A Two-grid Stabilization Method for Solving the Steady-state Navier-Stokes Equations

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We formulate a subgrid eddy viscosity method for solving the steady-state incompressible flow problem. The eddy viscosity does not act on the large flow structures. Optimal error estimates are obtained for velocity and pressure. The numerical illustrations agree completely with the theoretical results. Numer Methods Partial Differential Eq 22: 728–743, 2005

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I. INTRODUCTION

We consider herein the approximate solution of the steady-state Navier-Stokes problem:

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \qquad \text{in } \Omega,$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, $u : \Omega \to \mathbb{R}^d$ the fluid velocity, $p : \Omega \to \mathbb{R}$ the fluid pressure and f a prescribed body force. The kinematic viscosity, which is inversely proportional to the Reynolds number *Re*, is denoted by ν ($\nu > 0$).

In this article, we consider a subgrid eddy viscosity model as a numerical stabilization of a convection dominated and underresolved flow. This approach adds an artificial viscosity only on the fine scales and is referred to as a subgrid eddy viscosity model. We consider the classical finite element method for the spatial discretization. The resulting scheme involves two grids

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coupled to each other through the artificial viscosity term. Unlike the standard eddy viscosity method that is too diffusive, our method only adds diffusion on the small scales.

The general idea of using two-grid discretization to increase the *efficiency* of numerical methods was pioneered by J. Xu ([1], see also Marion and Xu [2]) and developed by Girault and Lions ([3, 4]). This two-grid discretization idea and previous work by Fortin et al. [5] on stabilizations in viscoelasticity are combined with the physical ideas of eddy viscosity models. This combination of ideas leads very naturally to the presented method.

The idea of the subgrid eddy viscosity model is inspired by earlier work of Guermond [6], in which the subgrid scale is augmented by bubble functions. The artificial viscosity is added only on the fine scales of the problem. This concept is generalized by Layton [7] for the stationary convection diffusion problem. In the work of Kaya and Layton [8], this model has been connected with another consistent stabilization technique, also known as variational multiscale method, introduced by Hughes [9]. The model has been analyzed for the time-dependent Navier-Stokes equations by John and Kaya [10] for the continuous finite element method and by Kaya and Rivière [11] for the discontinuous Galerkin method.

To motivate the method, we define the spaces $X := (H_0^1(\Omega))^d$ and $M := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$ and $L := \{\mathbb{L} \in (L^2(\Omega))^{d\times d}, \mathbb{L} = \mathbb{L}^T\}$ and consider a variational formulation of (1.1): find $u \in X$, $p \in M$, and $\mathbb{G} \in L$ such that

$$a(u, v) + c(u, u, v) - b(v, p) + (\nu_T \mathbb{D}(u), \mathbb{D}(v)) - (\nu_T \mathbb{G}, \mathbb{D}(v)) = (f, v), \qquad \forall v \in X,$$

$$b(u, q) = 0, \qquad \forall q \in M,$$

$$(\mathbb{G} - \mathbb{D}(u), \mathbb{L}) = 0, \qquad \forall \mathbb{L} \in L. \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the L^2 inner-product and the bilinear forms are defined below

$$a(v, w) := (2\nu \mathbb{D}(v), \mathbb{D}(w)), \qquad \forall v, w \in X,$$

$$c(z, v, w) := \tfrac{1}{2}(z \cdot \nabla v, w) - \tfrac{1}{2}(z \cdot \nabla w, v), \qquad \forall z, v, w \in X,$$

$$b(v, q) := (q, \nabla \cdot v), \qquad \forall v \in X, \forall q \in M. \tag{1.3}$$

Here, the stress tensor is defined by $\mathbb{D}(v) = 0.5(\nabla v + \nabla v^T)$ and the parameter $\nu_T > 0$ is the eddy viscosity parameter. In the continuous case, this method reduces to the standard Navier-Stokes equations. However, in the discrete case it leads to different discretizations. In this article, we consider multiscale finite element approximation of the Navier-Stokes equation based on the formulation (1.2).

Our approach can be understood as a large eddy simulation model but the point herein is to study it as a numerical stabilization. To our knowledge, this is the first article presenting error estimates for velocity and pressure in L^2 and numerical examples for this subgrid eddy viscosity model.

The outline of the article is as follows. In the next section, some notation and the finite element scheme are presented. In Section III, IV, and V, error estimates are given for velocity and pressure. The algorithm and numerical experiments are described in Section VI. Conclusions follow.

II. NOTATION AND SCHEME

We first recall some standard notation: $L^2(\Omega)$ denotes the space of square-integrable functions over Ω with norm $\|\cdot\|$ and inner-product $(\cdot,\cdot)$; $H^k(\Omega)$ denotes the standard Sobolev space with norm $\|\cdot\|_k$ and semi-norm $|\cdot|_k$ (Adams [12]). $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ of functions whose trace is zero on $\partial\Omega$; it is a Banach space with norm $|\cdot|_1$. Finally, the space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, and if $\langle\cdot,\cdot\rangle$ denotes the duality pairing, $H^{-1}(\Omega)$ is equipped with the negative norm

$$\|z\|_{-1} = \sup_{v\in H_0^1(\Omega)} \frac{|\langle z, v\rangle|}{\|v\|_1}.$$

The forms (1.3) defined in Section I, have the following properties. The bilinear form $a(\cdot,\cdot)$ is clearly coercive in X: there is a constant $C_1 > 0$ such that

$$a(v, v) = 2\nu\|\mathbb{D}(v)\|^2 \geq C_1\nu\|\nabla v\|^2, \qquad \forall v \in X, \tag{2.1}$$

owing to the Korn's inequality (Duvaut and Lions [13]). The trilinear form $c(\cdot,\cdot,\cdot)$ satisfies the following bound (Girault and Raviart [14] with Korn's inequality): there exist constants $\tilde{K}, K > 0$ such that

$$c(z, v, w) \leq \tilde{K}\|\nabla z\|\,\|\nabla v\|\,\|\nabla w\| \leq K\|\mathbb{D}(z)\|\,\|\mathbb{D}(v)\|\,\|\mathbb{D}(w)\|, \qquad \forall z, v, w \in X. \tag{2.2}$$

