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Problem Set #3

Use the variational inequality to solve the following ~~three~~ two problems. Be reasonably complete in your claims and your arguments.

1. Prove Gordan's Theorem:

Theorem (Gordon, 1873) Let $a_1, \ldots, a_m \in \Re^m$. *Then either*

I) *There exists* $z \in \Re^n$ *such that* $\langle a_i, z \rangle > 0,\ i = 1, \ldots, m,$ *or*

II) *There exist nonnegative* $\lambda_1, \ldots, \lambda_m$, *not all zero, such that* $\lambda_1 a_1 + \ldots + \lambda_m a_m = 0$.

Hint: Let $S = \left\{ \lambda_1 a_1 + \ldots + \lambda_m a_m : \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}$.

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & J(a) = \langle a, a \rangle \\ \text{subject to} & a \in S. \end{array} \qquad (*)$$

First, argue that Problem (*) always has a solution. Then suppose that (I) does not hold. Use the variational inequality necessary condition to show that the solution of Problem (*) must be $a^* = 0$. This establishes (II).

2. Consider a random sample $x_1, \ldots, x_m$ satisfying $x_1 < x_2 < \ldots < x_m$ and contained in an interval (a, b). Let $x_0 = a$ and $x_{m+1} = b$. Let $S_1(x_0, \ldots, x_{m+1})$ denote the subspace of $L^1[a, b]$ consisting of continuous functions which are linear on $[x_i, x_{i+1}]$, $i = 0, \ldots, m$ and vanish outside the interval (x_0, x_{m+1}). Find the maximum likelihood estimate corresponding to the subspace $S_1(x_0, \ldots, x_{m+1})$. (Give it in closed form.)

3. Use the Euler-Lagrange equation to find the extremals of

$$J(y)=\int_{x_0}^{x_1}\frac{\sqrt{1+y'(x)^2}}{x}dx;\ y(x_0)=\alpha,\ y(x_1)=\beta.$$

In solving the differential equation given by the Euler-Lagrange equation make the change of variables

$$y'=\tan(t)$$

the solution in parametric form is

$$\begin{aligned} x &= c_1\sin(t) \\ y &= c_2-c_1\cos(t) \end{aligned}$$

or eliminating t

$$x^2+(y-c_2)^2=c_1^2$$