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Coupling locally conservative methods for single phase flow

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This work presents the coupling of two locally conservative methods for elliptic problems: namely, the discontinuous Galerkin method and the mixed finite element method. The couplings can be defined with or without interface Lagrange multipliers. The formulations are shown to be equivalent. Optimal error estimates are given; penalty terms may or may not be included. In addition, the analysis for non-conforming grids is also discussed.

Keywords: multinumerics, non-conforming meshes, mixed finite elements, discontinuous Galerkin method, Lagrange multipliers, penalties, error estimates

1. Introduction

Local mass conservation is an essential feature for many transient simulations including subsurface flow models. Without this property, the mass error can accumulate, and the numerical solution exhibits increasing instability. Two efficient finite element methods satisfy the local mass conservation property: namely the discontinuous Galerkin (DG) method and the mixed finite element (MFE) method.

Because of their great flexibility, the DG methods have recently received much attention from the finite element community, and several schemes have been introduced and analyzed for elliptic equations in the last five years. Besides the local mass conservation, discontinuous finite element methods can handle unstructured and irregular grids, full tensor coefficients, and also can easily take full advantage of the *hp* adaptivity techniques. One should note that the scheme allows for non-conforming meshes, with several "hanging nodes". The DG methods we consider in the paper are based on the work of Wheeler [18], Oden et al. [11], and Rivière et al. [14,15]. The bilinear form is nonsymmetric and may or may not contain penalty terms. One can refer to [4,17] for further information.

The MFE methods are very popular among the computational scientists and engineers and a large number of papers have been dedicated to this method applied to elliptic problems [5]. From a practical point of view, the lower order Raviart–Thomas spaces are preferred. In [3,16] it was shown that in that case, the MFE are equivalent to a finite difference scheme, thus they are well suited for structured grids. Attempts have been made to apply the MFE on irregular grids [2]. However, the implementation as well as the use of higher order MFE approximation spaces can be complex in such cases.

The goal of this work is to present and analyze the coupling of DG and MFE methods for an elliptic equation. The advantage of this multinumerics approach lies in the ability of choosing a particular scheme for a particular subdomain. In the regions containing faults and highly variable permeability fields, the DG methods have the advantage to handle full tensors and locally refined unstructured grids. In the subdomains with tensor product grids, the MFE should be used. Each subdomain will be meshed, and we allow for non-conforming meshes in the DG region. The proposed coupling can be formulated with the addition of Lagrange multipliers, and thus can be run in parallel very efficiently.

Being able to couple different numerical methods is sometimes a challenging task. The mathematical literature contains a great number of articles on that subject, and we remark on work done on coupling discontinuous finite element methods with other numerical schemes. Another discontinuous method is referred to as the local discontinuous Galerkin (LDG) method [6,7,9]. In that case, the elliptic equation is written in a mixed form and both pressure and velocity are approximated. LDG was first coupled with the conforming finite element method by Alotto et al. [1] and Perugia and Schōtzau [12]. This work was extended to transport problems by Dawson and Proft [10]. Recently, Cockburn and Dawson [8] coupled the LDG with the MFE for elliptic equations and proved error estimates. Our work is the first paper on coupling the DG methods described and analyzed in [11,14,15] with another numerical method.

The outline of the paper is as follows: after this introduction, we describe in section 2 the model problem, some notation, and the appropriate approximation results. In section 3, we introduce the scheme defining the coupling. The error estimates are proven in section 4. A domain decomposition formulation is introduced in section 5. Extensions of the different schemes to penalty methods are presented in section 6, and we finish the paper with some concluding remarks.

2. Model problem and notation

Let Ω be a polygonal domain in $\mathbb{R}^d$, $d = 2, 3$, and let the boundary of the domain $\partial\Omega$ be the union of two disjoint sets Γ^{D} and Γ^{N}. We denote by $\boldsymbol{n}$ the unit normal vector to each edge of $\partial\Omega$ exterior to Ω. For f given in $L^2(\Omega)$, p_0 given in $H^{1/2}(\Gamma^{\mathrm{D}})$ and g given in $L^2(\Gamma^{\mathrm{N}})$, we consider the following elliptic problem:

$$-\nabla\cdot(K\nabla p) + \alpha p = f \quad \text{in } \Omega, \tag{1.1}$$

$$p = p_0 \quad \text{on } \Gamma^{\mathrm{D}}, \tag{1.2}$$

$$-K\nabla p\cdot\boldsymbol{n} = g \quad \text{on } \Gamma^{\mathrm{N}}. \tag{1.3}$$

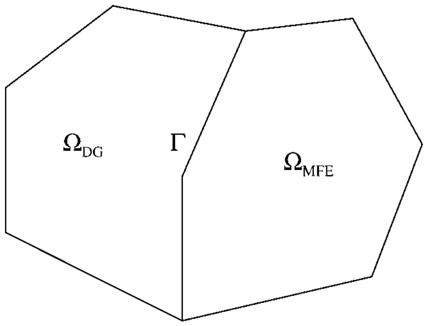

Figure 1. Example of subdomains.

For Darcy flow problems, p denotes the fluid pressure, K a permeability tensor, and f a general source function. We assume that K is symmetric positive definite and that α is a positive constant. We can rewrite (1.1)–(1.3) in mixed form by introducing the Darcy velocity $\boldsymbol{u}$, e.g.,

$$\boldsymbol{u} = -K\nabla p \quad \text{in } \Omega, \tag{2.1}$$
$$\nabla\cdot\boldsymbol{u} + \alpha p = f \quad \text{in } \Omega, \tag{2.2}$$
$$p = p_0 \quad \text{on } \Gamma^{\mathrm{D}}, \tag{2.3}$$
$$\boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{on } \Gamma^{\mathrm{N}}. \tag{2.4}$$

We subdivide Ω into non-degenerate triangles in 2D, tetrahedra in 3D, and denote by $\mathcal{E}_h^{\mathrm{DG}}$ (respectively $\mathcal{E}_h^{\mathrm{MFE}}$) the set of elements on which the DG method (respectively the MFE method) is applied. We also define $\Omega_{\mathrm{DG}} = \bigcup_{E\in\mathcal{E}_h^{\mathrm{DG}}} E$ and $\Omega_{\mathrm{MFE}} = \bigcup_{E\in\mathcal{E}_h^{\mathrm{MFE}}} E$. We assume that the partition on Ω_{MFE} is a conforming one but there is no restriction on the geometry of the decomposition of Ω_{DG}. Let Γ be the interface composed of edges in 2D, faces in 3D, shared by elements of Ω_{DG} and Ω_{MFE}. A simple illustration is given in the case of two subdomains in figure 1. A DG element edge at the interface Γ may consist of one or several MFE element edges. Let Γ_{DG} (respectively Γ_{MFE}) denote the skeleton of the mesh of Ω_{DG} (respectively Ω_{MFE}), that is the union of open segments that coincide with interior edges of elements. Let $\Gamma_{\mathrm{DG}}^{\mathrm{D}}$ (respectively $\Gamma_{\mathrm{MFE}}^{\mathrm{D}}$) be the union of Dirichlet edges that belong to $\partial\Omega_{\mathrm{DG}}$ (respectively $\partial\Omega_{\mathrm{MFE}}$). Similarly, we define $\Gamma_{\mathrm{DG}}^{\mathrm{N}}$ and $\Gamma_{\mathrm{MFE}}^{\mathrm{N}}$. We also associate with each segment (or face) e_a in Γ_{DG} a unit normal vector $\boldsymbol{n}_a$. For e_a in Γ, the vector $\boldsymbol{n}_a$ is outward to $\partial\Omega_{\mathrm{DG}}$ and is denoted by $\boldsymbol{n}_\Gamma$.

