THE EULER-LAGRANGE EQUATION AS A MULTIPLIER RULE

SPCV:

$$\min J(y) = \int_a^b f(x, y(x), y'(x))\,dx$$

$$\text{S.T.} \quad y(a) = \alpha \text{ and } y(b) = \beta.$$

A DIRECT MULTIPLIER APPROACH

$$L(y, \lambda_1, \lambda_2) = J(y) + \lambda_1 (y(a) - \alpha) + \lambda_2 (y(b) - \beta)$$

$$\therefore L'(y, \lambda_1, \lambda_2)(\eta) = J'(y)(\eta) + \lambda_1 \eta(a) + \lambda_2 \eta(b) = 0$$

$$J'(y)(\eta) = \lambda_1 \eta(a) + \lambda_2 \eta(b)$$

$$J'(y)(\eta) = \int_a^b (f_y \eta + f_{y'} \eta')$$

WHAT DO WE DO?

INTEGRATION by PARTS

$$\int_a^b (f_y \eta + f_{y'} \eta') = \int_a^b \left(f_y - \frac{d f_{y'}}{dx}\right) \eta$$

so

$$\int_a^b \left(f_y - \frac{d f_{y'}}{dx}\right) \eta = \lambda_1 \eta(a) + \lambda_2 \eta(b)$$

WHAT DO we do ?

INTRODUCE AN INNER PRODUCT and STATE IN TERM OF REPRESENTEES.

FIRST ATTEMPT: L^2

$$\langle u, v \rangle = \int_a^b uv$$

$$\nabla J(u) = f_y - \frac{d f_{y'}}{dx}$$

but $\nexists$ $\hat{y}$ s.t. $\int \hat{y} \eta = \eta(a)$ (or $\eta(b)$).

THE LINEAR FUNCTIONAL $\eta \to \eta(a)$ IS NOT CONTINUOUS, $\therefore$ NOT BOUNDED.

OK INTEGRATE BY PARTS IN OTHER DIRECTION

$$\int_a^b (f_y \eta + f_{y'} \eta') = \int_a^b (f_{y'} - \int_a^x f_y) \eta'$$

SO

$$\int_a^b (f_{y'} - \int_a^x f_y) \eta' = \lambda_1 \eta(a) + \lambda_2 \eta(b)$$

SECOND ATTEMPT H^1

$$\langle u, v \rangle = u(a)v(a) + \int_a^b u'v'$$

OBSERVE $\langle 1, \eta \rangle = \eta(a)$

$$\langle x-a+1, \eta \rangle = \eta(b)$$

$$\left((a-a+1)\eta(a) + \int_a^b \eta' \right) = \eta(a) + \eta(b) - \eta(a) = \eta(b)$$

$\therefore$ WE HAVE

$$\int_a^x (f_{y'} - \int_a^x f_y) = \lambda_1 + \lambda_2 x \qquad \text{Euler-Lagrange}$$

$$f_{y'} - \int_a^x f_y = \lambda_2 \qquad \text{E-L IN INTEGRAL FORM}$$

$$\frac{d\, f_{y'}}{dx} - f_y = 0 \qquad \text{E-L IN DERIVATIVE FORM}$$

COULD EULER HAVE THOUGHT OF THIS?

THERE WAS NO FUNCTIONAL ANALYSIS?

LET'S VOTE

YES	NO	DNV