We also recall the following property of c:

$$c(z, v, v) = 0, \qquad \forall z, v \in X. \tag{2.3}$$

We now introduce the finite element discretization of (1.2). Let τ^h and τ^H be two regular triangulations of the domain Ω, such that h (resp. H) denotes the maximum diameter of the elements in τ^h (resp. τ^H) and such that $h < H$. We will refer to the mesh obtained from τ^h as the *fine mesh* and the mesh obtained from τ^H as the *coarse mesh.* Let (X^h, M^h) be a pair of conforming finite element spaces satisfying the inf-sup condition: there exists a constant β independent of h such that

$$\inf_{q^h\in M^h}\sup_{v^h\in X^h} \frac{b(v^h, q^h)}{\|q^h\|\,\|\nabla v^h\|} \geq \beta > 0. \tag{2.4}$$

Examples of such compatible spaces are the mini-element spaces (Arnold et al. [15]), the Taylor-Hood spaces (Gunzburger [16]) and the continuous piecewise quadratics for the velocity space and discontinuous piecewise constants for the pressure space (Fortin [17]). We suppose the spaces (X^h, M^h) satisfy the following approximation properties for a given integer $k \geq 1$:

$$\inf_{v^h\in X^h} \{\|u - v^h\| + h\|\nabla(u - v^h)\|\} \leq Ch^{k+1}|u|_{k+1}, \qquad \forall u \in (H^{k+1}(\Omega))^d \cap X, \tag{2.5}$$

$$\inf_{q^h \in M^h} \|p - q^h\| \leq Ch^k|p|_k, \qquad \forall p \in H^k(\Omega) \cap M. \tag{2.6}$$

Let $L^H \subset L$ be a finite dimensional subspace of L containing discontinuous piecewise polynomials of degree $k - 1$. Let $P_{L^H} : L \to L^H$ be the L^2 orthogonal projection onto L^H. Thus, for $k \geq 1$ we have

$$(P_{L^H}\mathbb{L}, \mathbb{G}^H) = (\mathbb{L}, \mathbb{G}^H), \qquad \forall \mathbb{G}^H \in L^H, \forall \mathbb{L} \in L,$$

$$\|\mathbb{L} - P_{L^H}\mathbb{L}\| \leq CH^k|\mathbb{L}|_k, \qquad \forall \mathbb{L} \in L \cap (H^k(\Omega))^{d\times d}. \tag{2.7}$$

We will also use the fact that

$$\|I - P_{L^H}\| \leq 1. \tag{2.8}$$

Remark. For the error analysis given in the following two sections, only properties (2.7) and (2.8) are needed for the space L^H. For the numerical experiments, we will choose L^H to be a particular subspace, namely $L^H = \mathbb{D}(X^H)$, where X^H is the corresponding velocity space to X^h, but defined on the coarse mesh τ^H.

We propose the following finite element approximation of (1.2): find $(u^h, p^h) \in (X^h, M^h)$ satisfying

$$a(u^h, v^h) + c(u^h, u^h, v^h) - b(v^h, p^h) + g(u^h, v^h) = (f, v^h), \qquad \forall v^h \in X^h,$$

$$b(u^h, q^h) = 0, \qquad \forall q^h \in M^h, \tag{2.9}$$

where the bilinear form g is

$$g(v^h, w^h) = (\nu_T(I - P_{L^H})\mathbb{D}(v^h), (I - P_{L^H})\mathbb{D}(w^h)), \qquad \forall v^h, w^h \in X^h.$$

The eddy viscosity parameter $\nu_T > 0$ is to be defined later.

We can formulate another problem in the space of discrete divergence-free functions, denoted by V^h:

$$V^h := \{v^h \in X^h : (\nabla \cdot v^h, q^h) = 0, \forall q^h \in M^h\}. \tag{2.10}$$

Under the inf-sup condition (2.4), the formulation (2.9) is equivalent to the following problem: find $u^h \in V^h$ such that

$$a(u^h, v^h) + c(u^h, u^h, v^h) + g(u^h, v^h) = (f, v^h), \qquad \forall v^h \in V^h. \tag{2.11}$$

Our analysis is based on the assumption that the following global uniqueness condition holds:

$$K\|f\|_{-1} \leq C_1\nu^2, \tag{2.12}$$

where K is the constant of (2.2) and C_1 is the constant of (2.1). Recall [14] that under this condition (2.12), (1.2) has a unique solution $(u, p) \in (X, M)$. It is easy to show that under the condition (2.12) and the inf-sup condition (2.4), there exists a unique solution to (2.9).

Remark. We could also consider the following forms for the nonlinear term in (2.9):

$$c(z, v, w) = (z \cdot \nabla v, w) \qquad \text{(convective form)}$$

$$c(z, v, w) = -(z \cdot \nabla w, v) \qquad \text{(conservation form)}$$

In both cases, the analysis and error estimates remain the same.

Throughout the article, C is a generic constant that does not depend on ν, ν_T, h, and H, unless specified otherwise.

III. ERROR ESTIMATE FOR VELOCITY IN H_0^1

In this section, we first prove a stability result for the approximation of velocity for (2.9). We then prove an error estimate for the velocity in the energy norm.