Based on [14], we define for $s \geqslant 0$ and $m \geqslant 1$

$$W^{s,m}(\mathcal{E}_h^{\mathrm{DG}}) = \{\phi \in L^m(\Omega_{\mathrm{DG}}):\ \phi|_E \in W^{s,m}(E)\ \forall E \in \mathcal{E}_h^{\mathrm{DG}}\}$$

and we denote it by $H^s(\mathcal{E}_h^{\mathrm{DG}})$ when $m = 2$. We associate to $H^s(\mathcal{E}_h^{\mathrm{DG}})$ the "broken" norm $|||\phi|||_s^2 = \sum_E \|\phi\|_{s,E}^2$, where $\|\cdot\|_{s,E}$ is the usual Sobolev norm. We will use the usual notation $(\cdot,\cdot)_E$ and $\langle\cdot,\cdot\rangle_\Gamma$ for the L^2 inner product on E and Γ, respectively. We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h^{\mathrm{DG}})$, $s > 1/2$

$$\{\phi\} = \tfrac{1}{2}(\phi|_{E_1} + \phi|_{E_2}), \qquad [\phi] = (\phi|_{E_1}) - (\phi|_{E_2}), \quad \forall e_a = \partial E_1 \cap \partial E_2,$$
$$\{\phi\} = (\phi|_{E_1}), \qquad [\phi] = (\phi|_{E_1}), \qquad \forall e_a \in \partial E_1 \cap \partial\Omega_{\mathrm{DG}}.$$

We recall the definition of $H(div, \Omega_{\mathrm{MFE}})$:

$$H(div, \Omega_{\mathrm{MFE}}) = \left\{ \boldsymbol{v} \in \left(L^2(\Omega_{\mathrm{MFE}})\right)^d\colon\ \nabla\cdot\boldsymbol{v} \in L^2(\Omega_{\mathrm{MFE}}) \right\}.$$

We now define standard approximation spaces for both methods. In the case of DG, for r positive integer, we consider the space

$$\mathcal{D}_r = \left\{ \phi \in L^2(\Omega_{\mathrm{DG}})\colon\ \phi|_E \in \mathbb{P}_r(E)\ \forall E \subset \Omega_{\mathrm{DG}} \right\}, \tag{3}$$

where $\mathbb{P}_r(E)$ denotes the set of polynomials of degree less than or equal to r on each element E. We associate to Ω_{MFE} the standard Raviart–Thomas spaces of order k, defined by

$$\begin{aligned} \boldsymbol{V}_h = \big\{ &\boldsymbol{v} \in H(div, \Omega_{\mathrm{MFE}})\colon\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{p}^k(\boldsymbol{x}) + q^k(\boldsymbol{x})\boldsymbol{x},\ \forall \boldsymbol{x} \in E, \\ &\boldsymbol{p}^k \in \left(\mathbb{P}_k(E)\right)^d,\ q^k \in \mathbb{P}_k(E),\ \forall E \in \mathcal{E}_h^{\mathrm{MFE}},\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{MFE}}^{\mathrm{N}} \big\}, \\ W_h = \big\{ &w \in L^2(\Omega_{\mathrm{MFE}})\colon\ w|_E \in \mathbb{P}_k(E)\ \forall E \in \mathcal{E}_h^{\mathrm{MFE}} \big\}. \end{aligned}$$

We associate the norm $|||\cdot|||_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}}$ to the product space $\mathcal{D}_r \times \boldsymbol{V}_h \times W_h$ as defined below

$$\begin{aligned} |||(\phi, \boldsymbol{v}, w)|||^2_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}} = {}& |||K^{1/2}\nabla\phi|||^2_{0,\Omega_{\mathrm{DG}}} + \left\|\alpha^{1/2}\phi\right\|^2_{0,\Omega_{\mathrm{DG}}} \\ &+ \left\|K^{-1/2}\boldsymbol{v}\right\|^2_{0,\Omega_{\mathrm{MFE}}} + \left\|\alpha^{1/2}w\right\|^2_{0,\Omega_{\mathrm{MFE}}}. \end{aligned}$$

We modify the approximation result proved in [15], so that it can be applied to nonconforming grids.

Lemma 1. For h small enough, let $p \in H^s(\mathcal{E}_h^{\mathrm{DG}})$ for $s \geqslant 2$ and let $r \geqslant 2$. There exists an interpolant of p, $p^{\mathrm{I}} \in \mathcal{D}_r$ such that for each E in $\mathcal{E}_h^{\mathrm{DG}}$ and each edge (or face) e that is divided into disjoint open sets $\gamma^1, \ldots, \gamma^{s_e}$, the following properties hold:

$$\int_{\gamma^j} K\nabla\left(p - p^{\mathrm{I}}|_E\right)\cdot\boldsymbol{n}_E = 0, \qquad j = 1,\ldots,s_e, \tag{4}$$

$$\left\|\nabla^i\left(p - p^{\mathrm{I}}\right)\right\|_{0,\Omega_{\mathrm{DG}}} \leqslant Ch^{\mu-i}, \quad i = 0,1,2, \tag{5}$$

where $\boldsymbol{n}_E$ is a unit outward normal vector to E, $\mu = \min(r+1, s)$ and C independent of h.

Proof. We follow the construction of p^{I}, as given in [15]. We first show (4) and (5) in the case of a constant tensor $\overline{K}$ and for triangles and tetrahedra.

The case of triangles. Let E be a triangle with vertices a_1, a_2, a_3, opposite sides e_1, e_2, e_3, and unit exterior normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$. Let λ_1, λ_2, λ_3 be the barycentric coordinates of a_1, a_2, and a_3 in E. We assume that each edge e_i is divided into disjoint open sets $\gamma_i^1, \ldots, \gamma_i^{s_{e_i}}$. First, we will show that given f in $H^s(E)$ with $s \geqslant 2$, there is a polynomial q_1 in $\mathbb{P}_2(E)$ such that $\int_{\gamma_1^j} \overline{K}\nabla(q_1 - f) \cdot \boldsymbol{n}_1 = 0$, $j = 1, \ldots, s_{e_1}$, and $\int_{e_2} \overline{K}\nabla q_1 \cdot \boldsymbol{n}_2 = \int_{e_3} \overline{K}\nabla q_1 \cdot \boldsymbol{n}_3 = 0$. For this, consider the polynomial $q_1 = 4q_1(a_{12})\lambda_1(1-\lambda_1)$, where a_{12} is the midpoint of $e_3 = [a_1, a_2]$. It is easy to check that each component of $\nabla q_1 = 4q_1(a_{12})\nabla\lambda_1(1-2\lambda_1)$ has zero mean-value on e_2 and e_3, and $(\nabla\lambda_1)\lambda_1$ vanishes on e_1. Therefore, $q_1(a_{12})$ is determined by the conditions

$$4q_1(a_{12}) \int_{\gamma_1^j} \overline{K}\nabla\lambda_1 \cdot \boldsymbol{n}_1 = \int_{\gamma_1^j} \overline{K}\nabla f \cdot \boldsymbol{n}_1, \quad j = 1, \ldots, s_{e_1}.$$

But

$$\nabla\lambda_1 = -\frac{\boldsymbol{n}_1}{2}\frac{|e_1|}{|E|}. \tag{6}$$

Therefore,

$$-2q_1(a_{21})\big(\overline{K}\boldsymbol{n}_1, \boldsymbol{n}_1\big)\frac{|e_1||\gamma_1^j|}{|E|} = \int_{\gamma_1^j} \overline{K}\nabla f \cdot \boldsymbol{n}_1, \quad j = 1, \ldots, s_{e_1}.$$

By summing over j, we obtain

$$q_1(a_{21}) = -\frac{1}{2}\frac{|E|}{|e_1|^2}\frac{1}{(\overline{K}\boldsymbol{n}_1, \boldsymbol{n}_1)} \int_{e_1} \overline{K}\nabla f \cdot \boldsymbol{n}_1.$$

Hence,

$$\big|q_1(a_{12})\big| \leqslant \frac{C}{\gamma_0}\left|\int_{e_1} \overline{K}\nabla f \cdot \boldsymbol{n}_1\right| \leqslant C\frac{\gamma_1}{\gamma_0} h_E^{1/2} \left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}.$$

