

A COMBINED MIXED FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD FOR MISCIBLE DISPLACEMENT PROBLEM IN POROUS MEDIA

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Abstract A combined method with mixed finite element method for flow and discontinuous Galerkin method for transport is introduced for the coupled system of miscible displacement problem. The “cut-off” operator \mathcal{M} is introduced in the discontinuous Galerkin scheme in order to make the combined scheme converge. The optimal choice of penalty parameter β in DG scheme is derived to be $\beta = 1$. Error estimates in $L^2(H^1)$ and $L^\infty(L^2)$ for concentration and error estimate in $L^\infty(L^2)$ for velocity are derived, which are the optimal $L^2(H^1)$ rate of convergence for concentration, and optimal $L^\infty(L^2)$ rate of convergence for velocity. The uniform positive definitiveness and uniform Lipschitz continuity of dispersion/diffusion tensor are proved.

Keywords: discontinuous Galerkin method, mixed finite element method, miscible displacement, error estimate

1. Introduction

Numerical modeling of miscible displacement is important and interesting in Engineering. It involves a coupled system of non-linear partial differential equations. The need for accurate solutions to the coupled equations challenges numerical analysts to design new methods.

The mixed finite element methods [1, 6] gained great popularity in the last two decades for the reasons that they provide very accurate approximations of the primary unknown and its flux and they conserve mass locally on any element. The discontinuous Galerkin method gained even greater popularity recently for at least four reasons [10, 11]: 1) the flexibility inherent to it allows more general meshes construction and degree of non uniformity than permitted by the more conventional finite

element method; 2) it also conserves mass locally on any element; 3) it has, in general, less numerical diffusion and provides more accurate local approximations for problems with rough coefficients; 4) it is easy to implement. Traditional numerical methods were studied for solving the miscible displacement problem by Darlow, Ewing, Wheeler and Douglas [3, 4, 5, 7, 9, 12]. The formulation of discontinuous Galerkin for both of flow and transport subproblems is given by Riviere [11].

In this paper, a combined method with mixed finite element method for flow and discontinuous Galerkin method for transport is introduced and analyzed. This paper consists of four additional sections. Problem definition is given in section 2 and the formulation of the combined method is described in section 3. In section 4, the results and proofs of error estimates for the subproblems and coupled system are given. Conclusions are described in the last section.

2. Governing Equations

The displacement of one incompressible fluid by another in porous media is considered in this paper. Detailed discussion on physical theories of miscible displacement in porous media can be found in [2] or [8].

Let Ω denote a bounded domain in \mathbb{R}^d , ($n = 2, 3$) and Let J denote the time interval $(0, T_f]$. The classical equations governing the miscible displacement in porous media is as follows.

- Continuity equation

$$\nabla \cdot \mathbf{u} = q \quad (x, t) \in \Omega \times J \quad (1)$$

- Transport equation

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{u})\nabla c) = qc^* \quad (x, t) \in \Omega \times J \quad (2)$$

- Darcy velocity

$$\mathbf{u} = -\frac{K}{\mu} \nabla p \quad (x, t) \in \Omega \times J \quad (3)$$

- Dispersion/diffusion tensor

$$\mathbf{D}(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}| \{ \alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t (\mathbf{I} - \mathbf{E}(\mathbf{u})) \} \quad (4)$$

where,

$$d = \phi \tau D_m$$

- Constitutive relation

$$\mu = \mu(c)$$

where the dependent variables are p , the pressure in the fluid mixture, and \mathbf{u} , the Darcy velocity of the mixture (volume flowing across a unit cross-section per unit time), and c , the concentration of interested species measured in amount of species per unit volume of the fluid mixture. The permeability K of the medium measures the conductivity of the medium to fluid flow; the viscosity μ of the fluid measures the resistance to flow of the fluid mixture; ρ is the density of fluid mixture; the porosity ϕ is the fraction of the volume of the medium occupied by pores; $\mathbf{D}(\mathbf{u})$ is the dispersion/diffusion tensor, which has contributions from molecular diffusion and mechanical dispersion, and it can be calculated by equation (4), where $\mathbf{E}(\mathbf{u})$ is the tensor that projects onto the \mathbf{u} direction, whose (i, j) component is $(\mathbf{E}(\mathbf{u}))_{i,j} = \frac{u_i u_j}{|\mathbf{u}|^2}$; τ is the tortuosity coefficient; D_m is the molecular diffusivity; α_l and α_t are the longitudinal and transverse dispersivities, respectively. The commonly used constitutive relation is the quarter-power mixing rule $\mu(c) = (c\mu_s^{-0.25} + (1-c)\mu_o^{-0.25})^{-4}$, but we consider $\mu(c)$ to be a general nonlinear relation in this paper. The imposed external total flow rate q is a sum of sources (injection) and sinks (extraction), c^* is the injected concentration c_w if $q > 0$ and is the resident concentration c if $q < 0$.

The continuity equation (1) can be obtained by the mass conservation for the whole fluid mixture and the equation (3) is a formulation of Darcy's law. Combination of equations (1) and (3) will give the flow equation.

$$-\nabla \cdot \left(\frac{K}{\mu(c)} \nabla p \right) = q \quad (x, t) \in \Omega \times J \quad (5)$$

The flow equation (5) governs the fluid flow and gives the pressure field and Darcy velocity field if the concentration is given. It is elliptic if the concentration is considered to be given.

The transport equation (2) can be obtained by the mass conservation of the interested species. It governs the convection-diffusion transport process and gives the concentration profile provided the velocity field is given. It is parabolic but normally convection-dominated.

We assume Ω is a bounded domain with Lipschitz boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N = \bar{\Gamma}_{in} \cup \bar{\Gamma}_{out}$, where Γ_D is the Dirichlet boundary and Γ_N is the Neumann boundary for flow subproblem and $\Gamma_D \cap \Gamma_N = \phi$; Γ_{in} is the inflow boundary and Γ_{out} is the outflow/noflow boundary condition, defined as follows.

$$\begin{aligned}\Gamma_{in} &= \Gamma_{in}(t) = \{x \in \partial\Omega : \mathbf{u}(t) \cdot \nu < 0\} \\ \Gamma_{out} &= \Gamma_{out}(t) = \{x \in \partial\Omega : \mathbf{u}(t) \cdot \nu \geq 0\}\end{aligned}$$

where, ν denotes the unit outward normal vector to $\partial\Omega$. Though Γ_{in} and Γ_{out} can be time-dependent for some physical problem, we assume they are fixed at all the time in J for simplicity.

We consider following boundary condition for this problem.

$$p = p_B \quad (x, t) \in \Gamma_D \times J \quad (6)$$

$$\mathbf{u} \cdot \nu = u_B \quad (x, t) \in \Gamma_N \times J \quad (7)$$

$$(\mathbf{u}c - \mathbf{D}\nabla c) \cdot \nu = c_B \mathbf{u} \cdot \nu \quad t \in J, x \in \Gamma_{in}(t) \quad (8)$$

$$(-\mathbf{D}\nabla c) \cdot \nu = 0 \quad t \in J, x \in \Gamma_{out}(t) \quad (9)$$

The initial concentration is specified in the following way.

$$c(x, 0) = c_0(x) \quad x \in \Omega \quad (10)$$

In this paper we are only interested in the convergence result for velocity \mathbf{u} and concentration c . We give the convergence theorem only for the case $\partial\Omega = \Gamma_N$ (i.e. $\Gamma_D = \phi$). The convergence study for the case $\Gamma_D \neq \phi$ and for pressure p is still under progress.

3. Discontinuous Galerkin/Mixed Finite Element Scheme

3.1 Assumption

We consider the scheme of mixed finite element (MFE) method for flow subproblem and discontinuous Galerkin (DG) with interior penalty

term for transport subproblem. The scheme used for transport subproblem is referred to as the Non-symmetric Interior Penalty Galerkin (NIPG) [11].

For simplicity, we consider only two or three dimensional rectangular domain $\Omega = \prod_{i=1}^d (0, L_i)$, $d = 2$, or 3 and we only consider rectangular mesh. However, the results can be directly extended to logically rectangular domain/mesh by conforming mapping. Though we can choose separate domain partitions for flow and for transport problem, the same rectangular domain partition \mathcal{T}_h is considered here for both of flow and transport equations. We also assume the permeability tensor K is invertible and is uniformly positive definite and uniformly bounded above.

3.2 Notation

Let $\mathcal{T}_{h>0}$ be a quasi-uniform family of rectangular partition of Ω such that no element crosses the boundaries of Γ_D , Γ_N , Γ_{in} , or Γ_{out} , where h is the maximal element diameter. The set of all interior edges (for 2 dimensional domain) or faces (for 3 dimensional domain) for \mathcal{T}_h are denoted by E_h . On each edges or faces $e \in E_h$, a unit normal vector ν_e is arbitrarily fixed. The set of all edges or faces on Γ_{out} and on Γ_{in} for \mathcal{T}_h are denoted by $E_{h,out}$ and $E_{h,in}$, respectively, for which the normal vector ν_e coincides with the outward unit normal vector.