Lemma 3.1. *The finite element approximation of velocity for* (2.9) *is stable*:

$$\nu \|\mathbb{D}(u^h)\|^2 + \nu_T \|(I - P_{L^H})\mathbb{D}(u^h)\|^2 \leq \frac{1}{2\nu C_1} \|f\|_{-1}^2, \tag{3.1}$$

where C_1 is the coercivity constant in (2.1).

Proof. The result is easily obtained by setting $v^h = u^h$ in (2.11) and using (2.3), Cauchy Schwarz, Korn's and Young's inequalities. ∎

Remark. Lemma 3.1 directly implies that

$$\|\mathbb{D}(u^h)\| \leq \frac{1}{\nu\sqrt{2C_1}} \|f\|_{-1}. \tag{3.2}$$

Theorem 3.2. *Suppose the global uniqueness condition* (2.12) *holds. Then*,

$$\begin{aligned}\nu\|\mathbb{D}(u - u^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u - u^h)\|^2 \leq C \inf_{w^h \in V^h} \Big\{ \nu\|\mathbb{D}(u - w^h)\|^2 \\ + \frac{K^2}{\nu}(\|\nabla u\| + \|\nabla u^h\|)^2 \|\mathbb{D}(u - w^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u - w^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|^2 \Big\} \\ + C \inf_{q^h \in M^h} \frac{1}{\nu}\|p - q^h\|^2,\end{aligned}$$

where C is independent of ν, ν_T, h, and H.

Proof. We first derive an error equation by noting that the exact solution satisfies

$$a(u, v^h) + c(u, u, v^h) - b(v^h, p) + g(u, v^h) = (f, v^h) + g(u, v^h), \qquad \forall v^h \in X^h,$$

$$b(u^h, q^h) = 0, \qquad \forall q^h \in M^h, \tag{3.3}$$

and by subtracting (2.9) from (3.3):

$$\begin{aligned} a(u - u^h, v^h) + c(u, u, v^h) - c(u^h, u^h, v^h) - b(v^h, p - p^h) + b(u - u^h, q^h) \\ + g(u - u^h, v^h) = g(u, v^h), \qquad \forall v^h \in X^h, \forall q^h \in M^h. \end{aligned} \tag{3.4}$$

We now decompose the error $u - u^h = \eta - \phi^h$, with $\eta = u - w^h$ and $\phi^h = u^h - w^h$, where w^h is any function in V^h. Rearranging the terms of (3.4), choosing $v^h = \phi^h \in V^h$, we obtain

$$\begin{aligned} a(\phi^h, \phi^h) + g(\phi^h, \phi^h) = a(\eta, \phi^h) + c(u, u, \phi^h) - c(u^h, u^h, \phi^h) + g(\eta, \phi^h) \\ - b(\phi^h, p - q^h) - g(u, \phi^h), \qquad \forall q^h \in M^h. \end{aligned} \tag{3.5}$$

To bound the linear terms in the right-hand side of (3.5), we simply use Cauchy Schwarz inequality and Young's inequality. To bound the nonlinear convective terms we rewrite these terms as follows:

$$c(u, u, \phi^h) - c(u^h, u^h, \phi^h) = c(u, \eta, \phi^h) + c(\eta, u^h, \phi^h) - c(\phi^h, u^h, \phi^h). \tag{3.6}$$

Then, the term (3.6) is estimated by using (2.2), Young's inequality and (2.12) as

$$\begin{aligned} |c(u, u, \phi^h) - c(u^h, u^h, \phi^h)| &\le K(\|\nabla u\| + \|\nabla u^h\|)\|\mathbb{D}(\eta)\| \, \|\mathbb{D}(\phi^h)\| + K\|\nabla u^h\| \, \|\mathbb{D}(\phi^h)\|^2 \\ &\le \frac{CK^2}{C_1 \nu} (\|\nabla u\| + \|\nabla u^h\|)^2 \|\nabla \eta\|^2 + \frac{1}{4} \nu \|\mathbb{D}(\phi^h)\|^2. \end{aligned}$$

From (2.8), the last term in the right-hand side of (3.5), which characterizes the inconsistency error, is bounded by

$$|g(u, \phi^h)| \le \nu_T \|(I - P_{L^H})\mathbb{D}(u)\| \, \|(I - P_{L^H})\mathbb{D}(\phi^h)\|. \tag{3.7}$$

Combining all the bounds above gives

$$\begin{aligned} \nu\|\mathbb{D}(\phi^h)\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(\phi^h)\|^2 \le C\Big[\nu\|\mathbb{D}(\eta)\|^2 + \frac{K^2}{\nu} (\|\nabla u\| + \|\nabla u^h\|)^2 \|\mathbb{D}(\eta)\|^2 \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(\eta)\|^2 + \frac{1}{\nu} \|p - q^h\|^2 + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|^2 \Big]. \end{aligned}$$

The final result is easily obtained by using the triangle inequality

$$\nu\|\mathbb{D}(u-u^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(u-u^h)\|^2 \le C(\nu\|\mathbb{D}(u-w^h)\|^2 + \nu_T\|(I-P_{L^H})\mathbb{D}(u-w^h)\|^2+\nu\|\mathbb{D}(\phi^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(\phi^h)\|^2). \blacksquare$$

By appropriately choosing the parameters ν_T, H, and h, one can obtain an optimal error estimate, as stated in the following corollary.

