Part I. Improved Energy Estimates for Interior Penalty, Constrained and Discontinuous Galerkin Methods for Elliptic Problems

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Three Galerkin methods using discontinuous approximation spaces are introduced to solve elliptic problems. The underlying bilinear form for all three methods is the same and is nonsymmetric. In one case, a penalty is added to the form and in another, a constraint on jumps on each face of the triangulation. All three methods are locally conservative and the third one is not restricted. Optimal *a priori hp* error estimates are derived for all three procedures.
Keywords: discontinuous spaces, elliptic equations, error estimates, constrained spaces

1. Introduction

Over the last two decades there has been a collection of papers devoted to the use of approximation spaces with weak continuity for finite element approximations to elliptic and parabolic problems. The motivation for developing these methods was the flexibility afforded by local approximation spaces. These approaches allow meshes which are more general in their construction and degree of nonuniformity both in time and space than is permitted by the more conventional finite element methods. In general, numerical methods defined for discontinuous spaces have less numerical diffusion/dispersion and provide more accurate local approximations for problems with rough solutions. Another advantage that has recently become apparent is the application of domain decomposition algorithms for the discrete solution.

In this paper, we discuss three numerical algorithms for elliptic problems which employ discontinuous approximation spaces. The three methods are called the nonsymmetric interior penalty Galerkin method (NIPG), the nonsymmetric constrained Galerkin (NCG) method, and the discontinuous Galerkin (DG) method. The three algorithms are closely related in that the underlying bilinear form for all three is the same and is nonsymmetric. Moreover, for all three methods, one can employ an unusual space $\mathbb{P}_k$ on quadrilaterals which have dimension substantially lower than $\mathbb{Q}_k$. $\mathbb{P}_k$ is the set of polynomials in two variables of total degree k, and $\mathbb{Q}_k$ is the set of polynomials of degree k in *each* variable. In addition, in all three methods, the error for the mass conservation can be retrieved element by element. In that sense, all three methods are locally mass conservative. The main advantage of the DG method is that the error in the mass conservation is zero.

In the NIPG method the bilinear form of the interior penalty Galerkin method treated by Douglas and Dupont [5], Wheeler [7], Arnold [8], Darlow and Wheeler [9] is modified. In this paper, an optimal *hp* error estimate is obtained in H^1 and in L^2. In particular the NIPG method only requires a positive penalty rather than one bounded below by a problem-dependent constant as in the proofs described in [7].

The second approach is based on constraining the approximation spaces: jumps on each edge of the triangulation are required to have integral average zero. Here optimal *hp* estimates in H^1 and L^2 are derived.

The DG method with this bilinear form was first introduced by Baumann and Oden in [3],[11]. In [3], Baumann showed that the method is elementwise conservative and he proved a stability result in one dimension for polynomials of at least degree three. Numerical experiments showed that the method is robust and gives high-order accuracy where the solution is smooth. In this paper, theoretical optimal results are obtained for the DG method in H^1 for $n = 2$ and suboptimal for $n = 3$. To our knowledge these results represent the first convergence results for DG in higher dimensions.

This paper consists of four sections after this introduction. In §2, we list the notation, state the problem and describe the formulation of the three methods. In §3, §4 and §5, the proofs of the error estimates of the three methods described in §2 are respectively given. In the last section, we present some conclusions.

Part II of this paper describes computational results with the DG method.

2. Statement of the Problem. Finite Element Methods

2.1. Notation and Approximation Properties

Let Ω be a polygonal domain in $I\!R^n$. Everything here applies also to $n = 1$, but to simplify we consider only $n = 2$ or 3. Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω, where E_j is a triangle or a quadrilateral if $n = 2$, or a tetrahedron if $n = 3$. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \mathrm{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. However, for quadrilaterals, the requirement is a little bit stronger: the quadrilateral is convex and each of its subtriangle contains a ball of radius ρh_j. Let $h = \max\{h_j, \quad j = 1 \ldots N_h\}$. The quasiuniformity requirement is that there is $\tau > 0$ such that $\frac{h}{h_j} \leq \tau$ for all $j \in 1, \ldots, N_h$. This quasiuniformity assumption is used for deriving error estimates in terms of the degree of polynomials (i.e. for the p-versoin). For deriving error estimates in terms of h (i.e. for the h-version), we only need a regular subdivision. We denote the edges (resp. faces for $n = 3$) of $\mathcal{E}_h$ by $\left\{e_1, e_2, \ldots, e_{P_h}, e_{P_{h+1}}, \ldots, e_{M_h}\right\}$ where $e_k \subset \Omega, 1 \leq k \leq P_h$, and $e_k \subset \partial\Omega,\ P_h + 1 \leq k \leq M_h$. With each edge (or face) e_k , we associate a unit normal vector ν_k. For $k > P_h$, ν_k is taken to be the unit outward vector normal to $\partial\Omega$. For $s \geq 0$, let