For $s \geq 0$, define,

$$H^s(\mathcal{T}_h) = \left\{ \phi \in L^2(\Omega) : \phi|_R \in H^s(R), R \in \mathcal{T}_h \right\} \quad (11)$$

We now define the average jump for $\phi \in H^s(\mathcal{T}_h)$, $s > 1/2$. Let $R_i, R_j \in \mathcal{T}_h$ and $e = \partial R_i \cap \partial R_j \in E_h$ with ν_e exterior to R_i . Denote

$$\langle \phi \rangle = \frac{1}{2} \left(\phi|_{R_i} \right)|_e + \frac{1}{2} \left(\phi|_{R_j} \right)|_e \quad (12)$$

$$[\phi] = \left(\phi|_{R_i} \right)|_e - \left(\phi|_{R_j} \right)|_e \quad (13)$$

The usual Sobolev norm on Ω is denoted by $\|\cdot\|_{m,\Omega}$. The broken norms are defined, for positive integer m , as

$$\|\phi\|_m^2 = \sum_{R \in \mathcal{T}_h} \|\phi\|_{m,\Omega}^2 \quad (14)$$

The finite element space is taken to be

$$\mathcal{D}_r(\mathcal{T}_h) \equiv \left\{ \phi \in L^2(\Omega) : \phi|_R \in P_r(R), R \in \mathcal{T}_h \right\} \quad (15)$$

where $P_r(R)$ denotes the space of polynomials of (total) degree less than or equal to r on R .

Define

$$V \equiv H(\Omega; \text{div}) \equiv \left\{ \mathbf{u} \in (L^2(\Omega))^d : \text{div} \mathbf{u} \in L^2(\Omega) \right\} \quad (16)$$

$$W \equiv L^2(\Omega) \quad (17)$$

Let V^0 and V^N be the subspaces of V consisting of functions with normal trace on Γ_N (weakly) equal to zero and \mathbf{u}_B , respectively.

Let the approximating subspace $V_k(\mathcal{T}_h) \times W_k(\mathcal{T}_h)$ of $V \times W$ be the k -th ($k \geq 0$) order Raviart-Thomas space (RT_k) of the partition \mathcal{T}_h . For example, for three dimensional domain Ω , it is defined as

$$\begin{aligned} V_k(\mathcal{T}_h) &= \left\{ \mathbf{v} \in H(\Omega; \text{div}) : \mathbf{v}|_R \in Q_{k+1,k,k}(R) \times Q_{k,k+1,k}(R) \right. \\ &\quad \left. \times Q_{k,k,k+1}(R), R \in \mathcal{T}_h \right\} \\ W_k(\mathcal{T}_h) &= \left\{ w \in L^2(\Omega) : w|_R \in Q_{k,k,k}(R), R \in \mathcal{T}_h \right\} \end{aligned}$$

where, we denote by $Q_{i,j,k}(R)$ the space of polynomials of degree less than or equal to i (j, k) in the first (second, third) variable restricted to R .

Corresponding to V^0 and V^N , define their subspaces $V_k^0(\mathcal{T}_h) = V_k(\mathcal{T}_h) \cap V^0$ and

$$V_h^N(\mathcal{T}_h) = \left\{ \mathbf{v} \in V_k(\mathcal{T}_h) : (\mathbf{v} \cdot \boldsymbol{\nu}, \lambda)_{\Gamma_N} = 0 \quad \forall \lambda \in \Lambda_h \right\} \quad (18)$$

where $\Lambda_h \subset L^2(\partial\Omega)$ is the corresponding hybrid space of Lagrange multipliers for the pressure restricted to $\partial\Omega$, and we have, $\Lambda_h = V_h \cdot \boldsymbol{\nu}|_{\partial\Omega}$.

The inner product in $(L^2(\Omega))^d$ or $L^2(\Omega)$ is indicated by (\cdot, \cdot) and the inner product in boundary function space $L^2(\Gamma)$ is indicated by $(\cdot, \cdot)_\Gamma$. Denote

$$|\mathbf{u}| = |\mathbf{u}|_2 = \sqrt{\sum_{i=1}^d (\mathbf{u})_i^2} \quad (19)$$

$$\|\mathbf{u}\|_{(L^2(\Omega))^d} = \|(|\mathbf{u}|_2)\|_{L^2(\Omega)} \quad (20)$$

$$\|\mathbf{u}\|_{(L^\infty(\Omega))^d} = \|(|\mathbf{u}|_2)\|_{L^\infty(\Omega)} \quad (21)$$

3.3 Continuous in time scheme

Let us define the bilinear form $B(c, \psi; \mathbf{u})$ and the linear functional $L(\psi; \mathbf{u})$ as follows.

$$\begin{aligned}
 B(c, \psi; \mathbf{u}) &= \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}) \nabla c - c \mathbf{u}) \cdot \nabla \psi - \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}) \nabla c \cdot \nu_e \rangle [\psi] \\
 &+ \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}) \nabla \psi \cdot \nu_e \rangle [c] + \sum_{e \in E_h} \int_e c^* \mathbf{u} \cdot \nu_e [\psi] \\
 &+ \sum_{e \in E_{h, \text{out}}} \int_e c \mathbf{u} \cdot \nu_e \psi - \int_{\Omega} c q^- \psi + J_0^{\sigma, \beta}(c, \psi)
 \end{aligned} \tag{22}$$

where, $c^*|_e$ is the upwind value of concentration,

$$c^*|_e = \begin{cases} c|_{R_1} & \text{if } \mathbf{u} \cdot \nu_e > 0 \\ c|_{R_2} & \text{if } \mathbf{u} \cdot \nu_e < 0 \end{cases}$$

for $e = \partial R_1 \cap \partial R_2$ and ν_e is the outward unit normal vector to R_1 . Notice $\mathbf{u} \cdot \nu_e$ is continuous on the direction of ν_e , thus has well-defined value at the interface.

q^+ is the injection part of source term and q^- is the extraction part of source term,

$$\begin{aligned}
 q^+ &= \max(q, 0) \\
 q^- &= \min(q, 0)
 \end{aligned}$$

Of course, we have $q = q^+ + q^-$.

$J_0^{\sigma, \beta}(c, \psi)$ is the interior penalty term,

$$J_0^{\sigma, \beta}(c, \psi) = \sum_{e \in E_h} \frac{\sigma_e}{h_e^\beta} \int_e [c] [\psi] \tag{23}$$

where, σ is a discrete positive function that takes constant value σ_e on the edge or face e , and is bounded below by $\sigma_* > 0$ and above by σ^* , h_e denotes the size of e and $\beta \geq 0$ is a real number.

The linear functional $L(\psi)$ is defined as

$$L(\psi; \mathbf{u}) = \int_{\Omega} c_w q^+ \psi - \sum_{e \in E_{h, \text{in}}} \int_e c_B \mathbf{u} \cdot \nu_e \psi \tag{24}$$

The continuous in time DG/MFE scheme for approximating the solution of the equations (1), (2) and (3) is as follows.

Finding $\mathbf{u}_h \in L^\infty(J, V_k^N(\mathcal{T}_h))$, $p_h \in L^\infty(J, W_k(\mathcal{T}_h))$, $c_h \in L^\infty(J, \mathcal{D}_r(\mathcal{T}_h))$ such that,

$$\begin{aligned} (\mu(c_h) K^{-1} \mathbf{u}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h) &= - (p_B, \mathbf{v} \cdot \nu)_{\Gamma_D} & (25) \\ \forall \mathbf{v} \in V_k^0(\mathcal{T}_h) \quad \forall t \in J & \end{aligned}$$

$$(\nabla \cdot \mathbf{u}_h, w) = (q, w) \quad \forall w \in W_k(\mathcal{T}_h) \quad \forall t \in J \quad (26)$$

$$\begin{aligned} \left(\phi \frac{\partial c_h}{\partial t}, \psi \right) + B(c_h, \psi; \mathbf{u}_h^M) &= L(\psi; \mathbf{u}_h^M) & (27) \\ \forall \psi \in \mathcal{D}_r(\mathcal{T}_h) \quad \forall t \in J & \end{aligned}$$

$$(c_h, \psi) = (c_0, \psi) \quad \forall \psi \in \mathcal{D}_r(\mathcal{T}_h) \quad t = 0 \quad (28)$$

where the \mathbf{u}_h^M is defined as,

$$\mathbf{u}_h^M = \min(|\mathbf{u}_h|, M) \frac{\mathbf{u}_h}{|\mathbf{u}_h|} \quad (29)$$

where, M is a fixed positive real number and $|\mathbf{u}_h| = |\mathbf{u}_h|_2 = \sqrt{\sum_{i=1}^d (\mathbf{u}_h)_i^2}$.

The reason for using \mathbf{u}_h^M rather than \mathbf{u}_h for approximation of transport equation will be clear in the next section.

4. Error Estimates for Combined DG/MFE Approximation for Miscible Displacement Problem

4.1 Notations

Throughout this paper, C denotes a generic constant whose value may change with different occurrences.

We need three projection operators and their approximation properties.

Let P_h denote L^2 -projection of W onto $W_h = W_k(\mathcal{T}_h)$: for $p \in W$, $P_h p \in W_h$ is defined by

$$(P_h p - p, w) = 0, \quad \forall w \in W_h \quad (30)$$

Let Π_h denote the usual Raviart-Thomas projection $\Pi_h : V \rightarrow V_h$ satisfies the following properties [6],

$$(\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}), w) = 0, \quad \forall w \in W_h \quad (31)$$

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{(L^2(\Omega))^d} \leq C \|\mathbf{u}\|_{(H^j(\Omega))^d} h^j \quad 1 \leq j \leq k+1 \quad (32)$$

$$\nabla \cdot \Pi_h = P_h \nabla. \quad (33)$$

where, k is the order of the RT spaces.