Therefore, for $i = 0, 1, 2$,

$$\big\|\nabla^i q_1\big\|_{0,E} \leqslant C|E|^{1/2} h_E^{-i}\big|q_1(a_{12})\big| \leqslant C|E|^{1/2} h_E^{1/2-i}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}. \tag{7}$$

Similarly, we construct polynomials q_2 and q_3 in $\mathbb{P}_2(E)$ such that

$$\int_{\gamma_i^j} \overline{K}\nabla q_i \cdot \boldsymbol{n}_i = \int_{\gamma_i^j} \overline{K}\nabla f \cdot \boldsymbol{n}_i, \quad \text{for } i = 2, 3,\ j = 1, \ldots, s_{e_i},$$

$$\int_{e_1} \overline{K}\nabla q_2 \cdot \boldsymbol{n}_1 = \int_{e_3} \overline{K}\nabla q_2 \cdot \boldsymbol{n}_3 = 0,$$

$$\int_{e_1} \overline{K}\nabla q_3 \cdot \boldsymbol{n}_1 = \int_{e_2} \overline{K}\nabla q_3 \cdot \boldsymbol{n}_2 = 0,$$

and such that (7) hold for q_2 and q_3, with respect to e_2 and e_3. Let $q = q_1 + q_2 + q_3$, constructed with $f = p - \tilde{p}$, where $\tilde{p}$ is an approximation of p satisfying (5), and set $p^{\mathrm{I}} = q + \tilde{p}$. Then p^{I} satisfies (4), and we derive for $i = 0, 1, 2$:

$$\begin{aligned} |p^{\mathrm{I}} - p|_{i,E} &\leqslant |q|_{i,E} + |\tilde{p} - p|_{i,E} \\ &\leqslant C h_E^{\mu - i} \|p\|_{s,E} + |\tilde{p} - p|_{i,E}, \end{aligned}$$

which has the same order of approximation as $|\tilde{p} - p|_{i,E}$.

The case of tetrahedra. The situation is much the same for tetrahedra. Let E be a tetrahedron with vertices a_1, a_2, a_3, a_4, opposite faces e_1, e_2, e_3, e_4, and unit exterior normal vectors $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{n}_4$. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the barycentric coordinates of a_1, a_2, a_3, and a_4 in E. Again, we will show that given f in $H^s(E)$ with $s \geqslant 2$, there is a polynomial q_1 in $\mathbb{P}_2(E)$ such that $\int_{\gamma_1^j} K\nabla(q_1 - f) \cdot \boldsymbol{n}_1 = 0$, $j = 1, \ldots, s_{e_1}$, $\int_{e_i} K\nabla q_1 \cdot \boldsymbol{n}_i = 0$ for $i = 2, 3, 4$. For this, consider the polynomial

$$q_1 = q_1(a_1)\lambda_1(3\lambda_1 - 2).$$

It is easy to check that each component of ∇q_1:

$$\nabla q_1 = q_1(a_1)\nabla\lambda_1(6\lambda_1 - 2)$$

has zero mean-value on e_2, e_3, e_4, and $(\nabla\lambda_1)\lambda_1$ vanishes on e_1. Therefore, $q_1(a_1)$ is determined by the condition:

$$-2q_1(a_1)\int_{\gamma_1^j} \overline{K}\nabla\lambda_1 \cdot \boldsymbol{n}_1 = \int_{\gamma_1^j} \overline{K}\nabla f \cdot \boldsymbol{n}_1,$$

and since (6) is valid in 3D, we sum over j and obtain:

$$q_1(a_1) = \frac{|E|}{|e_1|^2}\frac{1}{(\overline{K}\boldsymbol{n}_1, \boldsymbol{n}_1)}\int_{e_1} \overline{K}\nabla f \cdot \boldsymbol{n}_1,$$

and for $i = 0, 1, 2$, we obtain the analogue of (7):

$$\left\|\nabla^i q_1\right\|_{0,E} \leqslant C|E|^{1/2}h_E^{-i}\left|q_1(a_1)\right| \leqslant C|E|^{1/2}h_E^{-i}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}.$$

The proof finishes exactly as the proof of corollary 5.2 in [15], where the case of a general tensor K is considered. □

We also recall the approximation results associated with the Π and L^2 projections associated with MFE [5].

Lemma 2. There is a projection operator $\Pi : H(div, \Omega_{\mathrm{MFE}}) \to \boldsymbol{V}_h$ with the properties:

$$\int_{\Omega_{\mathrm{MFE}}} \nabla \cdot (\Pi z - z)w = 0, \quad \forall w \in W_h, \tag{8}$$

$$\|\Pi z - z\|_{0,\Omega_{\rm MFE}} \leqslant Ch^{k+1}\|z\|_{k+1,\Omega_{\rm MFE}}, \tag{9}$$

$$\|\Pi z - z\|_{0,e_a} \leqslant Ch^{k+1}\|z\|_{k+1,\infty,e}, \quad \forall e_a \in \Gamma_{\rm MFE}, \tag{10}$$

$$\int_{e_a} (\Pi z - z)w = 0, \qquad \forall w \in W_h,\ \forall e_a \in \Gamma_{\rm MFE}. \tag{11}$$

Lemma 3. Let $\mathcal{P}_h : L^2(\Omega_{\rm MFE}) \to W_h$ be the L^2 projection operator. Then,

$$\int_{\Omega_{\rm MFE}} (\mathcal{P}_h q - q)w = 0, \qquad \forall w \in W_h, \tag{12}$$

$$\|\mathcal{P}_h q - q\|_{s,\Omega_{\rm MFE}} \leqslant Ch^{j-s}\|q\|_{j,\Omega_{\rm MFE}}, \quad 0 \leqslant s < j,\ 0 \leqslant j \leqslant k+1. \tag{13}$$

Finally, we recall the trace and inverse inequalities that hold on each element E and for any $e \in \partial E$:

$$\forall \phi \in H^1(E), \quad \|\phi\|_{0,e}^2 \leqslant C\big(h^{-1}\|\phi\|_{0,E}^2 + h\|\nabla\phi\|_{0,E}^2\big), \tag{14}$$

$$\forall \phi \in H^2(E), \quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e}^2 \leqslant C\big(h^{-1}\|\nabla\phi\|_{0,E}^2 + h\big\|\nabla^2\phi\big\|_{0,E}^2\big), \tag{15}$$

$$\forall \phi \in \mathbb{P}_r(E), \quad \|\phi\|_{0,e} \leqslant Ch^{-1/2}\|\phi\|_{0,E}, \tag{16}$$

$$\forall \phi \in \mathbb{P}_r(E), \quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e} \leqslant Ch^{-1/2}\|\nabla\phi\|_{0,E}. \tag{17}$$

3. Scheme

In this section, we discuss the variational problem and show that the coupled scheme is consistent and has a unique solution. We first define the bilinear forms $a_{\rm DG} : \mathcal{D}_r \times \mathcal{D}_r \to \mathbb{R}$, $a_{\rm MFE} : \boldsymbol{V}_h \times W_h \to \mathbb{R}$ and $b_{\rm MFE} : \boldsymbol{V}_h \times W_h \to \mathbb{R}$:

$$a_{\rm DG}(\psi,\phi) = \sum_{E\in\mathcal{E}_h^{\rm DG}} \int_E (K\nabla\psi\nabla\phi + \alpha\psi\phi)$$
$$- \sum_{e_a\in\Gamma_{\rm DG}\cup\Gamma_{\rm DG}^{\rm D}} \int_{e_a} \{K\nabla\psi\cdot\boldsymbol{n}_a\}[\phi] + \sum_{e_a\in\Gamma_{\rm DG}\cup\Gamma_{\rm DG}^{\rm D}} \int_{e_a} \{K\nabla\phi\cdot\boldsymbol{n}_a\}[\psi],$$

$$a_{\rm MFE}(\boldsymbol{v}, \boldsymbol{z}) = \int_{\Omega_{\rm MFE}} K^{-1}\boldsymbol{v}\cdot\boldsymbol{z},$$

$$b_{\rm MFE}(\boldsymbol{v}, w) = \int_{\Omega_{\rm MFE}} w\nabla\cdot\boldsymbol{v}.$$