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1 \ldots N_h\}.$$

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h),\ s > \frac{1}{2}$. Let $1 \leq k \leq P_h$; for $e_k = \partial E_i \cap \partial E_j$ with ν_k exterior to E_i, set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

We consider a matrix-valued function $K = (k_{ij})_{1 \leq i,j \leq n}$, $K \in L^\infty(\Omega)$ and a non-negative scalar function $\alpha \in L^\infty(\Omega)$. We assume that K is symmetric, positive definite in $\bar{\Omega}$ uniformly with respect to x. This means that if $\gamma_{\mathrm{min}} \leq \gamma_{\mathrm{max}}$ are the smallest and the largest eigenvalues of K, then there exist $\gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$\forall x \in \bar{\Omega}, \quad \gamma_{\mathrm{min}}(x) \geq \gamma_0, \quad \gamma_{\mathrm{max}}(x) \leq \gamma_1. \tag{2.1}$$

The L^2 inner product is denoted by $(.,.)$. The usual Sobolev norm of H^m on $E \subset I\!R^n$ is denoted by $\|\cdot\|_{m,E}$. We define the following broken norms for m positive integer:

$$|||\Phi|||_m = \left(\sum_{j=1}^{N_h} \|\Phi\|_{m,E_j}^2 \right)^{\frac{1}{2}}.$$

Let r be a positive integer. The finite element subspace is taken to be

$$\mathcal{D}_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j),$$

where $P_r(E_j)$ denotes the set of polynomials of (total) degree less than or equal to r on E_j.

We use the following *hp* approximation properties, proven in [1] and [2]. Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exists a constant C depending on s, τ, ρ but independent of ϕ, r and h and a sequence $z_r^h \in I\!P_r(E_j),\ r = 1, 2, \ldots$ such that for any $0 \leq q \leq s$

$$\|\phi - z_r^h\|_{q,E_j} \leq C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{s,E_j}, \qquad s \geq 0,$$

$$\|\phi - z_r^h\|_{0,\gamma_i} \leq C \frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{1}{2},$$

where $\mu = \min(r+1, s)$ and $\gamma_i \subset \partial E_j$. Using the same technique as in [1], it can be shown that we have the additional approximation result:

$$\|\phi - z_r^h\|_{1,\gamma_i} \leq C \frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{3}{2}.$$

As a corollary of the above results, we obtain the following global approximation property. Let $\phi \in H^s(\Omega)$ and let $\mathcal{E}_h$ be the above subdivision of Ω, made of triangles or tetrahedra. There exists $z_r^h \in \mathcal{D}_r(\mathcal{E}_h) \cap C^0(\bar{\Omega})$ such that for any $0 \leq q \leq s$,

$$|||\phi - z_r^h|||_{q,\Omega} \leq C \frac{h^{\mu-q}}{r^{s-q}} \|\phi\|_{s,\Omega}, \tag{2.2}$$

where $\mu = \min(r+1, s)$ and C is independent of ϕ, r, h and $\mathcal{E}_h$. Note that this result also holds if $z_r^h \in \mathcal{D}_r(\mathcal{E}_h)$.

