

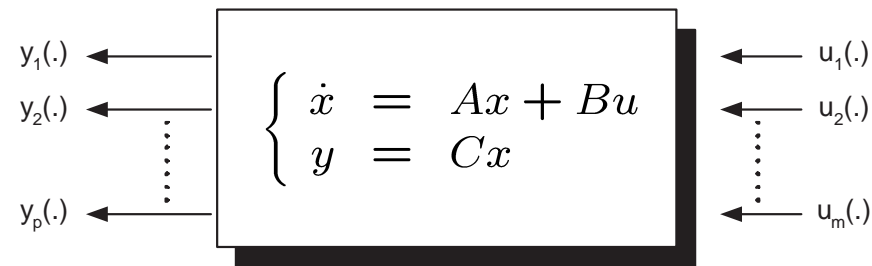
Model Reduction of Large-Scale Systems

CESAME (UCL) AND CSIT (FSU)

- Model Reduction using Multi-point Interpolation
- Structure Preserving Model Reduction
- H_∞ norm error estimation
- Application to the Building Model

Model reduction via interpolation

We consider input/output Linear Time Invariant systems:



They correspond in the Laplace Domain to a transfer function

$$T(s) \doteq C(sI_n - A)^{-1}B, \quad y(s) = T(s)u(s),$$

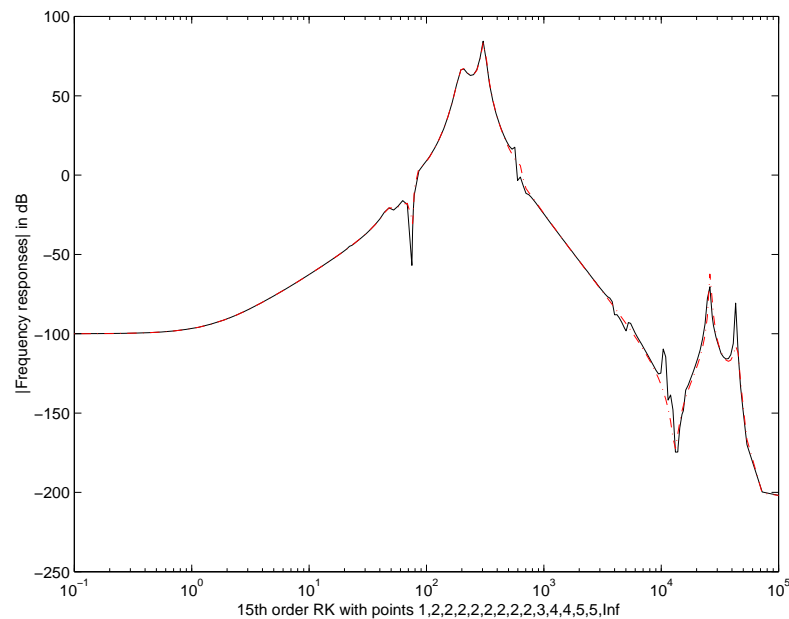
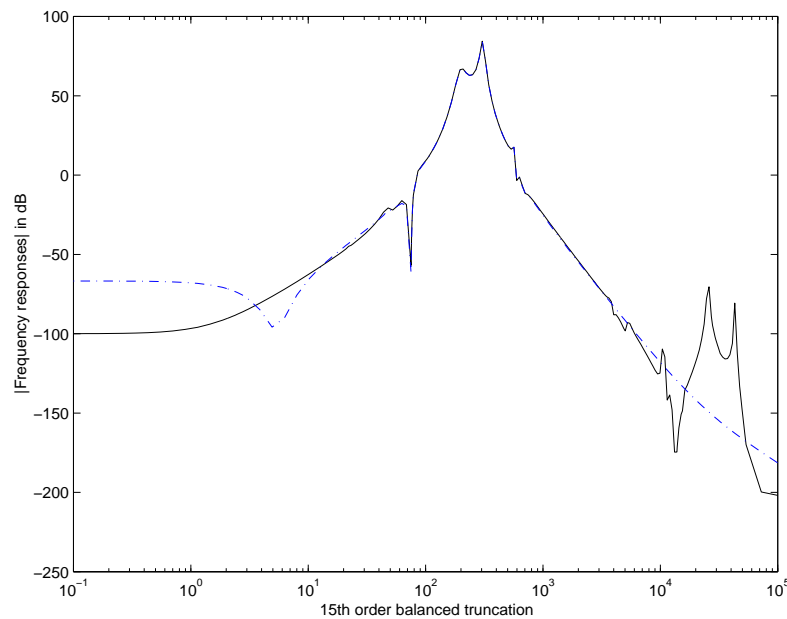
where n is the state space dimension.