Furthermore, we have [6],

$$\|P_h p - p\|_{L^2(\Omega)} \leq C \|p\|_{H^j(\Omega)} h^j \quad 0 \leq j \leq k + 1 \quad (34)$$

$$(\nabla \mathbf{v}, P_h p - p) = 0, \quad \forall \mathbf{v} \in V_h \quad (35)$$

Let \widehat{P}_h be the L^2 -projection of $H^s(\mathcal{T}_h)$ to $\mathcal{D}_r(\mathcal{T}_h)$ defined by,

$$\left(\widehat{P}_h c - c, \psi \right) = 0, \quad \forall \psi \in \mathcal{D}_r(\mathcal{T}_h) \quad (36)$$

We know,

$$\|\widehat{P}_h c - c\|_0 \leq C h^j \|c\|_j \quad 0 \leq j \leq r + 1 \quad (37)$$

Before we present the result, let's define some notation for the convenience of discussion.

Define the interpolation errors for velocity, pressure and concentration as

$$E_{\mathbf{u}}^I = \Pi_h \mathbf{u} - \mathbf{u} \quad (38)$$

$$E_p^I = P_h p - p \quad (39)$$

$$E_c^I = \widehat{P}_h c - c \quad (40)$$

Define the finite element solution error for velocity, pressure and concentration as

$$E_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h \quad (41)$$

$$E_p = p - p_h \quad (42)$$

$$E_c = c - c_h \quad (43)$$

Define the auxiliary error for velocity, pressure and concentration as

$$E_{\mathbf{u}}^A = E_{\mathbf{u}}^I + E_{\mathbf{u}} = \Pi_h \mathbf{u} - \mathbf{u}_h \quad (44)$$

$$E_p^A = E_p^I + E_p = P_h p - p_h \quad (45)$$

$$E_c^A = E_c^I + E_c = \widehat{P}_h c - c_h \quad (46)$$

4.2 *a priori* error estimate for flow subproblem

Following theorem describe the error estimate for the MFE approximation of flow subproblem by assuming that the error of concentration is given.

Theorem 4.1 (Error estimate for flow) *Assume that the equations (1), (2) and (3) with boundary conditions (6) through (9) and initial condition (10) has a solution. Also assume the permeability tensor K is invertible and is uniformly positive definite and uniformly bounded above; $\mu(c)$ is uniformly Lipschitz continuous with respect to c and $\mu(c)$ is uniformly bounded below and uniformly bounded above. Let k be the order of Raviart-Thomas space (RT_k) as defined in above notation subsection.*

Then, there exists a constant $C > 0$ independent of finite element size h and independent of exact solution $(\mathbf{p}, \mathbf{u}, c)$ such that the following inequality hold for any $t \in (0, T_f]$,

$$\|E_{\mathbf{u}}\|_{(L^2(\Omega))^d} \leq C \|\mathbf{u}\|_{(L^\infty(\Omega))^d} \|E_c\|_{L^2(\Omega)} + Ch^j \|\mathbf{u}\|_{(H^j(\Omega))^d} \quad (47)$$

where, $1 \leq j \leq k + 1$.

Moreover, we have, for any $t \in (0, T_f]$,

$$\|\nabla \cdot E_{\mathbf{u}}\|_{L^2(\Omega)} \leq Ch^j \|\nabla \cdot \mathbf{u}\|_{H^j(\Omega)} \quad (48)$$

where, $1 \leq j \leq k + 1$.

Proof. It is clear that if $(\mathbf{p}, \mathbf{u}, c)$ is solution of the equations (1), (2) and (3) with boundary conditions (6) through (9) and initial condition (10), then it satisfies the following formulation for any $t \in J$.

$$\begin{aligned} (\mu(c) K^{-1} \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= - (p_B, \mathbf{v} \cdot \boldsymbol{\nu})_{\Gamma_D} \quad \forall \mathbf{v} \in V_k^0(\mathcal{T}_h) \\ (\nabla \cdot \mathbf{u}, w) &= (q, w) \quad \forall w \in W_k(\mathcal{T}_h) \end{aligned}$$

Subtracting above equations by equations (25) and (26), respectively, we have,

$$\begin{aligned} (\mu(c) K^{-1} \mathbf{u} - \mu(c_h) K^{-1} \mathbf{u}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p - p_h) &= 0 \quad \forall \mathbf{v} \in V_k^0(\mathcal{T}_h) \\ (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w) &= 0 \quad \forall w \in W_k(\mathcal{T}_h) \end{aligned}$$

or

$$\left(\mu(c_h) K^{-1} (\mathbf{u} - \mathbf{u}_h), \mathbf{v} \right) - (\nabla \cdot \mathbf{v}, P_h p - p_h) = \left((\mu(c_h) - \mu(c)) K^{-1} \mathbf{u}, \mathbf{v} \right) \quad \forall \mathbf{v} \in V_k^0(\mathcal{T}_h)$$

$$(\nabla \cdot (\Pi_h \mathbf{u} - \mathbf{u}_h), w) = 0 \quad \forall w \in W_k(\mathcal{T}_h)$$

Now, let $\mathbf{v} = E_{\mathbf{u}}^A = \Pi_h \mathbf{u} - \mathbf{u}_h$ and $w = E_p^A = P_h p - p_h$, and add above two equations, we have

$$\left(\mu(c_h) K^{-1} E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) = \left((\mu(c_h) - \mu(c)) K^{-1} \mathbf{u}, E_{\mathbf{u}}^A \right) + \left(\mu(c_h) K^{-1} E_{\mathbf{u}}^I, E_{\mathbf{u}}^A \right) \quad (49)$$

Let us bound the left hand side of equation (49) from below:

$$\left(\mu(c_h) K^{-1} E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) \geq \frac{\mu_*}{k^*} \left\| E_{\mathbf{u}}^A \right\|_{(L^2(\Omega))^d}^2$$

where, we have used the fact that uniform positive definiteness and upper boundedness of K implies the uniform definiteness and upper boundedness of K^{-1} provided K^{-1} exists everywhere. $\mu_* > 0$ is the positive constant such that $\mu(c) \geq \mu_*$ for all c ; $1/k^*$ is the constant for uniform positive definiteness of K^{-1} .

Let us bound the right hand side of equation (49) from above:

$$\begin{aligned} \left((\mu(c_h) - \mu(c)) K^{-1} \mathbf{u}, E_{\mathbf{u}}^A \right) &\leq \frac{1}{k_*} \|\mathbf{u}\|_{(L^\infty(\Omega))^d} \|\mu(c_h) - \mu(c)\|_{L^2(\Omega)} \left\| E_{\mathbf{u}}^A \right\|_{L^2(\Omega)} \\ &\leq \frac{r_\rho}{k_*} \|\mathbf{u}\|_{(L^\infty(\Omega))^d} \|c_h - c\|_{L^2(\Omega)} \left\| E_{\mathbf{u}}^A \right\|_{L^2(\Omega)} \\ \left(\mu(c_h) K^{-1} E_{\mathbf{u}}^I, E_{\mathbf{u}}^A \right) &\leq \frac{\mu^*}{k_*} \left\| E_{\mathbf{u}}^I \right\|_{(L^2(\Omega))^d} \left\| E_{\mathbf{u}}^A \right\|_{(L^2(\Omega))^d} \end{aligned}$$

where, $r_\rho \geq 0$ is the Lipschitz constant for $\mu(c)$, i.e. $|\mu(c_1) - \mu(c_2)| \leq r_\rho |c_1 - c_2|$; μ^* is the upper boundedness constant for $\mu(c)$; $1/k_*$ is the upper boundedness constant for K^{-1} .

Combining above bounds for the left-hand side and right-hand side, we obtain

$$\left\| E_{\mathbf{u}}^A \right\|_{(L^2(\Omega))^d} \leq \frac{k^* r_\rho}{k_* \mu_*} \|\mathbf{u}\|_{(L^\infty(\Omega))^d} \|E_c\|_{L^2(\Omega)} + \frac{k^* \mu^*}{k_* \mu_*} \left\| E_{\mathbf{u}}^I \right\|_{(L^2(\Omega))^d}$$

Using approximation properties, we have,

$$\left\| E_{\mathbf{u}}^A \right\|_{(L^2(\Omega))^d} \leq C \|\mathbf{u}\|_{(L^\infty(\Omega))^d} \|E_c\|_{L^2(\Omega)} + Ch^j \|\mathbf{u}\|_{(H^j(\Omega))^d}$$

The first result follows by triangle inequality.

To show the second result, we only need to notice the following equality from error equation,

$$\nabla \cdot (\Pi_h \mathbf{u} - \mathbf{u}_h) = 0$$

The second result follows by using the approximation properties. \blacksquare

4.3 *a priori* error estimate for transport subproblem

Before we present the result of error estimate, let study the property of “cut-off” operator \mathcal{M} defined as

$$\mathcal{M}(\mathbf{u})(x) = \min(|\mathbf{u}(x)|, M) \frac{\mathbf{u}(x)}{|\mathbf{u}(x)|} \quad (50)$$

where, M is a fixed positive real number and $|\mathbf{u}| = |\mathbf{u}|_2 = \sqrt{\sum_{i=1}^d (\mathbf{u})_i^2}$.

The notation \mathbf{u}_h^M used in last section is $\mathbf{u}_h^M = \mathcal{M}(\mathbf{u}_h)$. Similarly, we denote $\mathbf{u}^M = \mathcal{M}(\mathbf{u})$. The “cut-off” operator \mathcal{M} is uniformly Lipschitz continuous in the following sense.

Lemma 4.2 (Property of operator \mathcal{M}) *The “cut-off” operator \mathcal{M} defined as in equation (50) is uniformly Lipschitz continuous,*

$$\|\mathcal{M}(\mathbf{u}) - \mathcal{M}(\mathbf{v})\|_{(L^\infty(\Omega))^d} \leq \|\mathbf{u} - \mathbf{v}\|_{(L^\infty(\Omega))^d} \quad (51)$$

Proof. We notice that for all $x \in \Omega$,

$$|\mathcal{M}(\mathbf{u}) - \mathcal{M}(\mathbf{v})|_2(x) \leq |\mathbf{u} - \mathbf{v}|_2(x)$$

which can be shown by separately studying the three cases for fixed x : 1) $|\mathbf{u}|_2(x) \leq M$, $|\mathbf{v}|_2(x) \leq M$; 2) $|\mathbf{u}|_2(x) \leq M$, $|\mathbf{v}|_2(x) > M$; 3) $|\mathbf{u}|_2(x) > M$, $|\mathbf{v}|_2(x) > M$.