We consider the following scheme: find $(P^{\rm DG}, \boldsymbol{U}^{\rm MFE}, P^{\rm MFE}) \in \mathcal{D}_r \times \boldsymbol{V}_h \times W_h$ such that

$$a_{\rm DG}\big(P^{\rm DG},\phi\big) = (f,\phi)_{\Omega_{\rm DG}} - \langle \boldsymbol{U}^{\rm MFE}\cdot\boldsymbol{n}_\Gamma, \phi\rangle_\Gamma$$
$$+ \int_{\Gamma_{\rm DG}^{\rm D}} K\nabla\phi\cdot\boldsymbol{n}p_0 - \int_{\Gamma_{\rm DG}^{\rm N}} g\phi, \quad \forall\phi\in\mathcal{D}_r, \tag{18.1}$$

$$a_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, \boldsymbol{v}\big) - b_{\mathrm{MFE}}\big(\boldsymbol{v}, P^{\mathrm{MFE}}\big) = -\langle \boldsymbol{v}\cdot\boldsymbol{n}, p_0\rangle_{\Gamma^{\mathrm{D}}_{\mathrm{MFE}}} + \big\langle \boldsymbol{v}\cdot\boldsymbol{n}_\Gamma, P^{\mathrm{DG}}\big\rangle_\Gamma, \quad \forall \boldsymbol{v}\in \boldsymbol{V}_h, \tag{18.2}$$

$$b_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, w\big) + \big(\alpha P^{\mathrm{MFE}}, w\big)_{\Omega_{\mathrm{MFE}}} = (f, w)_{\Omega_{\mathrm{MFE}}}, \quad \forall w\in W_h. \tag{18.3}$$

We approximate the pressure p by P^{DG} on Ω_{DG} and by P^{MFE} on Ω_{MFE}. The Darcy velocity is approximated by $-K\nabla P^{\mathrm{DG}}$ on Ω_{DG} and by $\boldsymbol{U}^{\mathrm{MFE}}$ on Ω_{MFE}.

Lemma 4. Let $(p, \boldsymbol{u})$ be the solution of (2.1)–(2.4). If $p|_{\Omega_{\mathrm{DG}}} \in H^2(\mathcal{E}_h^{\mathrm{DG}})$, then $(p|_{\Omega_{\mathrm{DG}}}, \boldsymbol{u}|_{\Omega_{\mathrm{MFE}}}, p|_{\Omega_{\mathrm{MFE}}})$ is a solution of (18.1)–(18.3).

Proof. We first show that $p|_{\Omega_{\mathrm{DG}}}$ satisfies (18.1). We multiply (1.1) by a test function ϕ, and we integrate on each element E and sum over all elements in $\mathcal{E}_h^{\mathrm{DG}}$.

$$\sum_{E\in\mathcal{E}_h^{\mathrm{DG}}}\int_E (K\nabla p\nabla\phi + \alpha p\phi) - \sum_{e_a\in\Gamma_{\mathrm{DG}}}\int_{e_a}(K\nabla p\cdot\boldsymbol{n}_a)[\phi] - \int_{\partial\Omega_{\mathrm{DG}}}(K\nabla p\cdot\boldsymbol{n})\phi = \int_{\Omega_{\mathrm{DG}}} f\phi.$$

Using the Neumann boundary conditions and the definition of the velocity, we can rewrite

$$\sum_{E\in\mathcal{E}_h^{\mathrm{DG}}}\int_E (K\nabla p\nabla\phi + \alpha p\phi) - \sum_{e_a\in\Gamma_{\mathrm{DG}}\cup\Gamma_{\mathrm{DG}}^{\mathrm{D}}}\int_{e_a}\{K\nabla p\cdot\boldsymbol{n}_a\}[\phi] = \int_{\Omega_{\mathrm{DG}}} f\phi - \int_{\Gamma_{\mathrm{DG}}^{\mathrm{N}}} g\phi - \int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}.$$

We then note that $[p] = 0$ and we add the Dirichlet boundary condition. Thus, we clearly have (18.1). Second, we show that (18.2) and (18.3) hold. We multiply (2.1) by $\boldsymbol{v}\in\boldsymbol{V}_h$ and integrate by parts the second term:

$$a_{\mathrm{MFE}}(\boldsymbol{u}, \boldsymbol{v}) - b_{\mathrm{MFE}}(\boldsymbol{v}, p) + \int_{\partial\Omega_{\mathrm{MFE}}} p\boldsymbol{v}\cdot\boldsymbol{n} = 0.$$

Using the Dirichlet boundary conditions and noting that $\boldsymbol{n} = -\boldsymbol{n}_\Gamma$, we obtain

$$a_{\mathrm{MFE}}(\boldsymbol{u}, \boldsymbol{v}) - b_{\mathrm{MFE}}(\boldsymbol{v}, p) = -\int_{\Gamma_{\mathrm{MFE}}^{\mathrm{D}}} p_0\boldsymbol{v}\cdot\boldsymbol{n} + \int_\Gamma p\boldsymbol{v}\cdot\boldsymbol{n}_\Gamma.$$

The equality (18.3) is obtained in a straightforward manner by multiplying (2.2) by a test function. □

Lemma 5. The solution to (18.1)–(18.3) exists and is unique.

Proof. Since (18.1), (18.2), and (18.3) yield a square system of linear equations in finite dimension, it suffices to show uniqueness of the solution. For that, we set $f = 0$ and $p_0 = 0$ and we choose $\phi = P^{\mathrm{DG}}$, $\boldsymbol{v} = \boldsymbol{U}^{\mathrm{MFE}}$, and $w = P^{\mathrm{MFE}}$ in (18.1)–(18.3).

We then obtain

$$a_{\mathrm{DG}}\big(P^{\mathrm{DG}}, P^{\mathrm{DG}}\big) = -\int_\Gamma \boldsymbol{U}^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma P^{\mathrm{DG}},$$

$$\big\|K^{-1/2}\boldsymbol{U}^{\mathrm{MFE}}\big\|^2_{0,\Omega_{\mathrm{MFE}}} - \int_{\Omega_{\mathrm{MFE}}} P^{\mathrm{MFE}}\nabla\cdot\boldsymbol{U}^{\mathrm{MFE}} = \int_\Gamma \boldsymbol{U}^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma P^{\mathrm{DG}},$$

$$\int_{\Omega_{\mathrm{MFE}}} P^{\mathrm{MFE}}\nabla\cdot\boldsymbol{U}^{\mathrm{MFE}} + \big\|\alpha^{1/2}P^{\mathrm{MFE}}\big\|^2_{0,\Omega_{\mathrm{MFE}}} = 0.$$

By adding the three equations above, we obtain:

$$\big|\!\big|\!\big|\big(P^{\mathrm{DG}}, \boldsymbol{U}^{\mathrm{MFE}}, P^{\mathrm{MFE}}\big)\big|\!\big|\!\big|^2_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}} = 0.$$

Since $\alpha > 0$, then P^{DG}, $\boldsymbol{U}^{\mathrm{MFE}}$ and P^{MFE} are zero everywhere. □

4. A priori error estimates

In this section, we state and prove our main result, that is the estimate of the error in the $|\!|\!|\cdot|\!|\!|_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}}$ norm.