Model Reduction via projection

Projection methods construct $Z, V \in \mathbb{R}^{n \times k}$ with $Z^T V = I_k$ such that

$$\hat{T}(s) \doteq CV(sI_k - Z^T AV)^{-1} Z^T B \approx T(s) \doteq C(sI_n - A)^{-1} B.$$

Different techniques produce different approximations



Balanced truncation versus multi-point rational interpolation

SISO Multi-point Interpolation

Padé interpolation expands $T(s)$ in a series

$$T(s) := m_{0,1}s + m_{0,2}s^2 + \dots + m_{0,d}s^d + \dots$$

and constructs $\hat{T}(s)$ such that $m_{0,i} = \hat{m}_{0,i}$ for $1 \leq i \leq d$

$$\hat{T}(s) := \hat{m}_{0,1}s + \hat{m}_{0,2}s^2 + \dots + \hat{m}_{0,d}s^d + \dots$$

Multipoint Padé interpolation does that for the interpolation set

$$I = \{(\sigma_1, d_1), \dots, (\sigma_r, d_r)\}$$

In other words, $\hat{T}(s)$ interpolates $T(s)$ at σ_i up to the d_i -th order

$$\left. \frac{d^{j-1}}{ds^{j-1}} \{T(s)\} \right|_{s=\sigma_i} = \left. \frac{d^{j-1}}{ds^{j-1}} \{\hat{T}(s)\} \right|_{s=\sigma_i}, \quad 1 \leq j \leq d_i$$

Projected system $\hat{T}(s) \doteq CV(sI_k - Z^T AV)^{-1} Z^T B$ interpolating $T(s)$

- can always be obtained via **Krylov** methods

$$\bigcup_{i=1}^r \mathcal{K}_{b_i} \left((A - \sigma_i I_N)^{-1}, (A - \sigma_i I_n)^{-1} B \right) \subseteq \text{Im}(V),$$

$$\bigcup_{j=1}^r \mathcal{K}_{c_j} \left((A - \sigma_j I_N)^{-T}, (A - \sigma_j I_n)^{-T} C^T \right) \subseteq \text{Im}(Z)$$

- or, equivalently, via the solution of **Sylvester Equations**

$$AVM^{(r)} - VN^{(r)} + BY = 0 \quad , \quad M^{(l)} Z^T A - N^{(l)} Z^T + XC = 0,$$

where $M^{(r)}$, $N^{(r)}$, $M^{(l)}$ and $N^{(l)}$ define the interpolation conditions

- is **universal** : every $\hat{T}(s)$ can be obtained via projection from $T(s)$.

MIMO multi-point interpolation

Consider the 2×2 case:

$$T(s) = \begin{bmatrix} t_{11}(s) & t_{12}(s) \\ t_{21}(s) & t_{22}(s) \end{bmatrix}, \quad \hat{T}(s) = \begin{bmatrix} \hat{t}_{11}(s) & \hat{t}_{12}(s) \\ \hat{t}_{21}(s) & \hat{t}_{22}(s) \end{bmatrix}.$$

Standard interpolation at $s = \sigma$ would imply

$$T(\sigma) = \hat{T}(\sigma) \iff \begin{cases} t_{11}(\sigma) = \hat{t}_{11}(\sigma), & t_{12}(\sigma) = \hat{t}_{12}(\sigma), \\ t_{21}(\sigma) = \hat{t}_{21}(\sigma), & t_{22}(\sigma) = \hat{t}_{22}(\sigma), \end{cases}$$

