

CAAM 452: Homework 1
Posted online on January 13
Due January 26 in class
Printout of codes are to be included

Problem 1 (10 points)

Show that the centered finite difference approximation $D^2u(x)$ approximates $u''(x)$ with an error $\mathcal{O}(h^2)$. Recall that

$$D^2u(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

Problem 2 (20 points)

Let $h = 1/(N+1)$. Define the tridiagonal matrix $A = (a_{ij})$ such that the only nonzero entries are

$$\begin{aligned} a_{ii} &= \frac{2}{h^2}, & 1 \leq i \leq N \\ a_{i,i+1} &= \frac{-1}{h^2}, & 1 \leq i \leq N-1 \\ a_{i,i-1} &= \frac{-1}{h^2}, & 2 \leq i \leq N \end{aligned}$$

Prove that the eigenpairs $(\lambda_k, \mathbf{u}_k)$ of A are

$$1 \leq k \leq N, \quad \lambda_k = \frac{2}{h^2}(1 - \cos(k\pi h)), \quad \mathbf{u}_k = \begin{pmatrix} \sin(k\pi h) \\ \sin(2k\pi h) \\ \sin(3k\pi h) \\ \vdots \\ \sin(Nk\pi h) \end{pmatrix}$$

Hint: you need to check that $A\mathbf{u}_k = \lambda_k\mathbf{u}_k$.

Problem 3 (30 points)

(a) Use the method of undetermined coefficients to set up the 5×5 Vandermonde system that would determine a fourth-order accurate finite difference approximation to $u''(x)$ based on 5 equally spaced points,

$$u''(x) = c_{-2}u(x-2h) + c_{-1}u(x-h) + c_0u(x) + c_1u(x+h) + c_2u(x+2h) + O(h^4).$$

(b) Compute the coefficients using the MATLAB code `fdstencil.m` available from the website, and check that they satisfy the system you determined in part (a).

(c) Test this finite difference formula to approximate $u''(1)$ for $u(x) = \cos(3x)$ with values of h from the array `hvals = logspace(-1, -4, 13)`. Make a table of the error vs. h and compare against the predicted error from the leading term of the expression printed by `fdstencil`.

Also produce a log-log plot of the absolute value of the error vs. h . Describe your results.

Problem 4 (40 points)

(a) Implement the finite difference method of second order for solving

$$\begin{aligned} u''(x) + cu(x) &= f(x), & 0 < x < 1 \\ u(0) &= \alpha, & u(1) = \beta \end{aligned}$$

The parameter c is a nonnegative real number.

(b) Verify your code on the following three examples

1. The exact solution is $u(x) = 1 - x$ and $c = 0$.
2. The exact solution is $u(x) = x$ and $c = 1$.
3. The exact solution is $u(x) = xe^{-x}$ and $c = 1$.

Consider a uniform partition of the interval $(0, 1)$ with M intervals. Make a table of the max-norm error vs. $h = 1/M$ for $M = 10, 20, 40, 80$. Obtain the numerical convergence rate.