Taking the essential superium on both sides of above equation, we get the result. \blacksquare

Thus we have,

$$|\mathbf{u}_h^M - \mathbf{u}^M| \leq |\mathbf{u}_h - \mathbf{u}|$$

and

$$\|\mathbf{u}_h^M\|_{(L^\infty(\Omega))^d} \leq M$$

If the exact solution \mathbf{u} is bounded, i.e. $\mathbf{u} \in (L^\infty(\Omega))^d$, we can pick M large enough such that $M \geq \|\mathbf{u}\|_{(L^\infty(\Omega))^d}$, then $\mathbf{u}^M = \mathbf{u}$.

Let's state and prove two lemmas for the properties of dispersion/diffusion tensor, which will be used to prove the theorem of error estimate for transport subproblem.

Lemma 4.3 (Uniform positive definiteness of $\mathbf{D}(\mathbf{u})$) *Let $\mathbf{D}(\mathbf{u})$ defined as in equation (4), where, $d_m(x) \geq 0$, $\alpha_l(x) \geq 0$ and $\alpha_t(x) \geq 0$ are nonnegative functions of $x \in \Omega$.*

Then

$$\mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c \geq (d_m + \min(\alpha_l, \alpha_t) |\mathbf{u}|) |\nabla c|^2 \quad (52)$$

In particular, if $d_m(x) \geq d_{m,} > 0$ uniformly in the domain Ω , then $\mathbf{D}(\mathbf{u})$ is uniformly positive definite and for all $x \in \Omega$, we have,*

$$\mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c \geq d_{m,*} |\nabla c|^2 \quad (53)$$

Proof. Notice that

$$\begin{aligned} \mathbf{D}(\mathbf{u})\nabla c \cdot \nabla c &= d_m \nabla c \cdot \nabla c + |\mathbf{u}| \{ \alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t (\mathbf{I} - \mathbf{E}(\mathbf{u})) \} \nabla c \cdot \nabla c \\ &= d_m |\nabla c|^2 + |\mathbf{u}| |\nabla c|^2 \alpha_l \cos^2(\theta) + |\mathbf{u}| |\nabla c|^2 \alpha_t (1 - \cos^2(\theta)) \\ &\geq (d_m + \min(\alpha_l, \alpha_t) |\mathbf{u}|) |\nabla c|^2 \end{aligned}$$

where θ is the angle between \mathbf{u} and ∇c , i.e.

$$\cos(\theta) = \frac{\mathbf{u} \cdot \nabla c}{|\mathbf{u}| |\nabla c|}$$

■

Lemma 4.4 (Uniform Lipschitz continuousness of $\mathbf{D}(\mathbf{u})$) *Let $\mathbf{D}(\mathbf{u})$ defined as in equation (4), where, $d_m(x) \geq 0$, $\alpha_l(x) \geq 0$ and $\alpha_t(x) \geq 0$ are nonnegative of domain $x \in \Omega$, and the dispersivity α_l and α_t is uniformly bounded, i.e. $\alpha_l(x) \leq \alpha_l^*$ and $\alpha_t(x) \leq \alpha_t^*$.*

Then

$$\|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})\|_{(L^2(\Omega))^{d \times d}} \leq k_D \|\mathbf{u} - \mathbf{v}\|_{(L^2(\Omega))^d} \quad (54)$$

where, $k_D = (4\alpha_t^* + 3\alpha_l^*) d^{3/2}$ is a fixed number ($d = 2$ or 3 is the dimension of domain Ω).

Proof. Notice that

$$|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_1 = \sum_{i=1}^d \max_{j=1, \dots, d} |(\mathbf{D}(\mathbf{u}))_{i,j} - (\mathbf{D}(\mathbf{v}))_{i,j}|$$

$$\begin{aligned}
&= \sum_{i=1}^d \max_{j=1, \dots, d} \left| \alpha_i \delta_{ij} (|\mathbf{u}|_2 - |\mathbf{v}|_2) + (\alpha_l - \alpha_t) \left(\frac{u_i u_j}{|\mathbf{u}|_2} - \frac{v_i v_j}{|\mathbf{v}|_2} \right) \right| \\
&\leq d \alpha_t (|\mathbf{u}|_2 - |\mathbf{v}|_2) + 3d |\alpha_l - \alpha_t| |\mathbf{u} - \mathbf{v}|_2 \\
&\leq (\alpha_t + 3 |\alpha_l - \alpha_t|) d |\mathbf{u} - \mathbf{v}|_2
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_2 &\leq \sqrt{d} |\mathbf{D}(\mathbf{u}) - \mathbf{D}(\mathbf{v})|_1 \\
&\leq (\alpha_t + 3 |\alpha_l - \alpha_t|) d^{3/2} |\mathbf{u} - \mathbf{v}|_2
\end{aligned}$$

where, we have used the property of matrix norm: for any matrix $A \in \mathbb{R}^{m \times n}$,

$$\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1$$

The result follows by integration. ■

We need a trace estimate inequality, which is stated in the following Lemma.

Lemma 4.5 *Let $\Omega = \prod_{i=1}^d (0, L_i)$ ($d = 2$, or 3), $\mathcal{T}_h > 0$ and $H^s(\mathcal{T}_h)$ ($s > 1/2$) defined as in the notation section. Then there exist a constant C (C depends only on the domain Ω) such that*

$$\sum_{e \subset \partial\Omega} \|\phi\|_{L^2(e)}^2 \leq \sum_{e \in E_h} \int_e [\phi]^2 + \varepsilon \|\nabla \phi\|_0^2 + \frac{C}{\varepsilon} \|\phi\|_0^2$$

holds for any $\varepsilon \in (0, 1)$ and any $\phi \in H^s(\mathcal{T}_h)$.

Proof. Denote $\Gamma_{i,+}$ and $\Gamma_{i,-}$ be the boundary faces of domain Ω such that the unit outward normal vector coincide with the positive and negative x_i direction, respectively. That is,

$$\Gamma_{i,+} = \{x \in \partial\Omega : \nu(x) = \mathbf{e}_i\} \quad i = 1, \dots, d$$

$$\Gamma_{i,-} = \{x \in \partial\Omega : \nu(x) = -\mathbf{e}_i\} \quad i = 1, \dots, d$$

We have,

$$\overline{\partial\Omega} = \bigcup_{i=1}^d (\overline{\Gamma_{i,+}} \cup \overline{\Gamma_{i,-}})$$

Similarly, denote $E_{h,i}$ the set of interior edges (faces) e with the unit normal vector ν_e being the positive or negative x_i direction. That is,

$$E_{h,i} = \{e \in E_h : \nu_e = \mathbf{e}_i \text{ or } \nu_e = -\mathbf{e}_i\} \quad i = 1, \dots, d$$

Obviously, $E_h = \bigcup_{i=1}^d E_{h,i}$.

Define a nice subspace of $H^s(\mathcal{T}_h)$ as

$$C^\infty(\mathcal{T}_h) = \left\{ \phi \in L^2(\Omega) : \phi|_R \in C^\infty(R) \cap H^s(R), R \in \mathcal{T}_h \right\}$$

Pick an arbitrary $\phi \in C^\infty(\mathcal{T}_h)$, let us bound the term $\|\phi\|_{L^2(\Gamma_{1,+})}^2 + \|\phi\|_{L^2(\Gamma_{1,-})}^2$.

Fix a point $(0, \zeta_2, \dots, \zeta_d) \in \Gamma_{1,-}$ such that

$$(0, \zeta_2, \dots, \zeta_d) \notin \bigcup_{e \in E_h} \bar{e}$$

We know $(L_1, \zeta_2, \dots, \zeta_d) \in \Gamma_{1,+}$ and

$$(L_1, \zeta_2, \dots, \zeta_d) \notin \bigcup_{e \in E_h} \bar{e}$$

thus $\phi(0, \zeta_2, \dots, \zeta_d)$ and $\phi(L_1, \zeta_2, \dots, \zeta_d)$ have well-defined values.