This is too restrictive \implies **Tangential interpolation**

- Left Tangential Interpolation :

$$\begin{aligned} \begin{bmatrix} x_1 & x_2 \end{bmatrix} T(\sigma) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \hat{T}(\sigma) \\ \iff \begin{cases} x_1 t_{11}(\sigma) + x_2 t_{21}(\sigma) &= x_1 \hat{t}_{11}(\sigma) + x_2 \hat{t}_{21}(\sigma) \\ x_1 t_{12}(\sigma) + x_2 t_{22}(\sigma) &= x_1 \hat{t}_{12}(\sigma) + x_2 \hat{t}_{22}(\sigma) \end{cases} \end{aligned}$$

- Right Tangential Interpolation :

$$T(\sigma) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \hat{T}(\sigma) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Two-Sided Tangential Interpolation :

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} T(\sigma) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \hat{T}(\sigma) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In general, Tangential Interpolation :

Left Tangential Interpolation conditions :

$$\left. \frac{d^{i-1}}{ds^{i-1}} \{x(s)T(s)\} \right|_{s=\sigma} = \left. \frac{d^{i-1}}{ds^{i-1}} \{x(s)\hat{T}(s)\} \right|_{s=\sigma},$$

Right Tangential Interpolation conditions :

$$\left. \frac{d^{i-1}}{ds^{i-1}} \{T(s)y(s)\} \right|_{s=\sigma} = \left. \frac{d^{i-1}}{ds^{i-1}} \{\hat{T}(s)y(s)\} \right|_{s=\sigma},$$

Two-sided Tangential Interpolation conditions :

$$\left. \frac{d^{f+g-1}}{ds^{f+g-1}} \{x^{(f)}(s)T(s)y^{(g)}(s)\} \right|_{s=\sigma} = \left. \frac{d^{f+g-1}}{ds^{f+g-1}} \{x^{(f)}(s)\hat{T}(s)y^{(g)}(s)\} \right|_{s=\sigma}.$$

Less powerful results in MIMO

$\hat{T}(s)$ interpolating $T(s)$ tangentially in directions contained in X, Y

- can always be obtained via **generalized Krylov** methods

$$\bigcup_{i=1}^r \mathcal{K}_{b_i} \left((A - \sigma_i I_N)^{-1}, (A - \sigma_i I_n)^{-1} B, Y \right) \subseteq \text{Im}(V),$$

$$\bigcup_{j=1}^r \mathcal{K}_{c_j} \left((A - \sigma_j I_N)^{-T}, (A - \sigma_j I_n)^{-T} C^T, X \right) \subseteq \text{Im}(Z)$$

