Theorem 0.1 *There exists $f : I\!R \to I\!R$ which is additive but not linear.*

Proof. Let $b^* \cup \{b_\beta : \beta \in B\}$ be a basis for $I\!R/Q$ (reals with rationals as scalar field). Also let S denote the (rational) linear span of $\{b_\beta : \beta \in B\}$. For $x \in I\!R$ we can write (uniquely)

$$x = \alpha(x)b^* + b(x) \quad \text{where} \quad \alpha(x) \in Q \quad \text{and} \quad b(x) \in S.$$

Define $f : I\!R \to I\!R$ by $f(x) = \alpha(x) \quad \forall x \in I\!R$. With x as above and

$$y = \alpha(y)b^* + b(x),$$

we have

$$x + y = (\alpha(x) + \alpha(y))b^* + b(x + y) \quad \text{with} \quad b(x + y) \in S.$$

Hence,

$$f(x + y) = \alpha(x) + \alpha(y) = f(x) + f(y).$$

Recall that, by definition, a linear function is homogeneous. Moreover, in this case homogeneity implies $f(x) = f(x \cdot 1) = f(1) \cdot x$. However, if $f(x) = ax$, then $f(b^*) = ab^* = 1$, so $a \neq 0$, and for non-zero $b \in S \quad f(b) = ab = 0$, so $a = 0$. This is a contradiction. Hence f is additive and not linear. ■