Corollary 3.3. *Under the assumption of Theorem* 3.2, *and under the regularity assumptions* $u \in H^{k+1}(\Omega)^d \cap X$ *and* $p \in H^k(\Omega) \cap M$, *there is a constant C independent of* ν, ν_T, h, *and H such that*

$$\nu\|\mathbb{D}(u-u^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(u-u^h)\|^2 \le Ch^{2k}|u|^2_{k+1}\left(\nu+\frac{1}{\nu}\left(1+\frac{1}{\nu}\right)^2+\nu_T\right) + \frac{C}{\nu}h^{2k}|p|^2_k + C\nu_T H^{2k}|u|^2_{k+1}.$$

Thus we obtain $\|\mathbb{D}(u-u^h)\| = \mathbb{O}(h^k)$ *if* $\nu_T H^{2k} = h^{2k}$. *In particular, this is satisfied for the following cases*:

$$k=1(\nu_T, H)=(h, h^{1/2}),$$

$$k=2(\nu_T, H)=(h, h^{3/4}) \qquad or\ (\nu_T, H)=(h^2, h^{3/4})$$

$$k=3(\nu_T, H)=(h, h^{5/6}) \qquad or\ (\nu_T, H)=(h^2, h^{2/3}).$$

IV. ERROR ESTIMATE FOR PRESSURE

This section is devoted to the estimation of the discrete pressure.

Theorem 4.1. *Suppose the global uniqueness condition* (2.12) *holds. Then the pressure error satisfies*

$$\|p-p^h\| \le C((\nu+1)\|\mathbb{D}(u-u^h)\| + \|\mathbb{D}(u-u^h)\|^2+\nu_T\|(I-P_{L^H})\mathbb{D}(u-u^h)\| + \nu_T\|(I-P_{L^H})\mathbb{D}(u)\|) + C\inf_{q^h\in M^h}\|p-q^h\|,$$

where C is independent of ν, ν_T, h, *and H.*

Proof. The proof follows the approach given by Crouzeix and Raviart [18]. Denoting the error in velocity $e = u - u^h$ and introducing an approximation $\tilde{p} \in M^h$ of the pressure in the error equation (3.4), we obtain

$$b(v^h, p^h-\tilde{p}) = b(v^h, p-\tilde{p}) - a(e, v^h) - (c(u, u, v^h) - c(u^h, u^h, v^h)) - g(e, v^h) + g(u, v^h), \qquad \forall v^h \in X^h. \tag{4.1}$$

To bound the linear terms in the right-hand side of (4.1), we apply Cauchy Schwarz inequality, Korn's inequality, and (2.8). The inconsistency term $g(u, v^h)$ is bounded as in (3.7). In view of Lemma 2.2 and Korn's inequality, the nonlinear terms are bounded as

$$|c(u, u, v^h) - c(u^h, u^h, v^h)| = |-c(e, e, v^h) + c(e, u, v^h) + c(u, e, v^h)| \\ \leq C(\|\mathbb{D}(e)\| + \|\nabla u\|)\|\mathbb{D}(e)\| \, \|\nabla v^h\|.$$

Combining all the bounds, then we have

$$|b(v^h, p^h - \tilde{p})| \leq C\{\|p - \tilde{p}\| + \nu\|\mathbb{D}(e)\| + (\|\mathbb{D}(e)\| + \|\nabla u\|)\|\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\| \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(u))\|\}\|\nabla v^h\|, \qquad \forall v^h \in X^h. \tag{4.2}$$

On the other hand, the inf-sup condition (2.4) implies that there exists a nontrivial $v^h \in X^h$, such that

$$(p^h - \tilde{p}, \nabla \cdot v^h) \geq \beta\|\nabla v^h\| \, \|p^h - \tilde{p}\|. \tag{4.3}$$

In view of (4.3), we have

$$\|p - p^h\| \leq \|p - \tilde{p}\| + \beta^{-1} \frac{|b(v^h, p^h - \tilde{p})|}{\|\nabla v^h\|}. \tag{4.4}$$

We conclude our proof by inserting (4.2) into (4.4):

$$\|p - p^h\| \leq C\|p - \tilde{p}\| + C(\nu\|\mathbb{D}(e)\| + \|\mathbb{D}(e)\|^2 + \|\mathbb{D}(e)\| \, \|\nabla u\| \\ + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|). \tag{4.5}$$

∎

Corollary 4.2. *The statement of Theorem* 3.2, *the approximation results* (2.5), (2.6), *and Corollary* 3.3 *imply that*

$$\|p - p^h\| \leq C(h^k + \nu_T H^k + \nu_T^{1/2}(\nu_T + 1)(h^k + H^k) + \nu_T h^k),$$

where C is independent of ν_T, h, and H.

Therefore, if $\nu_T = h^\alpha$, $H = h^\beta$, and $\alpha + 2\beta \geq 2k$, the error in the pressure is bounded by

$$\|p - p^h\| \leq Ch^k.$$

For instance, one can choose for $k = 1$, $(\nu_T, H) = (h, h^{1/2})$, or for $k = 2$, $(\nu_T, H) = (h^2, h^{1/2})$.

V. ERROR ESTIMATE FOR VELOCITY IN L^2

We now give an error estimate in L^2 for the velocity by using a duality argument [14]. We first consider the linearized adjoint problem of the Navier-Stokes equations: given $\xi \in L^2(\Omega)$, find $(\phi, \chi) \in (X, M)$ with

$$a(\phi, v) + c(u, v, \phi) + c(v, u, \phi) - b(v, \chi) + b(\phi, q) = (\xi, v), \qquad \forall (v, q) \in (X, M). \tag{5.1}$$

It is easy to show that under the condition (2.12), the Lax-Milgram theorem gives a unique solution (ϕ, χ) to (5.1). We also assume that the linearized adjoint problem is $H^2(\Omega)$ regular. This means that for any $\xi \in L^2(\Omega)$ there exists a unique pair (ϕ, χ) in $(X \cap H^2(\Omega)^d) \times (M \cap H^1(\Omega))$ such that the following inequality holds:

$$\|\phi\|_2 + \|\chi\|_1 \leq C\|\xi\|. \tag{5.2}$$

We now state the L^2 error estimate.