2.2. Problem and Nonsymmetric Bilinear Form

Let the boundary of the domain $\partial\Omega$ be the union of two disjoint sets Γ_D and Γ_N. We denote ν_D (respectively ν_N) the unit normal vector to each edge of Γ_D (respectively Γ_N) exterior to Ω. For f given in $L^2(\Omega)$, p_0 given in $H^{\frac{1}{2}}(\Gamma_D)$ and g given in $L^2(\Gamma_N)$, we consider the following elliptic problem:

$$-\nabla \cdot (K\nabla p) + \alpha p = f \quad \text{in} \quad \Omega, \tag{2.3a}$$

$$p = p_0 \quad \text{on} \quad \Gamma_D, \tag{2.3b}$$

$$K\nabla p \cdot \nu_N = g \quad \text{on} \quad \Gamma_N. \tag{2.3c}$$

With the above assumptions on K and α, problem (2.3) has a unique solution in $H^1(\Omega)$ when $|\Gamma_D| > 0$ or when $\alpha \neq 0$. On the other hand, when $\partial\Omega = \Gamma_N$ and $\alpha = 0$, problem (2.3) has a solution in $H^1(\Omega)$ which is unique up to an additive constant, provided $\int_\Omega f = -\int_{\partial\Omega} g$.

For K in $W^{1,4}(\mathcal{E}_h)$ and $\psi, \phi \in H^2(\mathcal{E}_h)$, we consider the non-symmetric bilinear form:

$$\begin{aligned} a_{NS}(\psi, \phi) = & \sum_{j=1}^{N_h} \int_{E_j} (K\nabla\psi\nabla\phi + \alpha\psi\phi) \\ & - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\psi \cdot \nu_k\} [\phi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \nu_k\} [\psi] \\ & - \int_{\Gamma_D} (K\nabla\psi \cdot \nu_D)\phi + \int_{\Gamma_D} (K\nabla\phi \cdot \nu_D)\psi. \end{aligned}$$

We define the linear form:

$$L(\phi) = (f, \phi) + \int_{\Gamma_D} (K\nabla\phi \cdot \nu_D)p_0 + \int_{\Gamma_N} \phi g$$

2.3. Finite Element Schemes

First, we introduce the following interior and boundary penalty term:

$$J_0^{\sigma,\beta}(\phi, \psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [\phi][\psi] + \sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} \phi\psi,$$

where σ is a discrete positive function that takes the constant value σ_k on the edge or face e_k, $|e_k|$ denotes the measure of e_k and $\beta \geq \frac{1}{2}$ is a real number. The Galerkin approximation $P^{NIPG} \in \mathcal{D}_r(\mathcal{E}_h)$ solves the following discrete problem:

$$a_{NS}(P^{NIPG}, v) + J_0^{\sigma,\beta}(P^{NIPG}, v) = L(v) + \sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} p_0 v, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

Lemma 2.1. Under the above assumptions on the data (including K), and if the solution p of (2.3) satisfies $p \in H^2(\mathcal{E}_h)$, then p satisfies

$$a_{NS}(p, v) + J_0^{\sigma,\beta}(p, v) = L(v) + \sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} p_0 v, \quad \forall v \in H^2(\mathcal{E}_h). \tag{2.4}$$

Conversely, if $p \in H^1(\Omega) \cap H^2(\mathcal{E}_h)$ satisfies (2.4), then p is the solution of (2.3).

Proof. First, suppose that the solution p of (2.3) belongs to $H^2(\mathcal{E}_h)$. Let v be an element in $H^2(\mathcal{E}_h)$. We multiply the first equation by v, integrate on E_j and sum over all j.

$$\sum_{j=1}^{N_h} \int_{E_j} K\nabla p \nabla v + \alpha p v - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla p \cdot \nu_k\}[v] - \int_{\partial\Omega} (K\nabla p \cdot \nu) v = (f, v).$$

Using the boundary conditions, we get:

$$\sum_{j=1}^{N_h} \int_{E_j} K\nabla p \nabla v + \alpha p v - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla p \cdot \nu_k\} [v] - \int_{\Gamma_D} (K\nabla p \cdot \nu_D) v = (f, v) + \int_{\Gamma_N} g v.$$

We add $\int_{\Gamma_D} (K\nabla v \cdot \nu_D) p$ and $\sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} p_0 v$ to both sides and we note that $[p] = 0$. We clearly have (2.4). Conversely, take $v \in \mathcal{D}(E_j)$; this gives $-\nabla \cdot K\nabla p + \alpha p = f$ in E_j, for any j. Next, let e_k be an interior edge or face and E_i and E_j the two elements adjacent to e_k. Take $v \in H_0^2(E_i \cup E_j)$, extended by zero outside, multiply the above equation by v, and use Green's formula:

$$\begin{aligned} \int_{E_i \cup E_j} K\nabla p \nabla v + \int_{E_i \cup E_j} \alpha p v \quad - \int_{e_k} [K\nabla p \cdot \nu_k] v &= \int_{E_i \cup E_j} f v \\ &= \int_{E_i \cup E_j} K\nabla p \nabla v + \int_{E_i \cup E_j} \alpha p v, \end{aligned}$$

since $[p] = 0$. Hence $\int_{e_k} [K\nabla p \cdot \nu_k] v = 0, \quad \forall v \in H_0^2(E_i \cup E_j)$ and therefore $[K\nabla p \cdot \nu_k]|_{e_k} = 0$. Since this holds for all e_k, it implies that $K\nabla p \in H(div; \Omega)$ and hence we have globally

$$-\nabla \cdot K\nabla p + \alpha p = f \quad \text{in} \quad \Omega.$$

To recover the boundary conditions, we multiply this last equation by $v \in H^2(\Omega)$, $\quad v|_{\Gamma_N} = 0$, apply Green's formula and compare with (2.4):

$$-\int_{\Gamma_D} (K\nabla p \cdot \nu_D)v = -\int_{\Gamma_D} (K\nabla p \cdot \nu_D)v + \int_{\Gamma_D} (K\nabla v \cdot \nu_D)p \\ - \int_{\Gamma_D} (K\nabla v \cdot \nu_D)p_0 - \sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} v(p_0 - p)$$

Thus, $p = p_0$ on Γ_D. Finally, choosing $v \in H^2(\Omega)$, $\quad v|_{\Gamma_D} = 0$, we find

$$-\int_{\Gamma_N} (K\nabla p \cdot \nu_N)v = -\int_{\Gamma_N} gv,$$

and this gives the other boundary condition. □

We note that on each element, the mass conservation for the NIPG method can be written as

$$\int_{E_j} \alpha P^{NIPG} - \int_{\partial E_j \setminus \Gamma_N} \{K\nabla P^{NIPG} \cdot \nu_{\partial E_j}\} \\ + \sum_{e_k \in \partial E_j \setminus \partial\Omega} \frac{\sigma_k}{|e_k|^\beta} \int_{e_k} [P^{NIPG}][1] = \int_{E_j} f + \int_{\partial E_j \cap \Gamma_N} g.$$

The constrained discrete space is defined as follows:

$$\mathcal{D}_r^*(\mathcal{E}_h) = \left\{ v \in \prod_{j=1}^{N_h} P_r(E_j) : \int_{e_k} [v] = 0 \quad \forall k = 1, \ldots, P_h \right\}.$$

The discrete approximation $P^{NCG} \in \mathcal{D}_r^*(\mathcal{E}_h)$ satisfies:

$$a_{NS}(P^{NCG}, v) = L(v), \qquad \forall v \in \mathcal{D}_r^*(\mathcal{E}_h). \tag{2.5}$$

The consistency of this scheme is a consequence of Lemma 2.1.

The Discontinuous Galerkin approximation $P^{DG} \in \mathcal{D}_r(\mathcal{E}_h)$ satisfies the formulation

$$a_{NS}(P^{DG}, v) = L(v), \qquad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{2.6}$$

The fact that this scheme is consistent with the problem (2.3) has been shown by Baumann [3].

Clearly the discrete solution of each of the three methods exists and is unique. Indeed, since it is a discrete problem, it suffices to show uniqueness of the solution. For instance for the NCG method, choose $f = 0$ and $v = P^{NCG}$. Thus, $\|K^{\frac{1}{2}}\nabla P^{NCG}\|_0 + \|\alpha^{\frac{1}{2}} P^{NCG}\|_0 = 0$. This easily implies that $P^{NCG} = 0$.

3. A priori error estimates for NIPG method

In this section, we derive *a priori* energy error estimates for the problem with mixed boundary conditions. In the case of the Neumann problem ($\Gamma_N = \partial\Omega$), the same result holds for any $\beta \geq 1$ in $2D$ and $\beta \geq \frac{1}{2}$ in $3D$. L^2 error estimates are also proved in the case of pure Neumann boundary conditions.