- or, equivalently, via the solution of **Sylvester Equations**

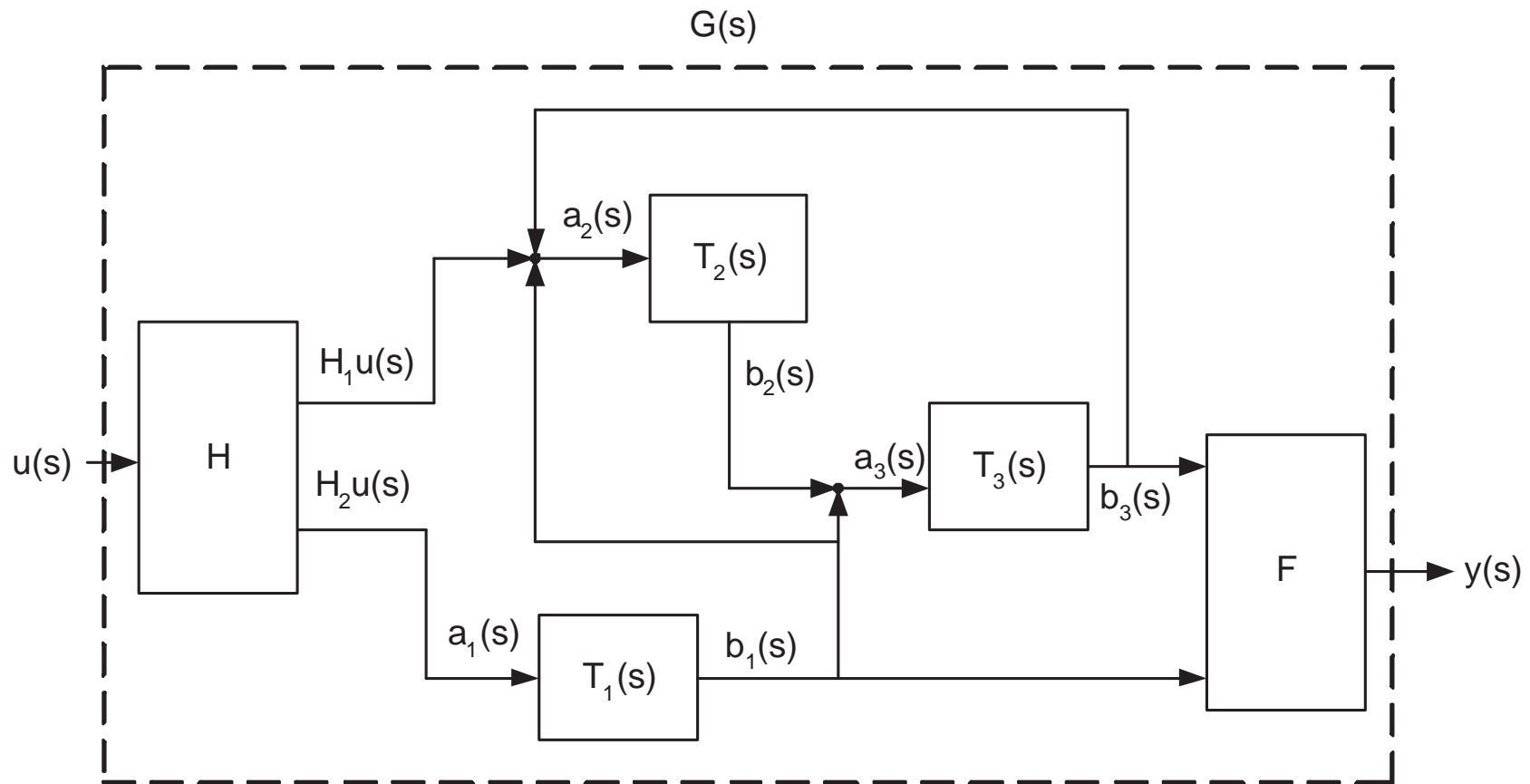
$$AVM^{(r)} - VN^{(r)} + BY = 0 \quad , \quad M^{(l)} Z^T A - N^{(l)} Z^T + XC = 0,$$

where $M^{(r)}, N^{(r)}, M^{(l)}$ and $N^{(l)}$ define the interpolation conditions

- is very general but **not universal anymore**

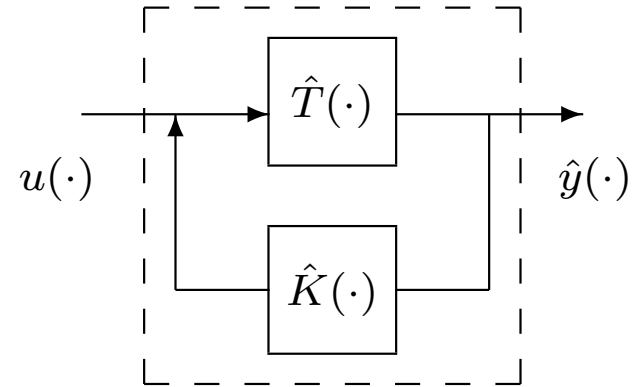
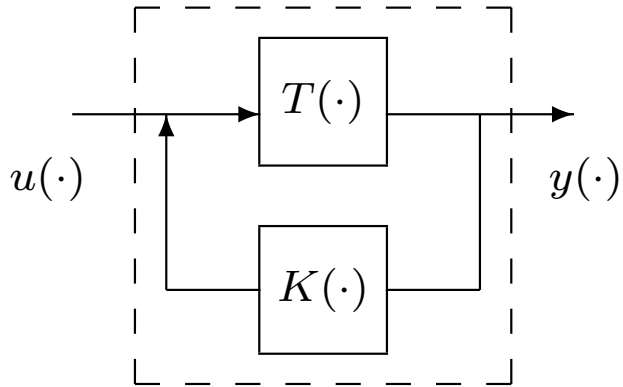
Model Reduction of Structured Systems

$G(s)$ is a set of subsystems $T_i(s)$ that interconnect:

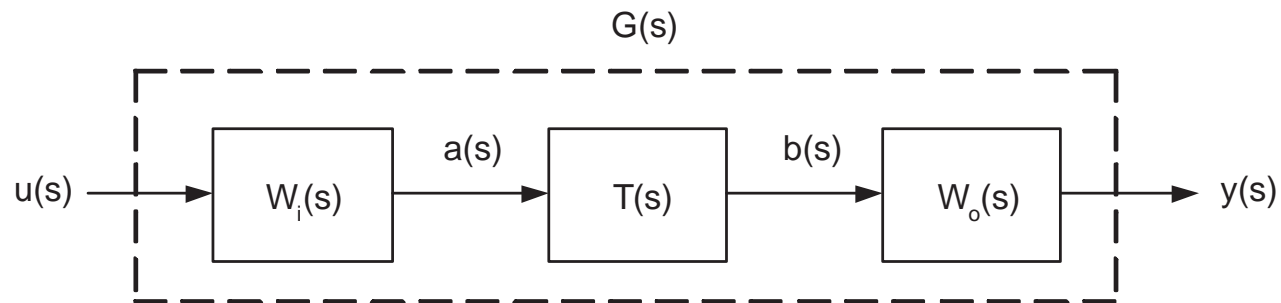


Examples:

- Controller Order Reduction



- Weighted System



New model reduction techniques for structured systems

- Interconnected System Balanced Truncation algorithm:
Project each subsystem $T_i(s)$ by keeping the dominant part of the state with respect to the mapping from $u(s)$ to $y(s)$ and not $a_i(s)$ to $b_i(s)$.
 - Generalization of Balanced Truncation for interconnected systems
 - Unify other existing structured model reduction techniques
 - $O(n^3)$ technique
- Structure preserving Krylov:
Project subsystems $T_i(s)$ in order that $G(s)$ and $\hat{G}(s)$ satisfy (tangential) interpolation conditions ($O(n)$ if sparsity).

Error estimation via Chandrasekhar equations

For $E(s) \doteq C_e(sI - A_e)^{-1}B_e \doteq T(s) - \hat{T}(s)$, compute $\gamma^* \doteq \|E(s)\|_\infty$

Estimate γ^* using the (Riccati difference) iterations

$$P_{i+1} = A_e^T P_i A_e + C_e^T C_e - K_i^T R_i^{-1} K_i,$$

where $K_i := B_e^T P_i A_e$, and $R_i = B_e^T P_i B_e - \gamma^2 I_m$

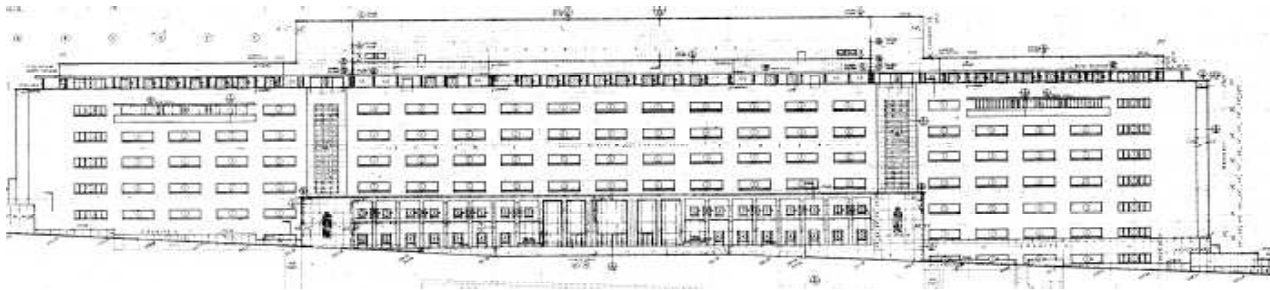
- Convergence behaviour and speed allows to estimate γ^* accurately.
- Use factored Chandrasekhar iteration for cheap (sparse) calculations :

$$\delta P_i = L_i^T \Sigma_2 L_i, \quad R_i = S_i^T \Sigma_1 S_i, \quad \text{and} \quad K_i = S_i^T \Sigma_1 G_i$$

- Construct a J -orthogonal matrix Q , i.e., $Q^T J Q = J$, such that

$$Q \begin{bmatrix} S_i & G_i \\ L_i B_e & L_i A_e \end{bmatrix} = \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}.$$

Application: Model Reduction of a Building



It is a second order system, i.e. it can be modelled as

$$\begin{cases} M\ddot{q}(t) + D\dot{q}(t) + Sq(t) = Bu(t) \\ y(t) = Cq(t) \end{cases}, \quad \text{where}$$

- $M \in \mathbb{C}^{n \times n}$ is the mass matrix
- $D \in \mathbb{C}^{n \times n}$ is the damping matrix
- $S \in \mathbb{C}^{n \times n}$ is the stiffness matrix

S-O models appear in mechanical engineering, circuit simulation,...