Theorem 5.1. *Assume that the solution of the dual problem* (5.1) *satisfies the stability estimate* (5.2). *Then, under the assumptions of Theorems* 3.2 *and* 4.1, *there exists a constant C independent of* ν_T, *h, and H such that*

$$\|u - u^h\| \leq Ch^{k+1}(1 + \nu_T^{1/2} + \nu_T + \nu_T^{3/2}) + C\nu_T H^{k+1} + ChH^k \nu_T^{1/2}(1 + \nu_T^{1/2} + \nu_T).$$

Proof. Consider the dual problem (5.1) with $\xi = e = u - u^h$, choose $v = e$, $q = p - p^h$, and subtract (3.4) to the resulting equation:

$$\begin{aligned}\|e\|^2 \leq{}& |a(\phi - v^h, e)| + |c(u, e, \phi) + c(e, u, \phi) - c(u, u, v^h) + c(u^h, u^h, v^h)| \\ &+ |b(e, \chi - q^h)| + |b(\phi - v^h, p - p^h)| + |g(\phi - v^h, e)| + |g(u, v^h)| + |g(\phi, e)| \leq C(\nu\|\mathbb{D}(e)\| \\ &+ \|p - p^h\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|)\|\mathbb{D}(\phi - v^h)\| + C\|\chi - q^h\|\,\|\mathbb{D}(e)\| \\ &+ \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|\,\|(I - P_{L^H})\mathbb{D}(v^h)\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|\,\|(I - P_{L^H})\mathbb{D}(\phi)\| \\ &+ |c(u, e, \phi) + c(e, u, \phi) - c(u, u, v^h) + c(u^h, u^h, v^h)|,\end{aligned} \tag{5.3}$$

owing to Cauchy-Schwarz, Korn's inequality, and (2.8). We then choose $(v^h, q^h) = (\tilde{\phi}, \tilde{\chi})$, where $\tilde{\phi}$, $\tilde{\chi}$ are the best approximations of (ϕ, χ) in (X^h, M^h). Using the approximation properties we have

$$\|\phi - \tilde{\phi}\|_1 \leq Ch\|\phi\|_2,$$

$$\|\chi - \tilde{\chi}\| \leq Ch\|\chi\|_1.$$

The Equation (5.3) becomes

$$\begin{aligned}\|e\|^2 \leq{}& Ch(\nu\|\mathbb{D}(e)\| + \|p - p^h\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|)\|\phi\|_2 + Ch\|\chi\|_1\|\mathbb{D}(e)\| \\ &+ \nu_T\|(I - P_{L^H})\mathbb{D}(u)\|\,\|(I - P_{L^H})\mathbb{D}(\tilde{\phi})\| + \nu_T\|(I - P_{L^H})\mathbb{D}(e)\|\,\|(I - P_{L^H})\mathbb{D}(\phi)\| \\ &+ |c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi})|.\end{aligned} \tag{5.4}$$

The consistency error term in the right-hand side of (5.4) is bounded by using (2.7):

$$\begin{aligned}\nu_T \|(I - P_{L^H})\mathbb{D}(u)\| \, \|(I - P_{L^H})\mathbb{D}(\tilde{\phi})\| &\leq \nu_T H^k |u|_{k+1} H \|\mathbb{D}(\tilde{\phi})\|_1 \\ &\leq C\nu_T H^{k+1}|u|_{k+1}\|\tilde{\phi}\|_2 \\ &\leq C\nu_T H^{k+1}|u|_{k+1}(\|\tilde{\phi} - \phi\|_2 + \|\phi\|_2) \\ &\leq C\nu_T H^{k+1}|u|_{k+1}\|\phi\|_2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu_T \|(I - P_{L^H})\mathbb{D}(e)\| \, \|(I - P_{L^H})\mathbb{D}(\phi)\| &\leq \nu_T H \|\mathbb{D}(\phi)\|_1 \|(I - P_{L^H})\mathbb{D}(e)\| \\ &\leq \nu_T H \|\phi\|_2 \|(I - P_{L^H})\mathbb{D}(e)\|.\end{aligned}$$

We now consider the nonlinear terms in (5.4). Adding and subtracting u^h, gives

$$\begin{aligned}c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi}) = c(e, e, \phi) + c(u, e, \phi - \tilde{\phi}) \\ + c(e, u, \phi - \tilde{\phi}) + c(e, e, \tilde{\phi} - \phi).\end{aligned}$$

Using Lemma 2.2 and Korn's inequality, we have

$$\begin{aligned}&|c(u, e, \phi) + c(e, u, \phi) - c(u, u, \tilde{\phi}) + c(u^h, u^h, \tilde{\phi})| \\ &\quad \leq C\|\mathbb{D}(e)\|^2\|\phi\|_1 + C\|\nabla u\| \, \|\mathbb{D}(e)\| \, \|\phi - \tilde{\phi}\|_1 + C\|\mathbb{D}(e)\|^2\|\phi - \tilde{\phi}\|_1 \\ &\quad \leq C(\|\mathbb{D}(e)\| + h)\|\mathbb{D}(e)\| \, \|\phi\|_2.\end{aligned}$$

Combining all bounds and using the stability property (5.2) gives

$$\begin{aligned}\|e\| \leq Ch((\nu + \nu_T)\|\mathbb{D}(e)\| + C\|p - p^h\|) + C\nu_T H^{k+1}|u|_{k+1} + C\|\mathbb{D}(e)\|(h + \|\mathbb{D}(e)\|) \\ + \nu_T H\|(I - P_{L^H})\mathbb{D}(e)\|.\end{aligned}$$

The final result is obtained by using Corollaries 3.3 and 4.2. ■

Corollary 5.2. *By choosing appropriately the parameters* ν_T *and* H, *the error estimate becomes*

$$\|u - u^h\| \leq Ch^{k+1}.$$

For instance, this result holds true if one chooses $(\nu_T, H) = (h, h^{1/2})$ *for* $k = 1$ *and* $(\nu_T, H) = (h^{2k}, h^{1/k})$ *for* $k \geq 2$.