Theorem 3.1. Under the assumptions of Lemma 2.1 and if $p \in H^s(\mathcal{E}_h)$ and $\beta = 1$ in $2D$ or $\beta = \frac{1}{2}$ in $3D$, we have: if $\alpha \equiv 0$, then

$$\|K^{\frac{1}{2}}\nabla(P^{NIPG} - p)\|_0 \leq C(\frac{1}{\sigma}, K)\frac{h^{\mu-1}}{r^{s-3}}\|p\|_s.$$

If $\alpha \geq \alpha_0 > 0$, then

$$\|P^{NIPG} - p\|_1 \leq C(\frac{1}{\sigma}, K, \|\alpha\|_\infty)\frac{h^{\mu-1}}{r^{s-3}}\|p\|_s,$$

where $\mu = \min(r+1, s)$, $r \geq 1$ and $s \geq 2$. Besides, we also have

$$J_0^{\sigma,\beta}(P^{NIPG} - p, P^{NIPG} - p) \leq C\frac{h^{2\mu-2}}{r^{2s-6}}\|p\|_s^2.$$

Proof. In all the proofs, C will be a generic constant with different values on different places, that is independent of h and r. From Lemma 2.1, we have

$$a_{NS}(p, v) + J_0^{\sigma,\beta}(p, v) = L(v) + \sum_{e_k \in \Gamma_D} \frac{\sigma_k}{|e_k|^\beta}\int_{e_k} p_0 v, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

Let $\tilde{p}$ be the interpolant of p having optimal hp-approximation errors and denote $\chi = P^{NIPG} - \tilde{p}$. We have

$$\begin{aligned}
a_{NS}(\chi,\chi) + J_0^{\sigma,\beta}(\chi,\chi) &= a_{NS}(p-\tilde{p},\chi) + J_0^{\sigma,\beta}(p-\tilde{p},\chi),\\
&= \sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p})\nabla\chi + \alpha(p-\tilde{p})\chi)\\
&\quad - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]\\
&\quad - \int_{\Gamma_D}(K\nabla(p-\tilde{p})\cdot\nu_D)\chi + \int_{\Gamma_D}(K\nabla\chi\cdot\nu_D)(p-\tilde{p})\\
&\quad + \sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|^\beta}\int_{e_k}[p-\tilde{p}][\chi] + \sum_{e_k\in\Gamma_D}\frac{\sigma_k}{|e_k|^\beta}\int_{e_k}(p-\tilde{p})\chi. \qquad (3.1)
\end{aligned}$$

The first two terms in (3.1) can be bounded in the following way:

$$|\sum_{j=1}^{N_h}\int_{E_j} K\nabla(p-\tilde{p})\cdot\nabla\chi| \leq C|\!|\!|\nabla(p-\tilde{p})|\!|\!|_0 |\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0,$$
$$\leq \frac{1}{6}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + C|\!|\!|\nabla(p-\tilde{p})|\!|\!|_0^2.$$
$$|\sum_{j=1}^{N_h}\int_{E_j} \alpha(p-\tilde{p})\chi| \leq C\|\alpha\|_\infty\|p-\tilde{p}\|_0\|\chi\|_0,$$
$$\leq \frac{1}{2}\|\chi\|_0^2 + C\|\alpha\|_\infty^2\|p-\tilde{p}\|_0^2.$$

The third term is bounded by

$$|\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi]| \leq \left(\frac{|e_k|^\beta}{\sigma_k}\right)^{\frac{1}{2}}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}\left(\frac{\sigma_k}{|e_k|^\beta}\right)^{\frac{1}{2}}\|[\chi]\|_{0,e_k},$$
$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi]| \leq \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi) + C\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\|_{0,e_k}^2.$$

Using a trace theorem and an approximation result, we get

$$\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\|_{0,e_k}^2 \leq C(\frac{1}{\sigma})\frac{h^{2\mu-3}|e_k|^\beta}{r^{2s-4}}|\!|\!|p|\!|\!|_s^2.$$

In the case of triangles or tetrahedra, we can choose a continuous $\tilde{p}$ and for the pure Neumann problem, there are no other terms, thus we can conclude. To bound the fourth term in (3.1), we consider the contribution from each interior edge. We assume that $e_k = \partial E^1 \cap \partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$ and denote $E_{12} = E^1 \cup E^2$.