Theorem 1. Let $s \geqslant 2$ and $k \geqslant 0$. Assume $p|_{\Omega_{\mathrm{DG}}} \in H^s(\mathcal{E}_h^{\mathrm{DG}})$, $p|_{\Omega_{\mathrm{MFE}}} \in H^{k+1}(\Omega_{\mathrm{MFE}})$ and $\boldsymbol{u}|_{\Omega_{\mathrm{MFE}}} \in (H^{k+1}(\Omega_{\mathrm{MFE}}))^2$ satisfy (2.1)–(2.4). Then, for $\mu = \min(r+1, s)$, and $r \geqslant 2$, there exists a constant C independent of h such that

$$\begin{aligned}&\big|\!\big|\!\big|\big(p - P^{\mathrm{DG}}, \boldsymbol{u} - \boldsymbol{U}^{\mathrm{MFE}}, p - P^{\mathrm{MFE}}\big)\big|\!\big|\!\big|_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}}\\ &\quad\leqslant C\big(h^{\mu-1}\|p\|_{s,\Omega_{\mathrm{DG}}} + h^{k+1}\big(\|\boldsymbol{u}\|_{k+1,\infty,\Gamma_{\mathrm{MFE}}} + \|\boldsymbol{u}\|_{k+1,\Omega_{\mathrm{MFE}}} + \|p\|_{k+1,\Omega_{\mathrm{MFE}}}\big)\big).\end{aligned}$$

Proof. Let p^{I} be the interpolant of $p|_{\Omega_{\mathrm{DG}}}$ as defined in lemma 2.1 and let $\Pi\boldsymbol{u}$ and $\mathcal{P}_h p$ be the Π and the L^2 projections defined in lemma 2.2 and lemma 2.3. Let us denote the numerical errors $\xi_{\mathrm{DG}} = P^{\mathrm{DG}} - p^{\mathrm{I}}$, $\xi_{\mathrm{MFE}} = P^{\mathrm{MFE}} - \mathcal{P}_h p$ and $\boldsymbol{\zeta} = \boldsymbol{U}^{\mathrm{MFE}} - \Pi\boldsymbol{u}$. We have the following error equations:

$$\begin{aligned}a_{\mathrm{DG}}(\xi_{\mathrm{DG}}, \phi) &= a_{\mathrm{DG}}\big(p - p^{\mathrm{I}}, \phi\big) - \big\langle\big(\boldsymbol{U}^{\mathrm{MFE}} - \boldsymbol{u}\big)\cdot\boldsymbol{n}_\Gamma, \phi\big\rangle_\Gamma,\\ a_{\mathrm{MFE}}(\boldsymbol{\zeta}, \boldsymbol{v}) - b_{\mathrm{MFE}}(\boldsymbol{v}, \xi_{\mathrm{MFE}}) &= a_{\mathrm{MFE}}(\boldsymbol{u} - \Pi\boldsymbol{u}, \boldsymbol{v}) - b_{\mathrm{MFE}}(\boldsymbol{v}, p - \mathcal{P}_h p)\\ &\quad + \big\langle P^{\mathrm{DG}} - p, \boldsymbol{v}\cdot\boldsymbol{n}_\Gamma\big\rangle_\Gamma,\\ b_{\mathrm{MFE}}(\boldsymbol{\zeta}, w) + (\alpha\xi_{\mathrm{MFE}}, w)_{\Omega_{\mathrm{MFE}}} &= b_{\mathrm{MFE}}(\boldsymbol{u} - \Pi\boldsymbol{u}, w) + \big(\alpha(p - \mathcal{P}_h p), w\big)_{\Omega_{\mathrm{MFE}}}.\end{aligned}$$

We now choose $\phi = \xi_{\mathrm{DG}}$, $\boldsymbol{v} = \boldsymbol{\zeta}$ and $w = \xi_{\mathrm{MFE}}$, and we add all the equations:

$$\begin{aligned}&\big|\!\big|\!\big|(\xi_{\mathrm{DG}}, \boldsymbol{\zeta}, \xi_{\mathrm{MFE}})\big|\!\big|\!\big|^2_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}}\\ &\quad= a_{\mathrm{DG}}\big(p - p^{\mathrm{I}}, \xi_{\mathrm{DG}}\big) + a_{\mathrm{MFE}}(\boldsymbol{u} - \Pi\boldsymbol{u}, \boldsymbol{\zeta})\\ &\qquad - b_{\mathrm{MFE}}(\boldsymbol{\zeta}, p - \mathcal{P}_h p) - \big\langle(\Pi\boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}_\Gamma, \xi_{\mathrm{DG}}\big\rangle_\Gamma\\ &\qquad + \big\langle P^{\mathrm{DG}} - p, \boldsymbol{\zeta}\cdot\boldsymbol{n}_\Gamma\big\rangle_\Gamma + b_{\mathrm{MFE}}(\boldsymbol{u} - \Pi\boldsymbol{u}, \xi_{\mathrm{MFE}}) + \big(\alpha(p - \mathcal{P}_h p), \xi_{\mathrm{MFE}}\big)_{\Omega_{\mathrm{MFE}}}\\ &\quad= R_1 + R_2 + \cdots + R_7.\end{aligned}$$

We note that the terms R_3 and R_6 are zero by the properties of $\mathcal{P}_h$ and Π operators. We will bound the terms R_1, R_2, and R_7 by following the approach in [15] and the standard MFE analysis. Finally, we will estimate the coupling terms R_4 and R_5. We have

$$R_1 = \sum_{E\in\mathcal{E}_h^{\mathrm{DG}}} \int_E K\nabla\big(p - p^{\mathrm{I}}\big)\nabla\xi_{\mathrm{DG}} + \alpha\big(p - p^{\mathrm{I}}\big)\xi_{\mathrm{DG}}$$
$$- \sum_{e_a\in\Gamma_{\mathrm{DG}}\cup\Gamma_{\mathrm{DG}}^{\mathrm{D}}} \int_{e_a} \big\{K\nabla\big(p - p^{\mathrm{I}}\big)\cdot \boldsymbol{n}_a\big\}[\xi_{\mathrm{DG}}] + \sum_{e_a\in\Gamma_{\mathrm{DG}}\cup\Gamma_{\mathrm{DG}}^{\mathrm{D}}} \int_{e_a} \{K\nabla\xi_{\mathrm{DG}}\cdot \boldsymbol{n}_a\}\big[p - p^{\mathrm{I}}\big].$$

We estimate the first two terms by using Cauchy–Schwarz and approximation result (5)

$$\left|\sum_E \int_E K\nabla\big(p - p^{\mathrm{I}}\big)\nabla\xi_{\mathrm{DG}}\right| \leqslant \big|\!\big|\!\big|K^{1/2}\nabla\big(p - p^{\mathrm{I}}\big)\big|\!\big|\!\big|_{0,\Omega_{\mathrm{DG}}} \big|\!\big|\!\big|K^{1/2}\nabla\xi_{\mathrm{DG}}\big|\!\big|\!\big|_{0,\Omega_{\mathrm{DG}}}$$
$$\leqslant \frac{1}{8}\big|\!\big|\!\big|K^{1/2}\nabla\xi_{\mathrm{DG}}\big|\!\big|\!\big|^2_{0,\Omega_{\mathrm{DG}}} + Ch^{2\mu-2}\|p\|^2_{s,\Omega_{\mathrm{DG}}},$$
$$\left|\sum_E \alpha\big(p - p^{\mathrm{I}}\big)\xi_{\mathrm{DG}}\right| \leqslant \big\|\alpha^{1/2}\big(p - p^{\mathrm{I}}\big)\big\|_{0,\Omega_{\mathrm{DG}}} \big\|\alpha^{1/2}\xi_{\mathrm{DG}}\big\|_{0,\Omega_{\mathrm{DG}}}$$
$$\leqslant \frac{1}{8}\big\|\alpha^{1/2}\xi_{\mathrm{DG}}\big\|^2_{0,\Omega_{\mathrm{DG}}} + Ch^{2\mu}\|p\|^2_{s,\Omega_{\mathrm{DG}}}.$$

We now consider the edge (or face) integral terms of R_1. We assume that $e_a = \partial E_a^1 \cap \partial E_a^2$ where E_a^1 and E_a^2 are two elements of $\mathcal{E}_h^{\mathrm{DG}}$. We note that e_a can be a subset of an edge (or face) of E_a^1 (see figure 2). Define $c_a = c_1 - c_2$, where $c_i = (1/|E_i|)\int_{E_i}\xi_{\mathrm{DG}}$, $i = 1, 2$. Then, we have

$$\big\|[\xi_{\mathrm{DG}}] - c_a\big\|_{0,e_a} \leqslant \big\|\xi_{\mathrm{DG}}|_{E_a^1} - c_1\big\|_{0,e_a} + \big\|\xi_{\mathrm{DG}}|_{E_a^2} - c_2\big\|_{0,e_a}$$
$$\leqslant Ch^{1/2}|\!|\!|\nabla\xi_{\mathrm{DG}}|\!|\!|_{0,E_a^1\cup E_a^2}. \tag{19}$$

