

CAAM 452: Homework 2
Posted online on February 2
Due February 14 in class
Printout of codes are to be included

Problem 1 (30 points)

The MATLAB script `poisson.m` solves the Poisson problem on the unit square $(0,1)^2$ with a grid size $h_x = h_y = h = 1/(N+1)$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x,y) = \exp(x+y/2)$, using Dirichlet boundary conditions and the right hand side $f(x,y) = 1.25 \exp(x+y/2)$.

1. Test this script by performing a grid refinement study to verify that it is second order accurate: consider $h = 1/10, 1/20, 1/40, 1/80$.
2. Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $h_x = h_y = h$. Test your modified script on the domain $(0,1) \times (1,3)$ and use the values $h = 1/10, 1/20, 1/40, 1/80$. Give the errors and convergence rates.
3. Further modify the code to allow $h_x \neq h_y$ and test the modified script on the domain $(0,1)^2$ with $h_x = 3h_y$. Let $h = \max(h_x, h_y)$. Give the errors and convergence rates for $h = 1/10, 1/20, 1/40, 1/80$.

Problem 2 (20 points)

1. Consider the finite difference method:

$$\frac{U_{i+2} - U_{i+1} - U_{i-1} + U_{i-2}}{3h^2} = f(x_i)$$

to solve the problem $u'' = f$ on a uniform grid $x_i = ih$ of the unit interval. What is the order of the local truncation error? Is this finite difference method consistent?

2. Consider the finite difference method:

$$\frac{U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + U_{i-1,j}}{h^2} = f(x_i, y_j)$$

to solve the problem $u_{xx} - u_{yy} = f$ on a uniform grid $x_i = ih, y_j = jh$ of the unit square. What is the order of the local truncation error? Is this finite difference method consistent?

Problem 3 (50 points)

- (a) Implement the finite element method using continuous piecewise linears for solving

$$\begin{aligned} -u''(x) &= f(x), 0 < x < 1 \\ u(0) &= 0, \quad u(1) = 0 \end{aligned}$$

- (b) Verify your code on the following two examples

1. The exact solution is $u(x) = x(1-x)$.
2. The exact solution is $u(x) = x(1-x)e^{-x^2}$

Consider a sequence of uniform meshes with $h = 1/4, 1/8, 1/16, 1/32$. Denote by u_h the finite element solution. For each mesh, plot the numerical solution and compute the following errors:

$$err0 = \left(\int_0^1 (u - u_h)^2 dx \right)^{1/2}, \quad err1 = \left(\int_0^1 (u' - u'_h)^2 dx \right)^{1/2}$$

Obtain the numerical convergence rates for *err0* and *err1*. Hint: to compute the errors, write the integral as a sum of integrals over each subinterval and use a quadrature rule (trapezoid rule for *err0* and midpoint rule for *err1*). For instance if $h = 1/(N + 1)$ and the grid nodes are $x_i = ih$, we can write

$$\int_0^1 (u - u_h)^2 dx = \sum_{i=1}^{N+1} \int_{x_{i-1}}^{x_i} (u - u_h)^2 dx \approx \sum_{i=1}^{N+1} \frac{h}{2} ((u(x_{i-1}) - u_h(x_{i-1}))^2 + (u(x_i) - u_h(x_i))^2)$$

(c) Modify the code written in (a) to solve the non-homogeneous boundary condition

$$\begin{aligned} -u''(x) &= f(x), 0 < x < 1 \\ u(0) &= \alpha, \quad u(1) = \beta \end{aligned}$$

Test your code for $h = 1/16$ and for the exact solutions given below

1. The exact solution is $u(x) = x$.
2. The exact solution is $u(x) = e^x$