - P_{k+1} u_n \| \leq \omega_k^{(n)}(r_0), \quad 0 \leq k \leq m - 1, \]
\[ \| P_k u_n - x^* \| \leq \sigma_k(\tilde{\omega}(r_0)), \quad 0 \leq k \leq m; \]
Theorem

Suppose that we have two nonlinear solvers
- \( \mathcal{M}, Z_1, \omega, \)
- \( \mathcal{N}, Z_0, \psi, \)
and consider \( \mathcal{M} -_R \mathcal{N}, \) meaning a single step of \( \mathcal{N} \) for each step of \( \mathcal{M} \).

Concretely, take \( \mathcal{M} \) to be the Newton iteration, and \( \mathcal{N} \) the Chord method. Then the assumptions of the theorem above are satisfied using \( Z = Z_1 \) and

\[
\omega(r) = \{\psi(r), \omega \circ \psi(r)\},
\]

giving us the existence of a solution, and both a priori and a posteriori bounds on the error.
**Example**

\[ f(x) = x^2 + (0.0894427)^2 \]

<table>
<thead>
<tr>
<th>n</th>
<th>( | x_{n+1} - x_n | )</th>
<th>( | x_{n+1} - x_n | - w(n)(r_0) )</th>
<th>( | x_n - x^* | - s(w(n)(r_0)) )</th>
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</tbody>
</table>
Matrix iterations also 1D scalar once you diagonalize

Pták’s nondiscrete induction and its application to matrix iterations, Liesen, IMA J. Num. Anal.,

\[ x^2 + r_0 - 1/4 \text{ with } x_0 = 1/2 \]
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