Based on property (4) satisfied by $p^{\mathrm{I}}|_{E_a^1}$ and $p^{\mathrm{I}}|_{E_a^2}$, we obtain

$$\int_{e_a}\big\{K\nabla\big(p - p^{\mathrm{I}}\big)\cdot\boldsymbol{n}_a\big\}[\xi_{\mathrm{DG}}] = \frac{1}{2}\int_{e_a}\big(K\nabla\big(p - p^{\mathrm{I}}|_{E_a^1}\big)\cdot\boldsymbol{n}_a\big)\big([\xi_{\mathrm{DG}}] - c_a\big)$$
$$+ \frac{1}{2}\int_{e_a}\big(K\nabla\big(p - p^{\mathrm{I}}|_{E_a^2}\big)\cdot\boldsymbol{n}_a\big)\big([\xi_{\mathrm{DG}}] - c_a\big)$$
$$\leqslant \frac{1}{2}\big\|K\nabla\big(p - p^{\mathrm{I}}|_{E_a^1}\big)\cdot\boldsymbol{n}_a\big\|_{0,e_a}\|[\xi_{\mathrm{DG}}] - c_a\|_{0,e_a}$$
$$+ \frac{1}{2}\big\|K\nabla\big(p - p^{\mathrm{I}}|_{E_a^2}\big)\cdot\boldsymbol{n}_a\big\|_{0,e_a}\big\|[\xi_{\mathrm{DG}}] - c_a\big\|_{0,e_a}.$$

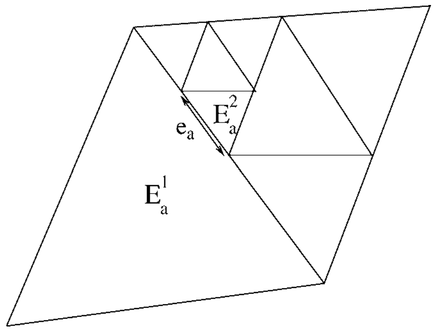

Figure 2. Example of nonconforming mesh.

Therefore by (19) and (5), we obtain

$$\left|\int_{e_a}\{K\nabla(p-p^{\mathrm{I}})\cdot\boldsymbol{n}_a\}[\xi_{\mathrm{DG}}]\right| \leqslant C|||\nabla\xi_{\mathrm{DG}}|||_{0,E_a^1\cup E_a^2}h^{\mu-3/2}h^{1/2}\|p\|_{s,E_a^1\cup E_a^2}.$$

If e_a is a Dirichlet face, i.e., $e_a \subset \partial E_a^1$, a similar bound is obtained with the constant $c_a = c_1$. Summing over all e_a yields:

$$\sum_{e_a\in\Gamma_{\mathrm{DG}}\cup\Gamma_{\mathrm{DG}}^{\mathrm{D}}}\int_{e_a}\{K\nabla(p-p^{\mathrm{I}})\cdot\boldsymbol{n}_a\}[\xi_{\mathrm{DG}}] \leqslant \frac{1}{8}\left|\left|\left|K^{1/2}\nabla\xi_{\mathrm{DG}}\right|\right|\right|^2_{0,\Omega_{\mathrm{DG}}} + Ch^{2\mu-2}\|p\|^2_{s,\Omega_{\mathrm{DG}}}.$$

Finally, by using the approximation result (5) and the inverse inequality (17), we deduce

$$\sum_{e_a\in\Gamma_{\mathrm{DG}}\cup\Gamma_{\mathrm{DG}}^{\mathrm{D}}}\int_{e_a}\{K\nabla\xi_{\mathrm{DG}}\cdot\boldsymbol{n}_a\}[p-p^{\mathrm{I}}] \leqslant \frac{1}{8}\left|\left|\left|K^{1/2}\nabla\xi_{\mathrm{DG}}\right|\right|\right|^2_{0,\Omega_{\mathrm{DG}}} + Ch^{2\mu-2}\|p\|^2_{s,\Omega_{\mathrm{DG}}}.$$

We then conclude that

$$|R_1| \leqslant \frac{1}{2}\left|\left|\left|K^{1/2}\nabla\xi_{\mathrm{DG}}\right|\right|\right|^2_{0,\Omega_{\mathrm{DG}}} + Ch^{2\mu-2}\|p\|^2_{s,\Omega_{\mathrm{DG}}}.$$

We bound R_2 and R_7 by using the Cauchy–Schwarz inequality and the approximation results (9) and (13).

$$\begin{aligned}
|R_2| &\leqslant C\|\boldsymbol{u}-\Pi\boldsymbol{u}\|_{0,\Omega_{\mathrm{MFE}}}\left\|K^{-1/2}\boldsymbol{\zeta}\right\|_{0,\Omega_{\mathrm{MFE}}}\\
&\leqslant \frac{1}{8}\left\|K^{-1/2}\boldsymbol{\zeta}\right\|^2_{0,\Omega_{\mathrm{MFE}}} + Ch^{2k+2}\|\boldsymbol{u}\|^2_{k+1,\Omega_{\mathrm{MFE}}},\\
|R_7| &\leqslant C\|p-\mathcal{P}_h p\|_{0,\Omega_{\mathrm{MFE}}}\left\|\alpha^{1/2}\xi_{\mathrm{MFE}}\right\|_{0,\Omega_{\mathrm{MFE}}}\\
&\leqslant \frac{1}{8}\left\|\alpha^{1/2}\xi_{\mathrm{MFE}}\right\|^2_{0,\Omega_{\mathrm{MFE}}} + Ch^{2k+2}\|p\|^2_{k+1,\Omega_{\mathrm{MFE}}}.
\end{aligned}$$

By adding and subtracting $\Pi\boldsymbol{u}$ and p^{I} in R_4 and R_5, we can write

$$R_4 + R_5 = -\left\langle(\Pi\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}_\Gamma, \xi_{\mathrm{DG}}\right\rangle_\Gamma + \langle p^{\mathrm{I}}-p, \boldsymbol{\zeta}\cdot\boldsymbol{n}_\Gamma\rangle_\Gamma.$$

Let us denote $\Gamma = \bigcup_a e_a^{\mathrm{MFE}}$, where each edge (or face) e_a^{MFE} belongs to an element of Ω_{MFE}. Let us assume that $e_a^{\mathrm{MFE}} \subset \partial E^{\mathrm{DG}}$ with $E^{\mathrm{DG}} \subset \Omega_{\mathrm{DG}}$ and let us define $c_a = (1/|E^{\mathrm{DG}}|)\int_{E^{\mathrm{DG}}} \xi_{\mathrm{DG}}$. Based on property (11) and the fact that the total number of faces e_a^{MFE} belonging to Γ is $\mathrm{O}(h^{-1})$, we have

$$\begin{aligned}
-\big\langle (\Pi \boldsymbol{u} - \boldsymbol{u})\cdot \boldsymbol{n}_\Gamma, \xi_{\mathrm{DG}}\big\rangle_\Gamma &= -\sum_a \int_{e_a^{\mathrm{MFE}}} (\Pi \boldsymbol{u} - \boldsymbol{u})\cdot \boldsymbol{n}_\Gamma (\xi_{\mathrm{DG}} - c_a)\\
&\leqslant \sum_a \|\xi_{\mathrm{DG}} - c_a\|_{0,e_a^{\mathrm{MFE}}} \big\|(\Pi \boldsymbol{u} - \boldsymbol{u})\cdot \boldsymbol{n}_\Gamma\big\|_{0,e_a^{\mathrm{MFE}}}\\
&\leqslant C\sum_a h^{1/2}\big\|K^{1/2}\nabla \xi_{\mathrm{DG}}\big\|_{0,E^{\mathrm{DG}}} h^{k+1}\|\boldsymbol{u}\|_{\infty,k+1,e_a^{\mathrm{MFE}}}\\
&\leqslant C h^{k+3/2}\big|\!\big|\!\big|K^{1/2}\nabla\xi_{\mathrm{DG}}\big|\!\big|\!\big|_{0,\Omega_{\mathrm{DG}}}\bigg(\sum_a \|\boldsymbol{u}\|^2_{\infty,k+1,e_a^{\mathrm{MFE}}}\bigg)^{1/2}\\
&\leqslant \frac{1}{8}\big|\!\big|\!\big|K^{1/2}\nabla\xi_{\mathrm{DG}}\big|\!\big|\!\big|^2_{0,\Omega_{\mathrm{DG}}} + C h^{2k+2}\|\boldsymbol{u}\|^2_{\infty,k+1,\Gamma}.
\end{aligned}$$