Define ϕ_0 as the average value:

$$\phi_0 = \frac{1}{L_1} \int_0^{L_1} \phi(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1$$

We know there exist at least one value $\chi \in (0, L_1)$ such that,

$$\phi(\chi_-, \zeta_2, \dots, \zeta_d) \leq \phi_0 \leq \phi(\chi_+, \zeta_2, \dots, \zeta_d)$$

or

$$\phi(\chi_+, \zeta_2, \dots, \zeta_d) \leq \phi_0 \leq \phi(\chi_-, \zeta_2, \dots, \zeta_d)$$

where, $\phi(\chi_-, \zeta_2, \dots, \zeta_d)$ and $\phi(\chi_+, \zeta_2, \dots, \zeta_d)$ are understood as

$$\phi(\chi_-, \zeta_2, \dots, \zeta_d) = \lim_{\delta \rightarrow 0^+} \phi(\chi - \delta, \zeta_2, \dots, \zeta_d)$$

$$\phi(\chi_+, \zeta_2, \dots, \zeta_d) = \lim_{\delta \rightarrow 0^+} \phi(\chi + \delta, \zeta_2, \dots, \zeta_d)$$

Integrating ϕ^2 along the line connecting $(0, \zeta_2, \dots, \zeta_d)$ and $(\chi, \zeta_2, \dots, \zeta_d)$ and the line connecting $(\chi, \zeta_2, \dots, \zeta_d)$ and $(L_1, \zeta_2, \dots, \zeta_d)$, we find,

$$\begin{aligned}
& \phi^2(0, \zeta_2, \dots, \zeta_d) + \phi^2(L_1, \zeta_2, \dots, \zeta_d) \\
\leq & 2\phi_0^2 + \int_0^{L_1} \left| \frac{\partial}{\partial \zeta_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) \right| d\zeta_1 \\
& + \sum_{\zeta \in (0, L_1)} \left| \phi^2(\zeta_+, \zeta_2, \dots, \zeta_d) - \phi^2(\zeta_-, \zeta_2, \dots, \zeta_d) \right|
\end{aligned}$$

but,

$$\begin{aligned}
& \int_0^{L_1} \left| \frac{\partial}{\partial \zeta_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) \right| d\zeta_1 \\
= & 2 \int_0^{L_1} |\phi(\zeta_1, \zeta_2, \dots, \zeta_d) \phi_{\zeta_1}(\zeta_1, \zeta_2, \dots, \zeta_d)| d\zeta_1 \\
\leq & 2 \left(\int_0^{L_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1 \int_0^{L_1} \phi_{\zeta_1}^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1 \right)^{1/2} \\
\leq & \frac{1}{\varepsilon} \int_0^{L_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1 + \varepsilon \int_0^{L_1} \phi_{\zeta_1}^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1
\end{aligned}$$

$$2\phi_0^2 \leq \frac{2}{L_1} \int_0^{L_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1$$

Thus,

$$\begin{aligned}
& \phi^2(0, \zeta_2, \dots, \zeta_d) + \phi^2(L_1, \zeta_2, \dots, \zeta_d) \\
\leq & \left(\frac{2}{L_1} + \frac{1}{\varepsilon} \right) \int_0^{L_1} \phi^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1 + \varepsilon \int_0^{L_1} \phi_{\zeta_1}^2(\zeta_1, \zeta_2, \dots, \zeta_d) d\zeta_1 \\
& + \sum_{\zeta \in (0, L_1)} \left| \phi^2(\zeta_+, \zeta_2, \dots, \zeta_d) - \phi^2(\zeta_-, \zeta_2, \dots, \zeta_d) \right|
\end{aligned}$$

Notice that above inequality holds for a.e. $(\zeta_2, \dots, \zeta_d) \in \prod_{i=2}^d (0, L_i)$. Now integrating above inequality on $\int_0^{L_2} d\zeta_2 \cdots \int_0^{L_d} d\zeta_d$, we have

$$\|\phi\|_{L^2(\Gamma_{1,-})}^2 + \|\phi\|_{L^2(\Gamma_{1,+})}^2 \leq \left(\frac{2}{L_1} + \frac{1}{\varepsilon} \right) \|\phi\|_0^2 + \varepsilon \|\phi_{\zeta_1}\|_0^2 + \sum_{e \in E_{h,1}} \int_e [\phi]^2$$

Similarly, for $i = 1, \dots, d$, we can have

$$\|\phi\|_{L^2(\Gamma_{i,-})}^2 + \|\phi\|_{L^2(\Gamma_{i,+})}^2 \leq \left(\frac{2}{L_i} + \frac{1}{\varepsilon} \right) \|\phi\|_0^2 + \varepsilon \|\phi_{\zeta_i}\|_0^2 + \sum_{e \in E_{h,i}} \int_e [\phi]^2$$

Summing above inequality for $i = 1, \dots, d$, we obtain,

$$\sum_{e \subset \partial\Omega} \|\phi\|_{L^2(e)}^2 \leq \sum_{e \in E_h} \int_e [\phi]^2 + \varepsilon \|\nabla \phi\|_0^2 + \left(\frac{d}{\varepsilon} + \sum_{i=1}^d \frac{2}{L_i} \right) \|\phi\|_0^2$$

Let

$$C = d + \sum_{i=1}^d \frac{2}{L_i}$$

C is fixed constant depending only on the size of domain Ω .

Notice that $\frac{d}{\varepsilon} + \sum_{i=1}^d \frac{2}{L_i} \leq \frac{C}{\varepsilon}$ for any $\varepsilon \in (0, 1)$, we obtain that

$$\sum_{e \subset \partial\Omega} \|\phi\|_{L^2(e)}^2 \leq \sum_{e \in E_h} \int_e [\phi]^2 + \varepsilon \|\nabla \phi\|_0^2 + \frac{C}{\varepsilon} \|\phi\|_0^2$$

holds for any $\varepsilon \in (0, 1)$ and any $\phi \in C^\infty(\mathcal{T}_h)$.

Using the fact that $C^\infty(\mathcal{T}_h)$ is dense in $H^s(\mathcal{T}_h)$, the lemma follows by density argument. \blacksquare

Now, we are ready for the error estimate for transport subproblem. Let us first derivate the error equation without any assumption.

It is clear that if $(\mathbf{p}, \mathbf{u}, c)$ is solution of the equations (1), (2) and (3) with boundary conditions (6) through (9) and initial condition (10), then it satisfies the following formulation for any $t \in J$.

$$\left(\phi \frac{\partial c}{\partial t}, \psi \right) + B(c, \psi; \mathbf{u}) = L(\psi; \mathbf{u}) \quad \forall \psi \in \mathcal{D}_r(\mathcal{T}_h)$$

Denote $\hat{c} = \widehat{P}_h c$ and $\hat{\mathbf{u}} = \Pi_h \mathbf{u}$. Notice that $[\hat{c} - c] = [\hat{c}]$ on any interior edge (face) $e \in E_h$ and that $\mathbf{u}^M = \mathbf{u}$ if we picked M large enough, then above equation can be written as, $\forall \psi \in \mathcal{D}_r(\mathcal{T}_h)$,

$$\begin{aligned} & \left(\phi \frac{\partial \hat{c}}{\partial t}, \psi \right) + \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}_h^M) \nabla \hat{c}) \cdot \nabla \psi + J_0^{\sigma, \beta}(\hat{c}, \psi) \\ = & \sum_{R \in \mathcal{T}_h} \int_R \hat{c} \mathbf{u}_h^M \cdot \nabla \psi + \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}_h^M) \nabla \hat{c} \cdot \nu_e \rangle [\psi] - \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}_h^M) \nabla \psi \cdot \nu_e \rangle [\hat{c}] \\ & - \sum_{e \in E_h} \int_e \hat{c}^* \mathbf{u}_h^M \cdot \nu_e [\psi] - \sum_{e \in E_{h, out}} \int_e \hat{c} \mathbf{u}_h^M \cdot \nu_e \psi + \int_\Omega \hat{c} q^- \psi + \int_\Omega c_w q^+ \psi \\ & - \sum_{e \in E_{h, in}} \int_e c_B \mathbf{u}_h^M \cdot \nu_e \psi + \left(\phi \frac{\partial \hat{c} - c}{\partial t}, \psi \right) + \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M)) \nabla \hat{c} \cdot \nabla \psi \\ & + \sum_{R \in \mathcal{T}_h} \int_R \mathbf{D}(\mathbf{u}^M) \nabla (\hat{c} - c) \cdot \nabla \psi + J_0^{\sigma, \beta}(\hat{c} - c, \psi) - \sum_{R \in \mathcal{T}_h} \int_R \hat{c} (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nabla \psi \end{aligned}$$

$$\begin{aligned}
& - \sum_{R \in \mathcal{T}_h} \int_R (\widehat{c} - c) \mathbf{u}^M \cdot \nabla \psi - \sum_{e \in E_h} \int_e \langle (\mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M)) \nabla \widehat{c} \cdot \nu_e \rangle [\psi] \\
& - \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}^M) \nabla (\widehat{c} - c) \cdot \nu_e \rangle [\psi] + \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}_h^M) \nabla \psi \cdot \nu_e \rangle [\widehat{c} - c] \\
& + \sum_{e \in E_h} \int_e \widehat{c}^* (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e [\psi] + \sum_{e \in E_h} \int_e (\widehat{c} - c)^* \mathbf{u}^M \cdot \nu_e [\psi] \\
& + \sum_{e \in E_{h,out}} \int_e \widehat{c} (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e \psi + \sum_{e \in E_{h,out}} \int_e (\widehat{c} - c) \mathbf{u}^M \cdot \nu_e \psi \\
& - \int_{\Omega} (\widehat{c} - c) q^- \psi + \sum_{e \in E_{h,in}} \int_e c_B (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e \psi
\end{aligned}$$