Model approximation via interpolation

$$T(s) = B^T (Ms^2 + Ds + S)^{-1} B,$$

$$C^T = B = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T,$$

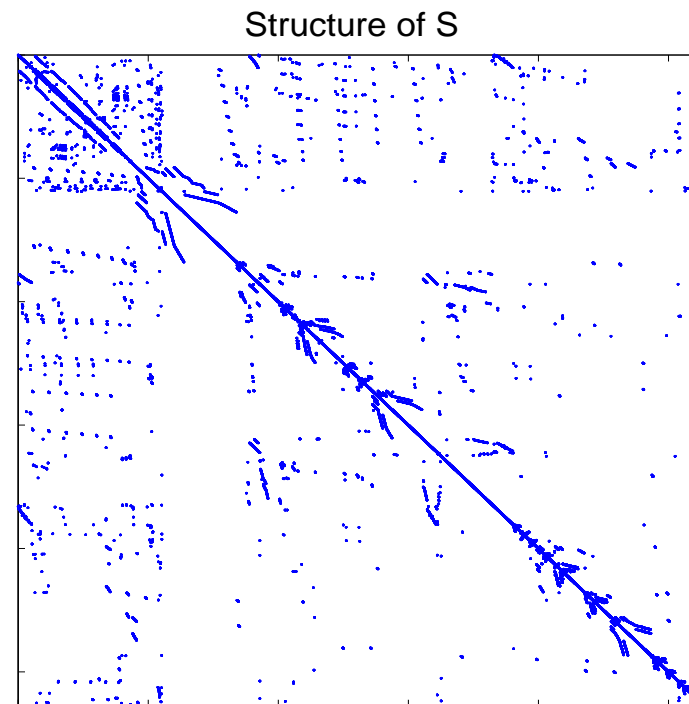
$$n = 26394,$$

can be described as an
interconnected system

M is diagonal,

S is symmetric and sparse

Too complex to compute $T(s)$
exactly!