We now bound the last term by using the approximation result (5) and the inverse estimate (16)

$$\begin{aligned}
\big|\langle p^{\mathrm{I}} - p, \boldsymbol{\zeta}\cdot \boldsymbol{n}_\Gamma\rangle_\Gamma\big| &\leqslant \sum_{e_a\in\Gamma} \big\|p^{\mathrm{I}} - p\big\|_{0,e_a}\|\boldsymbol{\zeta}\cdot \boldsymbol{n}_a\|_{0,e_a}\\
&\leqslant \sum_{e_a\in\Gamma} h^{\mu-1/2}\|p\|_{s,E_k^{\mathrm{DG}}} h^{-1/2}\|\boldsymbol{\zeta}\|_{0,E_k^{\mathrm{MFE}}}\\
&\leqslant \frac{1}{8}\big\|K^{-1/2}\boldsymbol{\zeta}\big\|^2_{0,\Omega_{\mathrm{MFE}}} + C h^{2\mu-2}\|p\|^2_{s,\Omega_{\mathrm{DG}}}.
\end{aligned}$$

The theorem is obtained by combining all the previous bounds and by using triangle inequalities. □

A straightforward application of theorem 4.1 gives the following result.

Corollary 1. With the assumptions of theorem 4.1, and if $r = k+1$, then

$$\big|\!\big|\!\big|\big(p - P^{\mathrm{DG}}, \boldsymbol{u} - \boldsymbol{U}^{\mathrm{MFE}}, p - P^{\mathrm{MFE}}\big)\big|\!\big|\!\big|_{\Omega_{\mathrm{DG}},\Omega_{\mathrm{MFE}}} \leqslant C h^{k+1}.$$

5. Application to domain decomposition

In this section, we consider a hybridized form of the method (18.1)–(18.3). Two Lagrange multipliers are defined on the interface: a flux multiplier is introduced in the DG variational form and a pressure multiplier is introduced on the interblock boundaries for the MFE variational form. The meshes at the interface can be non-matching. Standard domain decomposition algorithms may be employed. Let Λ_h^{F} and Λ_h^{D} be finite dimensional subspaces of $L^2(\Gamma)$. We then solve for $P^{\mathrm{DG}} \in \mathcal{D}_r$, $\boldsymbol{U}^{\mathrm{MFE}} \in \boldsymbol{V}_h$, $P^{\mathrm{MFE}} \in W_h$, $\lambda^{\mathrm{F}} \in \Lambda_h^{\mathrm{F}}$, and $\lambda^{\mathrm{D}} \in \Lambda_h^{\mathrm{D}}$ satisfying

$$a_{\mathrm{DG}}\big(P^{\mathrm{DG}}, \phi\big) = (f,\phi)_{\Omega_{\mathrm{DG}}} + \big\langle \lambda^{\mathrm{F}}, \phi\big\rangle_\Gamma + \int_{\Gamma_{\mathrm{DG}}^D} K\nabla\phi\cdot\boldsymbol{n}_a p_0 - \int_{\Gamma_{\mathrm{DG}}^{\mathrm{N}}} g\phi, \quad \forall\phi\in\mathcal{D}_r, \tag{20.1}$$

$$a_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, \boldsymbol{v}\big) - b_{\mathrm{MFE}}\big(\boldsymbol{v}, P^{\mathrm{MFE}}\big) = -\langle \boldsymbol{v}\cdot\boldsymbol{n}, p_0\rangle_{\Gamma_{\mathrm{MFE}}^{\mathrm{D}}} + \big\langle \boldsymbol{v}\cdot\boldsymbol{n}_\Gamma, \lambda^{\mathrm{D}}\big\rangle_\Gamma, \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h, \tag{20.2}$$

$$b_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, w\big) + \big(\alpha P^{\mathrm{MFE}}, w\big)_{\Omega_{\mathrm{MFE}}} = (f,w)_{\Omega_{\mathrm{MFE}}}, \quad \forall w\in W_h, \tag{20.3}$$

$$\big\langle \boldsymbol{U}^{\mathrm{MFE}}\cdot\boldsymbol{n}_\Gamma + \lambda^{\mathrm{F}}, \mu^{\mathrm{F}}\big\rangle_\Gamma = 0, \quad \forall\mu^{\mathrm{F}}\in\Lambda_h^{\mathrm{F}}, \tag{20.4}$$

$$\big\langle \lambda^{\mathrm{D}} - P^{\mathrm{DG}}, \mu^{\mathrm{D}}\big\rangle_\Gamma = 0, \quad \forall\mu^{\mathrm{D}}\in\Lambda_h^{\mathrm{D}}. \tag{20.5}$$

Lemma 6. There exists a unique solution to the problem (20.1)–(20.5) if one of the two following assumptions hold

$$\begin{array}{lll} \text{(a)} & \lambda^{\mathrm{F}} = -\boldsymbol{U}^{\mathrm{MFE}}\cdot\boldsymbol{n}_\Gamma, & \lambda^{\mathrm{D}} = P^{\mathrm{DG}}, \\ \text{(b)} & Tr(\mathcal{D}_r) \subset \Lambda_h^{\mathrm{F}}, & Tr(\boldsymbol{V}_h\cdot\boldsymbol{n}_\Gamma) \subset \Lambda_h^{\mathrm{D}}, \end{array}$$

where Tr denotes the trace operator on Γ.

Proof. Let $f = p_0 = g = 0$ and choose $\phi = P^{\mathrm{DG}}$, $\boldsymbol{v} = \boldsymbol{U}^{\mathrm{MFE}}$, and $w = P^{\mathrm{MFE}}$. We add the equations (20.1)–(20.3) and we obtain

$$a_{\mathrm{DG}}\big(P^{\mathrm{DG}}, P^{\mathrm{DG}}\big) + a_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, \boldsymbol{U}^{\mathrm{MFE}}\big) + \big(\alpha P^{\mathrm{MFE}}, P^{\mathrm{MFE}}\big)_{\Omega_{\mathrm{MFE}}} = \int_\Gamma \lambda^{\mathrm{F}} P^{\mathrm{DG}} + \int_\Gamma \boldsymbol{U}^{\mathrm{MFE}}\cdot\boldsymbol{n}_\Gamma\lambda^{\mathrm{D}}.$$

Therefore, based on assumption (a) or (b), (20.4) and (20.5), we can rewrite the previous equation

$$\big|\!\big|\!\big|\big(P^{\mathrm{DG}}, \boldsymbol{U}^{\mathrm{MFE}}, P^{\mathrm{MFE}}\big)\big|\!\big|\!\big|^2 = 0.$$

This implies that $P^{\mathrm{DG}} = \boldsymbol{U}^{\mathrm{MFE}} = P^{\mathrm{MFE}} = 0$. Therefore, we obtain

$$\big\langle\lambda^{\mathrm{F}}, \phi\big\rangle_\Gamma = 0, \quad \forall\phi\in\mathcal{D}_r,$$
$$\big\langle\lambda^{\mathrm{D}}, \boldsymbol{v}\cdot\boldsymbol{n}_\Gamma\big\rangle_\Gamma = 0, \quad \forall\boldsymbol{v}\in\boldsymbol{V}_h.$$