VI. NUMERICAL EXPERIMENTS

We first describe the algorithm used for handling the nonlinearity and the subgrid eddy viscosity term. We then present two numerical examples: one with a known analytical solution that allows for a numerical study of the convergence rates; and one benchmark problem. In both cases, the mini-element spaces of first order are used: the velocity space consists of continuous piecewise polynomials of degree 1 plus bubble functions and the pressure space consists of continuous piecewise linear polynomials.

A. Algorithm

To solve the nonlinear system a Newton method is used. Given $(u^{m-1}, p^{m-1}) \in X^h \times M^h$, we find $(u^m, p^m) \in X^h \times M^h$ satisfying

$$
\begin{aligned}
a(u^m, v^h) + \tfrac{1}{2}c(u^{m-1}, u^m, v^h) + \tfrac{1}{2}c(u^m, u^{m-1}, v^h) - \tfrac{1}{2}c(u^{m-1}, v^h, u^m) - \tfrac{1}{2}c(u^m, v^h, u^{m-1}) \\
- b(v^h, p^m) = (f, v^h) + \tfrac{1}{2}c(u^{m-1}, u^{m-1}, v^h) - \tfrac{1}{2}c(u^{m-1}, v^h, u^{m-1}) \\
- g(u^{m-1}, v^h), \ \forall v^h \in X^h,
\end{aligned}
$$

$$b(u^m, q^h) = 0, \qquad \forall q^h \in M^h. \tag{6.1}$$

This algorithm leads to a linear system of the form $Ax = b$ with A nonsymmetric. To solve this linear system we use the iterative conjugate gradient squared method of [19]. The stopping criteria of this Newton method is based on the absolute residual.

We now show that the extra stabilization term $g(u^{m-1}, v^h)$ requires a modification of the right-hand side of the linear system, that can be computed locally.

First, from (2.7), we can write

$$g(u^{m-1}, v^h) = \nu_T(\mathbb{D}(u^{m-1}), \mathbb{D}(v^h)) - \nu_T(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(v^h)).$$

In this decomposition, adding the first term is straight-forward, as it is similar to the diffusive term $a(u^{m-1}, v^h)$. The difficulty is to incorporate the second term, since it couples coarse and fine meshes. Denoting a basis of X^h by $\{\phi_j^h\}_{j=1}^{N^h}$, we want to compute $(P_{L^h}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_j^h))$, for all j. We first expand $P_{L^H}\mathbb{D}(u^{m-1}) \in L^H$ by using a basis $\{\psi_j^H\}_{j=1}^{N^H}$ of L^H:

$$P_{L^H}\mathbb{D}(u^{m-1}) = \sum_{j=1}^{N^H} \beta_j^{m-1}\psi_j^H. \tag{6.2}$$

The coefficients $\boldsymbol{\beta} = (\beta_j^{m-1})_j$ are obtained from the definition (2.7) of P_{L^H}:

$$S\boldsymbol{\beta} = (\mathbb{D}(u^{m-1}), \psi_j^H)_{1\le j\le N^H}, \tag{6.3}$$

where the matrix S is the mass matrix associated to L^H: $S_{ij} = (\psi_j^H, \psi_i^H)$. Since u^{m-1} belongs to X^h, there exists a vector $\boldsymbol{\alpha} = (\alpha_j^{m-1})_j$ such that

$$u^{m-1} = \sum_{j=1}^{N^h} \alpha_j^{m-1}\phi_j^h,$$

and Equation (6.3) becomes

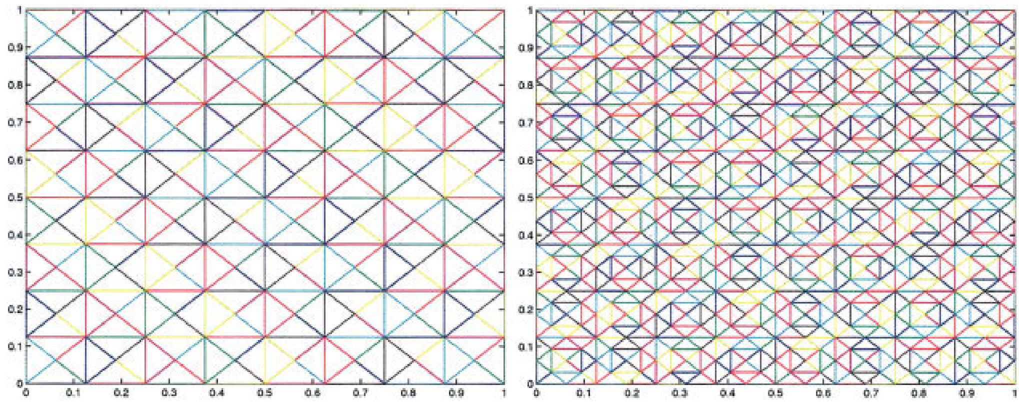

FIG. 1. Mesh $H = 1/8$ (left) and mesh with one refinement $h = 1/16$ (right). [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$$\boldsymbol{\beta} = S^{-1}R^T\boldsymbol{\alpha},$$

where R is the matrix coupling fine and large scales: $R_{ij} = (\psi_j^H, \mathbb{D}(\phi_i^h))$. Finally, we note that

$$(P_{L^H}\mathbb{D}(u^{m-1}), \mathbb{D}(\phi_i^h))_i = R\boldsymbol{\beta} = RS^{-1}R^T\boldsymbol{\alpha}.$$

Since L^h consists of discontinuous piecewise polynomials, the matrix S is block diagonal and the computation of $RS^{-1}R^T$ is done locally on each element of the coarse mesh τ^H.