Subtracting above equation by equation (27) and set $\psi = E_c^A$, we have,

$$\begin{aligned}
& \left(\phi \frac{\partial E_c^A}{\partial t}, E_c^A \right) + \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}_h^M) \nabla E_c^A) \cdot \nabla E_c^A \quad (55) \\
& + J_0^{\sigma, \beta} (E_c^A, E_c^A) \\
& = \sum_{R \in \mathcal{T}_h} \int_R E_c^A \mathbf{u}_h^M \cdot \nabla E_c^A - \sum_{e \in E_h} \int_e (E_c^A)^* \mathbf{u}_h^M \cdot \nu_e [E_c^A] \\
& - \sum_{e \in E_{h,out}} \int_e E_c^A \mathbf{u}_h^M \cdot \nu_e E_c^A + \int_{\Omega} E_c^A q^- E_c^A + \sum_{i=1}^{15} T_i
\end{aligned}$$

where,

$$\begin{aligned}
T_1 & = \left(\phi \frac{\partial \widehat{c} - c}{\partial t}, E_c^A \right) \\
T_2 & = \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M)) \nabla \widehat{c} \cdot \nabla E_c^A \\
T_3 & = \sum_{R \in \mathcal{T}_h} \int_R \mathbf{D}(\mathbf{u}^M) \nabla (\widehat{c} - c) \cdot \nabla E_c^A \\
T_4 & = J_0^{\sigma, \beta} (\widehat{c} - c, E_c^A) \\
T_5 & = - \sum_{R \in \mathcal{T}_h} \int_R \widehat{c} (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nabla E_c^A \\
T_6 & = - \sum_{R \in \mathcal{T}_h} \int_R (\widehat{c} - c) \mathbf{u}^M \cdot \nabla E_c^A
\end{aligned}$$

$$\begin{aligned}
 T_7 &= - \sum_{e \in E_h} \int_e \langle (\mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M)) \nabla \hat{c} \cdot \nu_e \rangle [E_c^A] \\
 T_8 &= - \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}^M) \nabla (\hat{c} - c) \cdot \nu_e \rangle [E_c^A] \\
 T_9 &= \sum_{e \in E_h} \int_e \langle \mathbf{D}(\mathbf{u}_h^M) \nabla E_c^A \cdot \nu_e \rangle [\hat{c} - c] \\
 T_{10} &= \sum_{e \in E_h} \int_e \hat{c}^* (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e [E_c^A] \\
 T_{11} &= \sum_{e \in E_h} \int_e (\hat{c} - c)^* \mathbf{u}^M \cdot \nu_e [E_c^A] \\
 T_{12} &= \sum_{e \in E_{h,out}} \int_e \hat{c} (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e E_c^A \\
 T_{13} &= \sum_{e \in E_{h,out}} \int_e (\hat{c} - c) \mathbf{u}^M \cdot \nu_e E_c^A \\
 T_{14} &= - \int_{\Omega} (\hat{c} - c) q^- E_c^A \\
 T_{15} &= \sum_{e \in E_{h,in}} \int_e c_B (\mathbf{u}_h^M - \mathbf{u}^M) \cdot \nu_e E_c^A
 \end{aligned}$$

The above error equation is difficult to analyze in general boundary condition and we thus assume that only Neumann boundary condition for flow subproblem is used. In this way, we can know the normal velocity on the boundary by exact value. The boundary condition for transport problem can still be either inflow or outflow/noflow.

Theorem 4.6 (Error estimate for transport) *Assume that the equations (1), (2) and (3) with boundary conditions (6) through (9) and initial condition (10) has a solution. Assume that only Neumann boundary condition is imposed for flow subproblem; the exact solution \mathbf{u} and c are smooth enough:*

$$\mathbf{u} \in C_B \left(\overline{\Omega} \times (0, T_f] \right) \cap L^\infty \left((0, T_f], H^{l+\frac{1}{2}}(\Omega) \right) \quad (56)$$

$$c \in C_B \left((0, T_f], W^{1,\infty}(\Omega) \right) \cap L^2 \left((0, T_f], H^m(\Omega) \right) \quad (57)$$

and

$$\frac{\partial c}{\partial t} \in L^2((0, T_f], H^n(\Omega)) \quad (58)$$

We also assume that porosity ϕ is time-dependent and is uniformly bounded below and above; the parameter β in interior penalty term for DG formulation is assume to be $\beta = 1$; the extraction part of source term satisfies $q^- \in (\Pi_{R \in \mathcal{T}_h} W^{s,1}(R))'$ for all of the partitions \mathcal{T}_h used, where $0 \leq s < 1$ (see remark 4.7 for the explanation of this assumption). Assume M is picked large enough such that $M \geq \|\mathbf{u}\|_{(L^\infty(\Omega))^d}$ for all $t \in (0, T_f]$.

Then, there exist a constant $C > 0$ independent of finite element size h and a constant $h_0 > 0$ such that the following inequality hold for any $\tau \in (0, T_f]$ and for any $h \leq h_0$,

$$\begin{aligned} & \|E_c\|_0^2(\tau) + \int_0^\tau \|\nabla E_c\|_0^2(t) dt \\ & \leq C \int_0^\tau \|E_c\|_0^2 + C \int_0^\tau \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + Ch^{\min(2k+2, 2l, 2r, 2m-2, 2n)} \end{aligned} \quad (59)$$

where, $k \geq 0$, $r \geq 1$ are the order of Raviart-Thomas space and discontinuous space, respectively, defined in above notation subsection; l, m, n describe the regularity order of solution $\mathbf{u}, c, \partial c/\partial t$ respectively, as defined in equations (56), (57) and (58).

Proof. Let us first relax the assumption $\beta = 1$ for a while so that we can also show that $\beta = 1$ is indeed the optimal choice for the value of parameter β . Using the lemma (4.3), let's bound the left hand side of the error equation (55) from below,

$$\begin{aligned} & \left(\phi \frac{\partial E_c^A}{\partial t}, E_c^A \right) + \sum_{R \in \mathcal{T}_h} \int_R (\mathbf{D}(\mathbf{u}_h^M) \nabla E_c^A) \cdot \nabla E_c^A + J_0^{\sigma, \beta} (E_c^A, E_c^A) \\ & \geq \frac{1}{2} \frac{\partial}{\partial t} \|\sqrt{\phi} E_c^A\|_0^2 + d_{m,*} \|\nabla E_c^A\|_0^2 + J_0^{\sigma, \beta} (E_c^A, E_c^A) \end{aligned}$$

where, we have used the uniform positive definiteness of the dispersion/diffusion tensor from Lemma (4.3).

Let's bound from above the right hand side of the error equation (55). The first term is straightforward.

$$\begin{aligned}
 \sum_{R \in \mathcal{T}_h} \int_R E_c^A \mathbf{u}_h^M \cdot \nabla E_c^A &\leq M \sum_{R \in \mathcal{T}_h} \int_R |E_c^A| |\nabla E_c^A| \\
 &\leq M \sum_{R \in \mathcal{T}_h} \|E_c^A\|_{L^2(R)} \|\nabla E_c^A\|_{(L^2(R))^d} \\
 &\leq \frac{C}{\varepsilon} \sum_{R \in \mathcal{T}_h} \|E_c^A\|_{L^2(R)}^2 + \varepsilon \sum_{R \in \mathcal{T}_h} \|\nabla E_c^A\|_{(L^2(R))^d}^2 \\
 &= \frac{C}{\varepsilon} \|E_c^A\|_0^2 + \varepsilon \|\nabla E_c^A\|_0^2
 \end{aligned}$$

where, ε is a small positive constant.

The second term is a little tricky.

$$\begin{aligned}
 - \sum_{e \in E_h} \int_e (E_c^A)^* \mathbf{u}_h^M \cdot \nu_e [E_c^A] &\leq M \left| \sum_{e \in E_h} \int_e (E_c^A)^* [E_c^A] \right| \\
 &\leq M \sum_{e \in E_h} \|(E_c^A)^*\|_{L^2(e)}^2 \|[E_c^A]\|_{L^2(e)}^2 \\
 &\leq M \sum_{e \in E_h} \left(\frac{\varepsilon}{M} \frac{\sigma_e}{h_e^\beta} \|[E_c^A]\|_{L^2(e)}^2 + \frac{Ch_e^\beta}{\varepsilon} \|(E_c^A)^*\|_{L^2(e)}^2 \right) \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{Ch^\beta}{\varepsilon} \sum_{R \in \mathcal{T}_h} h^{-1} \|E_c^A\|_{L^2(R)}^2 \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{C}{\varepsilon} \|E_c^A\|_0^2
 \end{aligned}$$

where, we have used the assumption $\beta \geq 1$.

The third term can be bounded by using Lemma (4.5).

$$\begin{aligned}
 - \sum_{e \in E_{h, \text{out}}} \int_e E_c^A \mathbf{u}_h^M \cdot \nu_e E_c^A &\leq M \sum_{e \in E_{h, \text{out}}} \|E_c^A\|_{L^2(e)}^2 \\
 &\leq \frac{Mh^\beta}{\sigma_*} J_0^{\sigma, \beta} (E_c^A, E_c^A) + \varepsilon \|\nabla E_c^A\|_0^2 + \frac{C}{\varepsilon} \|E_c^A\|_0^2
 \end{aligned}$$

The fourth term comes from the extraction wells, and it can be bounded as follows.

$$\int_\Omega E_c^A q^- E_c^A \leq \|q^-\|_{(\Pi_{R \in \mathcal{T}_h} W^{s, 1}(R))'} \sum_{R \in \mathcal{T}_h} \left\| (E_c^A)^2 \right\|_{W^{s, 1}(R)}$$

$$\begin{aligned}
&\leq C \sum_{R \in \mathcal{T}_h} \|E_c^A\|_{H^s(R)}^2 \leq C \sum_{R \in \mathcal{T}_h} \|E_c^A\|_{L^2(R)}^{2(1-s)} \|E_c^A\|_{H^1(R)}^{2s} \\
&\leq \sum_{R \in \mathcal{T}_h} \left(\frac{C}{\varepsilon^{s/(1-s)}} \|E_c^A\|_{L^2(R)}^2 + \varepsilon \|E_c^A\|_{H^1(R)}^2 \right) \\
&\leq \frac{C}{\varepsilon^{s/(1-s)}} \|E_c^A\|_0^2 + \varepsilon \|\nabla E_c^A\|_0^2
\end{aligned}$$

where, we have used the fact,

$$a^{1-s} b^s \leq \left(\frac{a}{\varepsilon^{1-s}} + b\varepsilon \right) \quad a > 0 \quad b > 0 \quad 0 \leq s < 1$$

