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Theorem 0.1 *Consider $\ell_0, \ell_1, \ldots, \ell_m$ linear functionals on the vector space X. Then the following statements are equivalent.*

(i) $\ell_1(x) = \cdots = \ell_m(x) = 0 \quad \Rightarrow \quad \ell_0(x) = 0 \quad \forall \quad x \in X$

(ii) $\exists\ t_1, \ldots, t_m$ *satisfying* $\ell_0 = t_1\ell_1 + \cdots + t_m\ell_m$.

Proof. It is immediate that $(ii) \Rightarrow (i)$. We now show that $(i) \Rightarrow (ii)$. Let $S = \cap_{i=1}^{q} \mathcal{N}(\ell_i)$ where $\mathcal{N}$ denotes nullspace. Consider the quotient space $X/S = \{x + S : x \in X\}$. Define $L : X/S \to I\!R^m$ by

$$L(x+S) = \begin{pmatrix} \ell_1(x+S) \\ \vdots \\ \ell_m(x+S) \end{pmatrix} = \begin{pmatrix} \ell_1(x) \\ \vdots \\ \ell_m(x) \end{pmatrix}.$$

Clearly, L is linear and one-to one. Let $\Re(L)$ denote the range space of L and define $\hat{\ell} : \Re(L) \subset I\!R^m \to I\!R$ by

$$\hat{\ell}(L(x+S)) = \ell_0(x+S) = \ell_0(x).$$

The latter equality follows from our hypothesis. Also $\hat{\ell}$ is well-defined since $L(x+S) = 0 \Rightarrow \ell_0(x) = 0$. Next, extend $\hat{\ell}$ to all of $I\!R^m$ by linearity. We know that $\hat{\ell}$ can be represented via the Euclidean inner product on $I\!R^m$, i.e., $\exists$ $t = (t_1, t_2, \ldots, t_m)^T \in I\!R^m$ such that $\hat{\ell}(y) =< t, y > \forall y \in I\!R^m$. Hence,

$$\ell_0(x) = \hat{\ell}(L(x+S)) = \hat{\ell} \begin{pmatrix} \ell_1(x) \\ \vdots \\ \ell_m(x) \end{pmatrix} = t_1\ell_1(x) + \cdots + t_m\ell_m(x) \quad \forall \quad x \in X$$

i.e.,

$$\ell_0 = t_1\ell_1 + \cdots + t_m\ell_m.$$

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