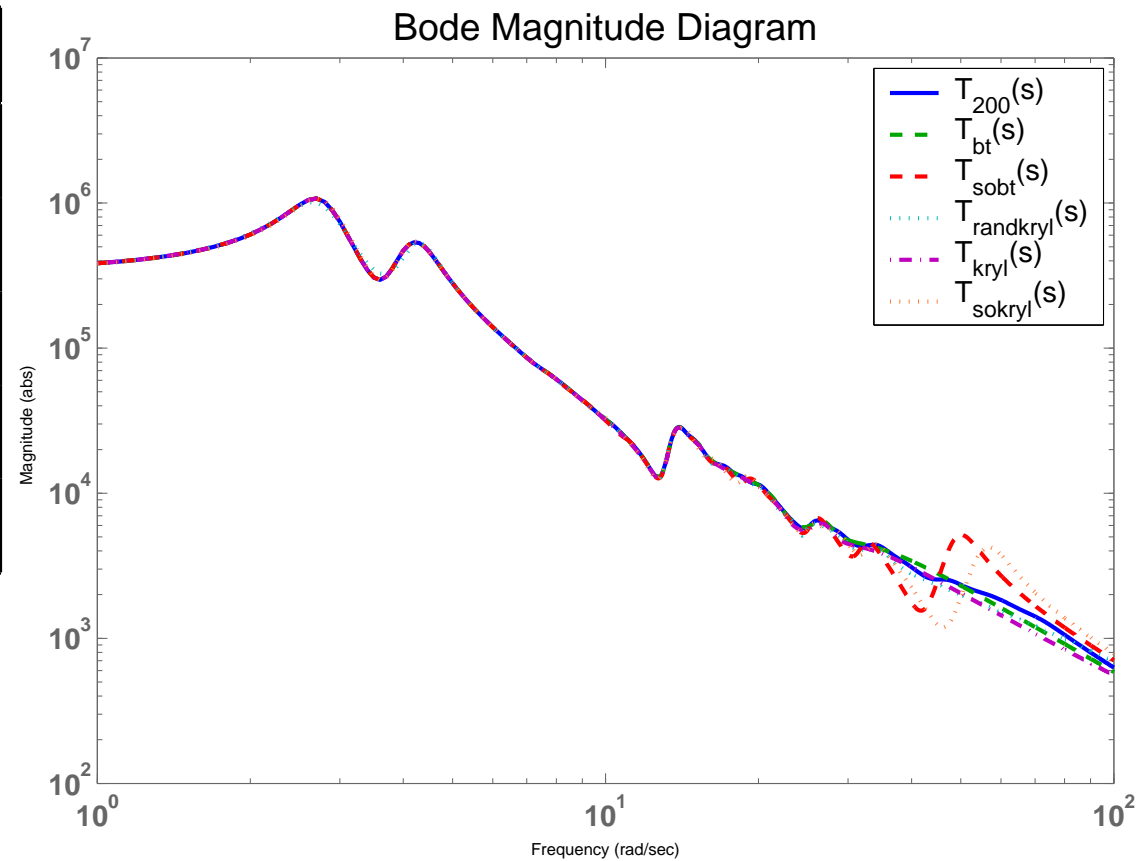


non zero elements = 278 904

Comparison of Different Reduced Order Systems

$\hat{T}(s)$	$\frac{\ T_{200}(s) - \hat{T}(s)\ _{\infty}}{\ T_{200}(s)\ _{\infty}}$
BT (20)	$4.3 \cdot 10^{-4}$
SOBT (40)	$2.6 \cdot 10^{-4}$
KRYL (20)	$8.3 \cdot 10^{-4}$
SOKRYL (28)	$5.8 \cdot 10^{-2}$
KRYLBIS (20)	$7 \cdot 10^{-2}$

Simulations



Contributions

Tangential interpolation

Gallivan, Vandendorpe, Van Dooren, “Model reduction of MIMO systems via tangential interpolation”, SIMAX 2004

Gallivan, Vandendorpe, Van Dooren, “Sylvester equations and projection-based model reduction”, JCAM 2003.

Gallivan, Vandendorpe, Van Dooren, “Model reduction via truncation: an interpolation point of view”, LAA 2003.

Thesis Vandendorpe, 2004

Structured model reduction

Chahlaoui, Lemonnier, Vandendorpe, Van Dooren, “Second-order balanced truncation”, accepted for LAA

Vandendorpe, Van Dooren, “Model reduction of interconnected systems”, MTNS, 2004.

Chahlaoui, Lemonnier, Vandendorpe, Van Dooren, “Second-order structure preserving model reduction”, MTNS, 2004.

Chandrasekhar and Riccati iterations

Gallivan, Rao, Van Dooren, “Singular Riccati equations stabilizing large-scale systems”, accepted for LAA.

Chahlaoui, Gallivan, Van Dooren, “The H_∞ -norm calculation for large sparse systems”, MTNS, 2004.

Time-varying model reduction

Chahlaoui, Van Dooren, “Model reduction of time-varying systems”, Oberwolfach book, 2005

Thesis Chahlaoui.

Surveys etc.

Antoulas, Sorensen, Gallivan, Van Dooren, Grama, Hoffmann, Sameh, “Model reduction and real-time dynamic data driven systems”, DAREMA book, 2005

Antoulas, Sorensen, Gallivan, Van Dooren, Grama, Hoffmann, Sameh, “Model reduction of large-scale dynamical systems”, ICCS, 2004.

Chahlaoui, Gallivan, Vandendorpe, Van Dooren, “Model Reduction of Second Order Systems”, Oberwolfach book, 2005.

Chahlaoui, Van Dooren, “Benchmark examples for model reduction of linear time invariant dynamical systems”, Oberwolfach book, 2005

Vandendorpe, Van Dooren, “Projection of state space realizations”, Unsolved Problems book, 2004

Future Work: automatic strategies for finding good interpolation points