Now let us bound the terms T_1 through T_{15} .

$$\begin{aligned}
T_1 &\leq \phi^* \left\| \frac{\partial E_c^I}{\partial t} \right\|_{L^2(\Omega)} \|E_c^A\|_{L^2(\Omega)} \leq \frac{\phi^*}{2} \left\| \frac{\partial E_c^I}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{\phi^*}{2} \|E_c^A\|_{L^2(\Omega)}^2 \\
T_2 &\leq \|\nabla \hat{c}\|_{(L^\infty(\Omega))^3} \sum_{R \in \mathcal{T}_h} \left\| \mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M) \right\|_{(L^2(R))^{d \times d}} \|\nabla E_c^A\|_{(L^2(R))^d} \\
&\leq C \|\nabla c\|_{(L^\infty(\Omega))^3} \sum_{R \in \mathcal{T}_h} \left\| \mathbf{u}_h^M - \mathbf{u}^M \right\|_{(L^2(R))^d} \|\nabla E_c^A\|_{(L^2(R))^d} \\
&\leq \sum_{R \in \mathcal{T}_h} \left(\frac{C}{\varepsilon} \|E_{\mathbf{u}}\|_{(L^2(R))^d}^2 + \varepsilon \|\nabla E_c^A\|_{(L^2(R))^d}^2 \right) \\
&\leq \frac{C}{\varepsilon} \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + \varepsilon \|\nabla E_c^A\|_0^2
\end{aligned}$$

where, we have used the facts of $\|\nabla \hat{c}\|_{(L^\infty(\Omega))^3} \leq C \|\nabla c\|_{(L^\infty(\Omega))^3} \leq C$ and Lemmas (4.2) and (4.4) for bounding the term T_2 .

$$\begin{aligned}
T_3 &\leq C \sum_{R \in \mathcal{T}_h} \int_R \nabla E_c^I \cdot \nabla E_c^A \leq \sum_{R \in \mathcal{T}_h} \left(\frac{C}{\varepsilon} \|\nabla E_c^I\|_{(L^2(R))^d}^2 + \varepsilon \|\nabla E_c^A\|_{(L^2(R))^d}^2 \right) \\
&\leq \frac{C}{\varepsilon} \|\nabla E_c^I\|_0^2 + \varepsilon \|\nabla E_c^A\|_0^2 \\
T_4 &= \sum_{e \in E_h} \frac{\sigma_e}{h_e^\beta} \int_e [E_c^I] [E_c^A] \\
&\leq \sum_{e \in E_h} \frac{\sigma_e}{h_e^\beta} \left(\varepsilon \| [E_c^A] \|_{L^2(e)}^2 + \frac{C}{\varepsilon} \| [E_c^I] \|_{L^2(e)}^2 \right)
\end{aligned}$$

$$\begin{aligned}
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{C}{\varepsilon h^\beta} \sum_{e \in E_h} \left\| [E_c^I] \right\|_{L^2(e)}^2 \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{C}{\varepsilon h^\beta} \sum_{R \in \mathcal{T}_h} \left(h^{-1} \|E_c^I\|_{L^2(R)}^2 + h \|\nabla E_c^I\|_{(L^2(R))^d}^2 \right) \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{C}{\varepsilon h^{\beta+1}} \|E_c^I\|_0^2 + \frac{C}{\varepsilon h^{\beta-1}} \|\nabla E_c^I\|_0^2
 \end{aligned}$$

T_5 can be bounded similarly as the term T_3 ,

$$\begin{aligned}
 T_5 &\leq \|\widehat{c}\|_{L^\infty(\Omega)} \sum_{R \in \mathcal{T}_h} \left\| \mathbf{u}_h^M - \mathbf{u}^M \right\|_{(L^2(R))^d} \left\| \nabla E_c^A \right\|_{(L^2(R))^d} \\
 &\leq \frac{C}{\varepsilon} \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + \varepsilon \|\nabla E_c^A\|_0^2
 \end{aligned}$$

where, we have used that fact $\|\widehat{c}\|_{L^\infty(\Omega)} \leq C \|c\|_{L^\infty(\Omega)} \leq C$.

$$\begin{aligned}
 T_6 &\leq M \sum_{R \in \mathcal{T}_h} \|E_c^I\|_{L^2(R)}^2 \|\nabla E_c^A\|_{(L^2(R))^d}^2 \\
 &\leq \sum_{R \in \mathcal{T}_h} \left(\frac{C}{\varepsilon} \|E_c^I\|_{L^2(R)}^2 + \varepsilon \|\nabla E_c^A\|_{(L^2(R))^d}^2 \right) \\
 &\leq \frac{C}{\varepsilon} \|E_c^I\|_0^2 + \varepsilon \|\nabla E_c^A\|_0^2
 \end{aligned}$$

The term T_7 is quite tricky,

$$\begin{aligned}
 T_7 &= \|\nabla \widehat{c}\|_{(L^\infty(\Omega))^d} \sum_{e \in E_h} \left\| \mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M) \right\|_{(L^2(e))^{d \times d}} \left\| [E_c^A] \right\|_{L^2(e)} \\
 &\leq \sum_{e \in E_h} \left(\varepsilon \frac{\sigma_e}{h_e^\beta} \left\| [E_c^A] \right\|_{L^2(e)}^2 + \frac{Ch^\beta}{\varepsilon} \left\| \mathbf{D}(\mathbf{u}_h^M) - \mathbf{D}(\mathbf{u}^M) \right\|_{(L^2(e))^{d \times d}}^2 \right) \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{Ch^\beta}{\varepsilon} \sum_{e \in E_h} \|\mathbf{u}_h - \mathbf{u}\|_{(L^2(e))^d}^2 \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{Ch^\beta}{\varepsilon} \sum_{e \in E_h} \|\mathbf{u}_h - \widehat{\mathbf{u}}\|_{(L^2(e))^d}^2 + \frac{Ch^\beta}{\varepsilon} \sum_{e \in E_h} \|\widehat{\mathbf{u}} - \mathbf{u}_h\|_{(L^2(e))^d}^2 \\
 &\leq \varepsilon J_0^{\sigma, \beta} (E_c^A, E_c^A) + \frac{Ch^{\beta-1}}{\varepsilon} \sum_{R \in \mathcal{T}_h} \|\mathbf{u}_h - \widehat{\mathbf{u}}\|_{(L^2(R))^d}^2 + \frac{C}{\varepsilon} h^{\min(2k+\beta+1, 2l+\beta-1)}
 \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon J_0^{\sigma,\beta} (E_c^A, E_c^A) + \frac{Ch^{\beta-1}}{\varepsilon} \sum_{R \in \mathcal{T}_h} \left(\|E_{\mathbf{u}}\|_{(L^2(R))^d}^2 + \|E_{\mathbf{u}}^I\|_{(L^2(R))^d}^2 \right) \\
&\quad + \frac{C}{\varepsilon} h^{\min(2k+\beta+1, 2l+\beta-1)} \\
&\leq \varepsilon J_0^{\sigma,\beta} (E_c^A, E_c^A) + \frac{Ch^{\beta-1}}{\varepsilon} \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + \frac{C}{\varepsilon} h^{\min(2k+\beta+1, 2l+\beta-1)}
\end{aligned}$$

where, we have used the facts of $\|\nabla \hat{c}\|_{(L^\infty(\Omega))^d} \leq C \|\nabla c\|_{(L^\infty(\Omega))^d} \leq C$ and the approximation properties for \mathbf{u} .

The boundedness of T_8 can be shown by using penalty term,

$$\begin{aligned}
T_8 &\leq C \sum_{e \in E_h} \|\nabla(\hat{c} - c)\|_{(L^2(e))^d} \left\| [E_c^A] \right\|_{L^2(e)} \\
&\leq \varepsilon J_0^{\sigma,\beta} (E_c^A, E_c^A) + \frac{C}{\varepsilon} h^{\beta-1} \|\nabla E_c^I\|_0^2 \\
T_9 &\leq C \sum_{e \in E_h} \left\| \nabla E_c^A \right\|_{(L^2(e))^d} \|\hat{c} - c\|_{L^2(e)} \\
&\leq \varepsilon \|\nabla E_c^A\|_0^2 + \frac{C}{\varepsilon h^2} \|E_c^I\|_0^2 + \frac{C}{\varepsilon} \|\nabla E_c^I\|_0^2
\end{aligned}$$

The term T_{10} is similar as T_7

$$\begin{aligned}
T_{10} &\leq \|\hat{c}\|_{L^\infty(\Omega)} \sum_{e \in E_h} \|\mathbf{u}_h - \mathbf{u}\|_{(L^2(e))^d}^2 \left\| [E_c^A] \right\|_{L^2(e)} \\
&\leq \varepsilon J_0^{\sigma,\beta} (E_c^A, E_c^A) + \frac{Ch^{\beta-1}}{\varepsilon} \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + \frac{C}{\varepsilon} h^{\min(2k+\beta+1, 2l+\beta-1)} \\
T_{11} &\leq C \sum_{e \in E_h} \left\| (E_c^I)^* \right\|_{L^2(e)} \left\| [E_c^A] \right\|_{L^2(e)} \\
&\leq \varepsilon J_0^{\sigma,\beta} (E_c^A, E_c^A) + \frac{Ch^{\beta-1}}{\varepsilon} \|E_c^I\|_0^2 + \frac{Ch^{\beta+1}}{\varepsilon} \|\nabla E_c^I\|_0^2
\end{aligned}$$