We conclude that λ^{F} and λ^{D} are zero. □

Theorem 2. Let $(P_1^{\mathrm{DG}}, \boldsymbol{U}_1^{\mathrm{MFE}}, P_1^{\mathrm{MFE}})$ be solution to (18.1)–(18.3). Let $(P_2^{\mathrm{DG}}, \boldsymbol{U}_2^{\mathrm{MFE}}, P_2^{\mathrm{MFE}}, \lambda^{\mathrm{F}}, \lambda^{\mathrm{D}})$ be solution to (20.1)–(20.5). Then, if either (a) or (b) holds, we have

$$P_1^{\mathrm{DG}} = P_2^{\mathrm{DG}}, \qquad \boldsymbol{U}_1^{\mathrm{MFE}} = \boldsymbol{U}_2^{\mathrm{MFE}}, \qquad P_1^{\mathrm{MFE}} = P_2^{\mathrm{MFE}},$$
$$\lambda^{\mathrm{F}} = -\boldsymbol{U}_1^{\mathrm{MFE}}\cdot\boldsymbol{n}_\Gamma, \qquad \lambda^{\mathrm{D}} = P_1^{\mathrm{DG}}.$$

Proof. By substracting the equations of the non-mortar formulation from the associated equations of the mortar formulation, we can write for any $(\phi, \boldsymbol{v}, w) \in \mathcal{D}_r \times \boldsymbol{V}_h \times W_h$:

$$
\begin{aligned}
&a_{\mathrm{DG}}\big(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}, \phi\big) = -\big\langle \boldsymbol{U}_1^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma, \phi\big\rangle_\Gamma - \big\langle \lambda^{\mathrm{F}}, \phi\big\rangle_\Gamma,\\
&a_{\mathrm{MFE}}\big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}, \boldsymbol{v}\big) - b_{\mathrm{MFE}}\big(\boldsymbol{v}, P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}\big) = \big\langle \boldsymbol{v} \cdot \boldsymbol{n}_\Gamma, P_1^{\mathrm{DG}} - \lambda^{\mathrm{D}}\big\rangle_\Gamma,\\
&b_{\mathrm{MFE}}\big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}, w\big) + \big(\alpha\big(P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}\big), w\big)_{\Omega_{\mathrm{MFE}}} = 0.
\end{aligned}
$$

We set $\phi = P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}$, $\boldsymbol{v} = \boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}$, and $w = P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}$, and we add the equations:

$$
\begin{aligned}
&\big|\!\big|\!\big|\big(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}, \boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}, P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}\big)\big|\!\big|\!\big|^2\\
&\quad = -\big\langle \boldsymbol{U}_1^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma, P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}\big\rangle_\Gamma - \big\langle \lambda^{\mathrm{F}}, P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}\big\rangle_\Gamma\\
&\qquad + \big\langle\big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}\big) \cdot \boldsymbol{n}_\Gamma, P_1^{\mathrm{DG}} - \lambda^{\mathrm{D}}\big\rangle_\Gamma.
\end{aligned}
\tag{21}
$$

It is easily shown that if assumption (a) holds, then the right-hand side of (21) is equal to zero. We now assume that assumption (b) holds. Since $(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}})|_\Gamma \in \Lambda^{\mathrm{F}}$, we can rewrite

$$
\begin{aligned}
&\big|\!\big|\!\big|\big(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}, \boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}, P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}\big)\big|\!\big|\!\big|^2\\
&\quad = -\int_\Gamma \big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}\big) \cdot \boldsymbol{n}_\Gamma\big(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}\big) + \int_\Gamma \big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}\big) \cdot \boldsymbol{n}_\Gamma\big(P_1^{\mathrm{DG}} - \lambda^{\mathrm{D}}\big)\\
&\quad = \int_\Gamma \big(\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}\big) \cdot \boldsymbol{n}_\Gamma\big(P_2^{\mathrm{DG}} - \lambda^{\mathrm{D}}\big).
\end{aligned}
$$

Since $((\boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}) \cdot \boldsymbol{n}_\Gamma)|_\Gamma \in \Lambda^{\mathrm{D}}$, we conclude

$$
\big|\!\big|\!\big|\big(P_1^{\mathrm{DG}} - P_2^{\mathrm{DG}}, \boldsymbol{U}_1^{\mathrm{MFE}} - \boldsymbol{U}_2^{\mathrm{MFE}}, P_1^{\mathrm{MFE}} - P_2^{\mathrm{MFE}}\big)\big|\!\big|\!\big|^2 = 0.
$$

Therefore, we are left with

$$
\begin{aligned}
\big\langle \lambda^{\mathrm{F}} + \boldsymbol{U}_1^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma, \phi\big\rangle_\Gamma &= 0, \quad \forall \phi \in \mathcal{D}_r,\\
\big\langle \lambda^{\mathrm{D}} - P_1^{\mathrm{DG}}, \boldsymbol{v} \cdot \boldsymbol{n}_\Gamma\big\rangle_\Gamma &= 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h,
\end{aligned}
$$

which finishes the proof. □

As noted above, the scheme (18.1)–(18.3) is a particular case of the formulation (20.1)–(20.5).

6. Penalty formulations

A variation of the DG method discussed in this paper, is the Non-symmetric Interior Penalty Galerkin (NIPG) method. The analysis of NIPG for elliptic problems can be found in [14,15]. The bilinear form of this method differs from the bilinear form of DG by the addition of penalty terms. The local mass error is known and can be taken into account to obtain a locally mass conservative velocity field [13]. The penalty terms have the following form

$$J_0^\beta(\phi, \psi) = \sum_{e_a \in \Gamma_{\mathrm{NIPG}}} \frac{\sigma_k}{|e_a|^\beta} \int_{e_a} [\phi][\psi],$$

where $|\cdot|$ denotes the measure. Unlike the DG method, the NIPG method is stable for linears, and the penalty parameter σ_k needs only to be positive. The coupling of NIPG with MFE is defined as follows:

$$\begin{aligned}
&a_{\mathrm{DG}}\big(P^{\mathrm{NIPG}}, \phi\big) + J_0^\beta\big(P^{\mathrm{NIPG}}, \phi\big) \\
&\quad = (f, \phi)_{\Omega_{\mathrm{DG}}} - \big\langle \boldsymbol{U}^{\mathrm{MFE}} \cdot \boldsymbol{n}_\Gamma, \phi\big\rangle_\Gamma + \int_{\Gamma_{\mathrm{DG}}^{\mathrm{D}}} K\nabla\phi \cdot \boldsymbol{n}_a p_0 - \int_{\Gamma_{\mathrm{DG}}^{\mathrm{N}}} g\phi, \quad \forall \phi \in \mathcal{D}_r, \\
&a_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, \boldsymbol{v}\big) - b_{\mathrm{MFE}}\big(\boldsymbol{v}, P^{\mathrm{MFE}}\big) \\
&\quad = -\langle \boldsymbol{v} \cdot \boldsymbol{n}, p_0\rangle_{\Gamma_{\mathrm{MFE}}^{\mathrm{D}}} + \big\langle \boldsymbol{v} \cdot \boldsymbol{n}_\Gamma, P^{\mathrm{NIPG}}\big\rangle_\Gamma, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \\
&b_{\mathrm{MFE}}\big(\boldsymbol{U}^{\mathrm{MFE}}, w\big) + \big(\alpha P^{\mathrm{MFE}}, w\big)_{\Omega_{\mathrm{MFE}}} = (f, w)_{\Omega_{\mathrm{MFE}}}, \quad \forall w \in W_h.
\end{aligned}$$

The results proved in section 4 hold true.

The flexibility of discontinuous finite element methods allows for the use of different degrees of approximation in different elements. The user can specify the degree and can also choose the regions where the DG, NIPG, or MFE methods would be used.

7. Conclusions

In this paper, we present a new class of locally conservative multinumeric approaches based on discontinuous and mixed finite element methods for nonconforming andunstructured grids. We show stability and convergence of the methods and introduce a formulation with Lagrange multipliers for an efficient parallel implementation.

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