B. Convergence Rates

We consider the exact solution $u = (u_1, u_2)$ and p of problem (1.1) on the domain $\Omega = [0, 1] \times [0, 1]$, defined by

$$u_1(x, y) = 2x^2(x-1)^2y(y-1)(2y-1), \qquad u_2(x, y) = -y^2(y-1)^22x(x-1) \times (2x-1), \qquad p(x, y) = y.$$

The fluid viscosity is $\nu = 10^{-2}$, which gives a Reynolds number of the order 10^2. All nonlinear systems are solved with Newton method with stopping criteria 10^{-6}. From Corollary 3.3, we choose $\nu_T = h$ and H such that $H^2 \leq h$. The theoretical analysis then predicts a convergence rate $\mathbb{O}(h)$ for the velocity in the energy norm, $\mathbb{O}(h^2)$ for the velocity in the L^2 norm, and $\mathbb{O}(h)$ for the pressure. The domain is subdivided into triangles. First, the coarse mesh is chosen such that $H =$

TABLE I. Numerical errors and degrees of freedom.

Meshes	N^h	L^2	Rate	H_0^1	Rate	L^2 pressure	Rate
$H = 1/2$, $h = 1/4$	218	0.0069		0.0509		4.3269e-04	
$H = 1/4$, $h = 1/8$	882	0.0017	2.0211	0.0241	1.0786	2.4448e-04	0.8236
$H = 1/8$, $h = 1/16$	3554	3.9446e-04	2.1076	0.0108	1.1580	9.6978e-05	1.3340
$H = 1/16$, $h = 1/32$	14274	8.1066e-05	2.2827	0.0046	1.2313	3.3879e-05	1.5173
$H = 1/32$, $h = 1/64$	57218	1.6313e-05	2.3131	0.0020	1.2016	1.1026e-05	1.6195

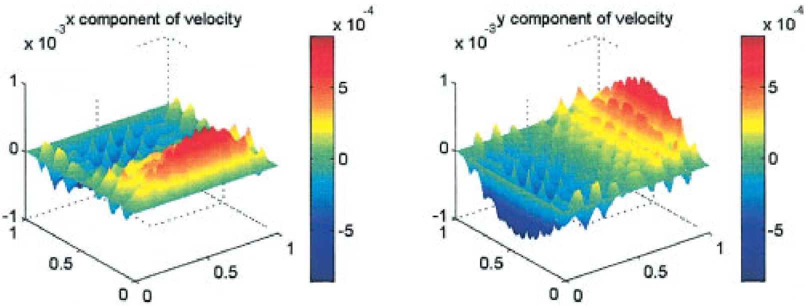

FIG. 2. Difference of the exact solution and computed solution for $(H, h) = (1/8, 1/16)$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

1/2 and the fine mesh is a refinement of the coarse mesh, so that $h = 1/4$ (here, $h = H^2$). Other pairs of meshes are obtained by successive uniform refinements (see Fig. 1 for the case $H = 1/8$ and $h = 1/16$). We choose $L^H = \mathbb{D}(X^H)$. In particular, if $\mathcal{F}_E$ denotes the affine mapping from the reference element to the physical element E, we can write

$$L^H = \{\mathbb{L} : \mathbb{L}|_E = \hat{\mathbb{L}}\circ\mathcal{F}_E,\ \forall\hat{\mathbb{L}} \in \hat{L}^H,\ \forall E \in \mathcal{T}^H\},$$

$$\hat{L}^H = \text{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}\frac{\partial b}{\partial x} & \frac{1}{2}\frac{\partial b}{\partial y}\\ \frac{1}{2}\frac{\partial b}{\partial y} & 0\end{pmatrix}, \begin{pmatrix}0 & \frac{1}{2}\frac{\partial b}{\partial x}\\ \frac{1}{2}\frac{\partial b}{\partial x} & \frac{\partial b}{\partial y}\end{pmatrix}\right\},$$

where b denotes the bubble function defined by: $b(x, y) = 27xy(1 - x - y)$. We note that a simpler choice for L^H is the space of piecewise constant symmetric tensors, i.e., $\hat{L}^H$ contains only the first three tensors in the definition above. In that case, the asymptotic numerical convergence rates are similar. By choosing $L^H = \mathbb{D}(X^H)$, we enrich the space to improve the accuracy of the solution. Table I gives the numerical errors and convergence rates obtained on successively refined meshes. These results agree with the optimal theoretical convergence rates. Figure 2

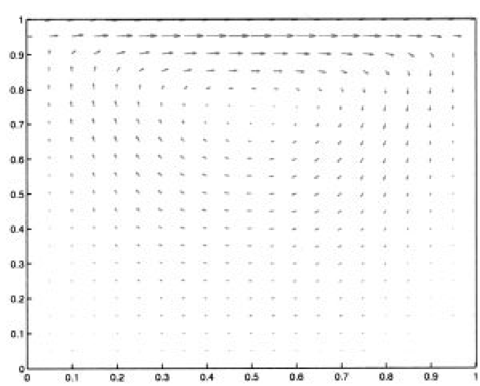

FIG. 3. For $Re = 1$, velocity vectors for subgrid eddy viscosity method.

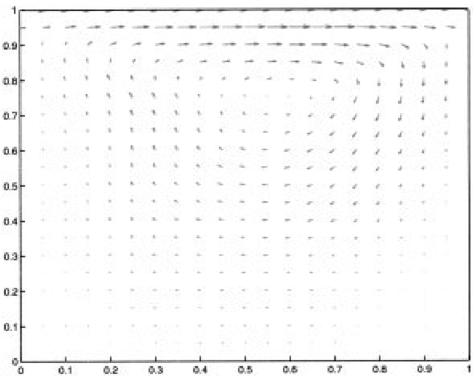

FIG. 4. For $Re = 100$, velocity vectors for subgrid eddy viscosity method.

represents the graph of the difference of the exact solution and computed solution for the case $(H, h) = (1/8, 1/16)$.