Recall we assume that only Neumann boundary condition for flow subproblem are used, we do not have the term T_{12} and T_{15} if we use the exact value of \mathbf{u} on the boundary in the DG formulation. That is,

$$T_{12} = T_{15} = 0$$

$$T_{13} \leq C \sum_{e \in E_{h,out}} \int_e (\hat{c} - c) E_c^A \leq C \sum_{e \in E_{h,out}} \left\| E_c^I \right\|_{L^2(e)} \left\| E_c^A \right\|_{L^2(e)}$$

$$\begin{aligned}
 &\leq \sum_{e \in E_{h,out}} \left(\frac{C}{h} \|E_c^I\|_{L^2(e)}^2 + h \|E_c^A\|_{L^2(e)}^2 \right) \\
 &\leq \sum_{R \in \mathcal{T}_h} \frac{C}{h^2} \left(\|E_c^I\|_{L^2(R)}^2 + h^2 \|\nabla E_c^I\|_{L^2(R)}^2 \right) + C \sum_{R \in \mathcal{T}_h} \|E_c^A\|_{L^2(R)}^2 \\
 &\leq \frac{C}{h^2} \|E_c^I\|_0^2 + C \|\nabla E_c^I\|_0^2 + C \|E_c^A\|_0^2
 \end{aligned}$$

The term T_{14} can be bounded similarly as the term $\int_{\Omega} E_c^A q^- E_c^A$ above,

$$\begin{aligned}
 T_{14} &\leq \left| \int_{\Omega} (\hat{c} - c) q^- E_c^A \right| \\
 &\leq \frac{C}{\varepsilon s / (1-s)} \|E_c^A\|_0^2 + \varepsilon \|\nabla E_c^A\|_0^2 + C \|E_c^I\|_0^2 + C \|\nabla E_c^I\|_0^2
 \end{aligned}$$

Combining the above bounds for both hand sides of the error equation (55), choosing ε small enough, we have, for $\forall h \leq (\frac{\sigma_*}{3M})^{1/\beta}$,

$$\begin{aligned}
 &\frac{1}{2} \frac{\partial}{\partial t} \|\sqrt{\phi} E_c^A\|_0^2 + \frac{d_{m,*}}{2} \|\nabla E_c^A\|_0^2 + \frac{1}{2} J_0^{\sigma,\beta} (E_c^A, E_c^A) \\
 &\leq C \|E_c^A\|_0^2 + (C + Ch^{\beta-1}) \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 \\
 &\quad + C \|\partial E_c^I / \partial t\|_0^2 + \left(C + \frac{C}{h^{\beta+1}} + \frac{C}{h^2} + Ch^{\beta-1} \right) \|E_c^I\|_0^2 \\
 &\quad + \left(C + \frac{C}{h^{\beta-1}} + Ch^{\beta-1} + Ch^{\beta+1} \right) \|\nabla E_c^I\|_0^2 + Ch^{\min(2k+\beta+1, 2l+\beta-1)}
 \end{aligned}$$

We can find the best choice of β is indeed $\beta = 1$, then,

$$\begin{aligned}
 &\frac{1}{2} \frac{\partial}{\partial t} \|\sqrt{\phi} E_c^A\|_0^2 + \frac{d_{m,*}}{2} \|\nabla E_c^A\|_0^2 + \frac{1}{2} J_0^{\sigma,\beta} (E_c^A, E_c^A) \\
 &\leq C \|E_c^A\|_0^2 + C \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 \\
 &\quad + C \|\partial E_c^I / \partial t\|_0^2 + \frac{C}{h^2} \|E_c^I\|_0^2 + C \|\nabla E_c^I\|_0^2 + Ch^{\min(2k+2, 2l)}
 \end{aligned}$$

Now, we integrate with respect to time between 0 to τ ($0 \leq \tau \leq T_f$) and we have,

$$\begin{aligned}
 &\|\sqrt{\phi} E_c^A\|_0^2(\tau) + \frac{d_{m,*}}{2} \int_0^\tau \|\nabla E_c^A\|_0^2(t) dt + \frac{1}{2} \int_0^\tau J_0^{\sigma,\beta} (E_c^A, E_c^A) \\
 &\leq \|\sqrt{\phi} E_c^A\|_0^2(0) + C \int_0^\tau \|E_c^A\|_0^2 + C \int_0^\tau \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2
 \end{aligned}$$

$$+C \int_0^\tau \|\partial E_c^I / \partial t\|_0^2 + \frac{C}{h^2} \int_0^\tau \|E_c^I\|_0^2 + C \int_0^\tau \|\nabla E_c^I\|_0^2 + CT_f h^{\min(2k+2, 2l)}$$

Notice that $\|\sqrt{\phi} E_c^A\|_0^2(0) = \|\sqrt{\phi} E_c^I\|_0^2(0)$, and that ϕ is uniformly bounded below and above, using the approximation results for $c \in \mathcal{D}_r(\mathcal{T}_h)$, we have,

$$\begin{aligned} & \|E_c^A\|_0^2(\tau) + \int_0^\tau \|\nabla E_c^A\|_0^2(t) dt + \int_0^\tau J_0^{\sigma, \beta}(E_c^A, E_c^A) \\ & \leq C \int_0^\tau \|E_c^A\|_0^2 + C \int_0^\tau \|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 + Ch^{\min(2k+2, 2l, 2r, 2m-2, 2n)} \end{aligned}$$

The theorem follows by the triangle inequality. ■

Remark 4.7 *In the above theorems, we don't make assumption about the regularity of injection part of source term q^+ and injection concentration c_w , which can have singular value such as point source (the delta function). But we make assumption for the extraction part of source term $q^- \in (\Pi_{R \in \mathcal{T}_h} W^{s,1}(R))'$ for all of the partitions \mathcal{T}_h used ($0 \leq s < 1$). This means that q^- can be more singular than $L^2(\Omega)$, but can not be too singular as the delta function. Roughly speaking, it is like $q^- \in (W^{s,1}(\Omega))'$ except we also need that the position for singularity of q^- can not be on the interior and boundary edges or faces for any partition \mathcal{T}_h .*

Remark 4.8 *We don't have assumption on the inflow boundary concentration c_B , which can have singular value.*

4.4 *A priori* error estimate for the coupled system of flow and transport

The error estimate for the coupled system of flow and transport will be easy once we get sharp error control on both subproblems.

Theorem 4.9 (Error estimate for coupled system of flow and transport) *Let the assumption in Theorems (4.1) and (4.6) holds.*

Then, there exist a constant $C > 0$ independent of finite element size h and a constant $h_0 > 0$ such that the following inequality hold for any $h \leq h_0$,

$$\|E_{\mathbf{u}}\|_{L^\infty((0, T_f); (L^2(\Omega))^d)}^2 \leq Ch^{\min(2k+2, 2l, 2r, 2m-2, 2n)} \quad (60)$$

$$\|E_c\|_{L^\infty((0,T_f);L^2(\Omega))}^2 + \|\nabla E_c\|_{L^2((0,T_f);(L^2(\Omega))^d)}^2 \leq Ch^{\min(2k+2,2l,2r,2m-2,2n)} \quad (61)$$

where, $k \geq 0$, $r \geq 1$ are the order of Raviart-Thomas space and discontinuous space, respectively, defined in above notation subsection; l, m, n describe the regularity order of solution $\mathbf{u}, c, \partial c/\partial t$ respectively, as defined in equations (56), (57) and (58).

Proof. Combining Theorems (4.1) and (4.6), we have,

$$\|E_c\|_0^2(\tau) + \int_0^\tau \|\nabla E_c\|_0^2(t) dt \leq C \int_0^\tau \|E_c\|_0^2 + Ch^{\min(2k+2,2l,2r,2m-2,2n)}$$

Using the Gronwall's inequality, we have the error result for E_c .

To get the bound for $E_{\mathbf{u}}$, we substitute the error result for E_c into the following result, which comes from Theorems (4.1).

$$\|E_{\mathbf{u}}\|_{(L^2(\Omega))^d}^2 \leq C \|E_c\|_{L^2(\Omega)}^2 + Ch^{\min(2k+2,2l+1)}$$

■

Remark 4.10 *If we let $r = k + 1$, and if the exact solution \mathbf{u}, c are smooth enough, then the Theorem (4.9) gives the optimal L^2 (H^1) rate of convergence for concentration, and also gives optimal L^∞ (L^2) rate of convergence for velocity.*

Remark 4.11 *The error estimate for the couple system of flow and transport is not applied for the case of having Dirichlet boundary condition for flow. It is difficult to bound the error for the couple system for the case of having Dirichlet boundary condition for flow, because we don't yet have sharp control on the error of velocity \mathbf{u}_h in the boundary edge or face.*

5. Conclusion

A combined method with mixed finite element method for flow and discontinuous Galerkin method for transport is introduced for the coupled system of miscible displacement problem. The “cut-off” operator \mathcal{M} is introduced in the discontinuous Galerkin (DG) scheme in order to make the combined scheme converge. The property of “cut-off” operator \mathcal{M} is given. The optimal choice of penalty parameter β in DG scheme is derived to be $\beta = 1$. The Neumann boundary condition for flow subproblem is assumed for getting the error estimate of the coupled

system. Error estimates in $L^2(H^1)$ and $L^\infty(L^2)$ for concentration and error estimate in $L^\infty(L^2)$ for velocity are derived, which are the optimal $L^2(H^1)$ rate of convergence for concentration, and optimal $L^\infty(L^2)$ rate of convergence for velocity. The uniform positive definitiveness and uniform Lipschitz continuity of dispersion/diffusion tensor computed by Engineering standard formula are proved. The injection part of source term is allowed to have arbitrarily singular value and the extraction part of source term to have value singular to some degree. The result extended to general boundary condition for flow subproblem is an ongoing research.

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