CAAM 552: Homework 3
Posted online on September 22
Due October 6 in class

All problems are taken from the textbook.

Problem 1 (20 points)

Prove that any function in $L_0^2(\Omega) = \{\phi \in L^2(\Omega) : \quad \int_\Omega \phi = 0\}$ can be written as a limit of functions in $\mathcal{D}_0(\Omega) = \{\phi \in \mathcal{C}_0^\infty(\Omega) : \quad \int_\Omega \phi = 0\}$. (Hint: pick a sequence in $\mathcal{C}_0^\infty(\Omega)$ that converges in L^2 and modify it to get a mean-zero sequence.)

Problem 2 (20 points)

Let $a_{ij} \in L^\infty(\Omega)$, let $b_i \in L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Define

$$Au(x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\big(a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\big) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i}(x) + b_0(x)u(x)$$

Consider the problem:

$$Au = f, \quad \text{in} \quad \Omega \tag{1}$$

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\nu_j = 0, \quad \text{on} \quad \partial\Omega \tag{2}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit normal vector on $\partial\Omega$. Define

$$\forall u, v \in H^1(\Omega) : \quad a(u,v) = \int_\Omega \big(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} v + b_0 uv\big)$$

Prove that the variational problem $a(u,v) = (f,v) \quad \forall v \in H^1(\Omega)$ is equivalent to problem (1)-(2), under a certain assumption on u.

Problem 3 (20 points)

Suppose that Ω is bounded and that $f_j \to f$ in $L^p(\Omega)$. Show that

$$\int_\Omega f_j(x)dx \to \int_\Omega f(x)dx, \quad \text{as } j \to \infty$$

Problem 4 (20 points)

Let $a, b \in \mathbb{R}$. Prove that

$$ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$$

Apply this to show that

$$a + b \le \big((1+\epsilon)a^2 + (1+1/\epsilon)b^2\big)^{1/2}$$

Problem 5 (20 points)

Let $\Omega = [0,1]$. Prove that $\{v \in \mathcal{C}^1([0,1]) : \quad v(0) = 0\}$ is dense in the set $\{v \in H^1(\Omega) : \quad v(0) = 0\}$. (Hint: use the density of $\mathcal{C}^1(\bar{\Omega})$ in $H^1(\Omega)$ and Sobolev's embedding theorem.)