C. Driven Cavity Problem

The second problem is the driven cavity problem, in which fluid is enclosed in a square box, with an imposed velocity of unity in the horizontal direction on the top boundary, and a no slip condition on the remaining walls. This problem has been widely used as test case for validating incompressible fluid dynamic algorithms. Since most examples of physical interest have corners, corner singularities for two-dimensional fluid flows are very important. We will compare our results to those obtained by Ghia et al. [20] and Akin [21].

We consider the flow for different Reynolds numbers on a fixed mesh where $H = 1/8$, $h = 1/16$ with Newton stopping criteria 10^{-6}. The same basis functions $\hat{L}^H$ are chosen as in Section B. For low Reynolds number ($Re = 1$), the flow has only one vortex located above the center (Fig. 3). When the Reynolds number increases to $Re = 100$, the flow pattern starts to form reverse circulation cells in lower corners (Fig. 4). These results agree with those found in [21], where a much finer mesh was used. In addition, for $Re = 2500$, we compare the velocity vectors for subgrid eddy viscosity and artificial viscosity model. Figure 5 shows that the main eddy of

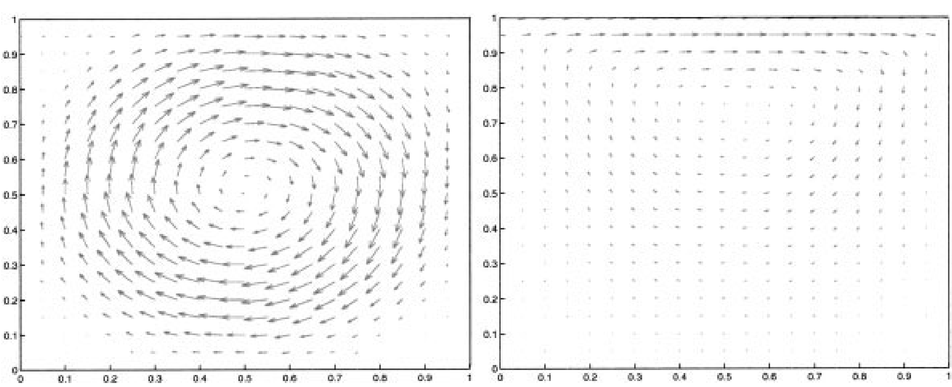

FIG. 5. Velocity vectors for $Re = 2500$ subgrid eddy viscosity model (left) and artificial viscosity model (right).

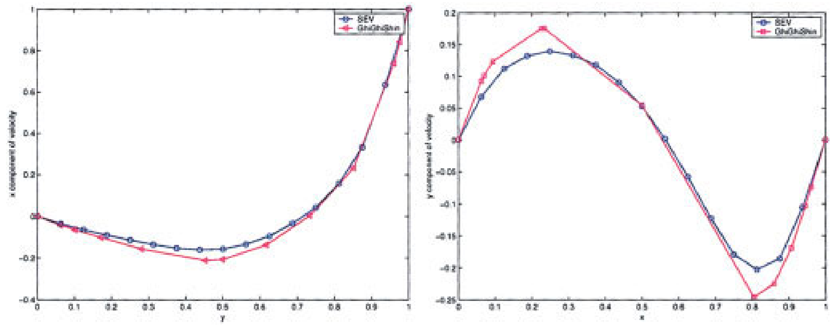

FIG. 6. Vertical and horizontal midlines for $Re = 100$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

artificial eddy viscosity model is too small and its center is too close to the upper lid. On the other hand, with the higher Reynolds number the subgrid eddy viscosity model reproduces the main eddy well and steady flow pattern becomes more complex with reverse circulation cells in both lower corners.

We also draw the x component of velocity along the vertical centerline and y component of velocity along the horizontal centerlines for $Re = 100$ and $Re = 400$. We compare the results obtained by [20], where the algorithm is based on time-dependent streamfunction using coupled implicit and multigrid methods. Their results are used as benchmark data as basis for comparison. Figures 6 and 7 show that our results using the subgrid eddy viscosity method agree with data of Ghia et al. [20], obtained on a much finer mesh ($h = 1/129$).

VII. CONCLUDING REMARKS

In this article, we presented and analyzed a two-grid method for solving the steady-state Navier-Stokes equations. This method has the advantage of adding diffusion only on the large scales. Numerical tests showed that the new stabilization technique gives comparable results on benchmark problems. The simulation of this model applied to the time-dependent Navier-Stokes is currently under investigation.

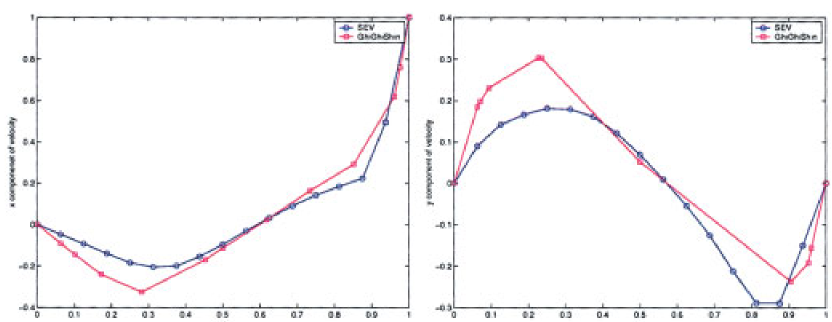

FIG. 7. Vertical and horizontal midlines for $Re = 400$. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

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