$$|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]| \leq \|\{K\nabla\chi\cdot\nu_k\}\|_{0,e_k}\|[p-\tilde{p}]\|_{0,e_k},$$
$$\leq C\frac{r^2}{h}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_{0,E_{12}}(\|p-\tilde{p}\|_{0,E_{12}} + h|\!|\!|\nabla(p-\tilde{p})|\!|\!|_{0,E_{12}}),$$
$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]| \leq \frac{1}{6}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + C\frac{h^{2\mu-2}}{r^{2s-6}}|\!|\!|p|\!|\!|_s^2.$$

Let $e_k \in \Gamma_D$. We take care of the boundary terms by the use of the trace theorem:

$$|\int_{\Gamma_D}(K\nabla(p-\tilde{p})\cdot\nu_D)\chi| \leq \sum_{e_k\in\Gamma_D}\left(\frac{|e_k|^\beta}{\sigma_k}\right)^{\frac{1}{2}}\|K\nabla(p-\tilde{p})\|_{0,e_k}\|\chi\|_{0,e_k}\left(\frac{\sigma_k}{|e_k|^\beta}\right)^{\frac{1}{2}},$$

$$|\int_{\Gamma_D}(K\nabla(p-\tilde{p})\cdot\nu_D)\chi| \leq \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi) + C\sum_{e_k\in\Gamma_D}\frac{|e_k|^\beta}{\sigma_k}\|K\nabla(p-\tilde{p})\cdot\nu_k\|_{0,e_k}^2,$$
$$\leq \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi) + C\frac{h^{2\mu-3}|e_k|^\beta}{r^{2s-4}}|\!|\!|p|\!|\!|_s^2.$$

$$|\int_{\Gamma_D}(K\nabla\chi\cdot\nu_D)(p-\tilde{p})| \leq \sum_{e_k\in\Gamma_D}\|K\nabla\chi\cdot\nu_k\|_{0,e_k}\|p-\tilde{p}\|_{0,e_k},$$
$$\leq \frac{1}{6}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + C\frac{h^{2\mu-2}}{r^{2s-6}}|\!|\!|p|\!|\!|_s^2.$$

The terms involving the penalty term in (3.1) are bounded by

$$|\sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|^\beta}\int_{e_k}[p-\tilde{p}][\chi]| \leq C\sum_{k=1}^{P_h}\frac{\sigma_k}{|e_k|^\beta}\|[p-\tilde{p}]\|_{0,e_k}^2 + \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi),$$
$$\leq C\frac{h^{2\mu-1}|e_k|^{-\beta}}{r^{2s-2}}|\!|\!|p|\!|\!|_s^2 + \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi).$$

$$|\sum_{e_k\in\Gamma_D}\frac{\sigma_k}{|e_k|^\beta}\int_{e_k}(p-\tilde{p})\chi| \leq C\sum_{e_k\in\Gamma_D}\frac{\sigma_k}{|e_k|^\beta}\|p-\tilde{p}\|_{0,e_k}^2 + \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi),$$
$$\leq C\frac{h^{2\mu-1}|e_k|^{-\beta}}{r^{2s-2}}|\!|\!|p|\!|\!|_s + \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi).$$

By combining the bounds together, we get

$$|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + \|\alpha^{\frac{1}{2}}\chi\|_0^2 + J_0^{\sigma,\beta}(\chi,\chi) \leq C\frac{h^{2\mu-2}}{r^{2s-6}}|\!|\!|p|\!|\!|_s^2 + C\|\alpha\|_\infty^2\frac{h^{2\mu}}{r^{2s}}|\!|\!|p|\!|\!|_s^2,$$
$$+C\left(\frac{h^{2\mu-3}|e_k|^\beta}{r^{2s-4}} + \frac{h^{2\mu-1}|e_k|^{-\beta}}{r^{2s-2}}\right)|\!|\!|p|\!|\!|_s^2.$$

Thus, if $\beta = 1$ in $2D$ or $\beta = \frac{1}{2}$ in $3D$, we obtained optimal convergence rates. In the case of the pure Neumann problem, it is easy to show that the result still holds true for any $\beta \geq 1$ in $2D$ and any $\beta \geq \frac{1}{2}$ in $3D$.

□

In the following theorem, we assume that $\Gamma_N = \partial\Omega$ and that the subdivision of Ω consists of triangles or tetrahedra.

Theorem 3.2. Assume that Ω is convex and K sufficiently smooth so that for any $f \in L^2(\Omega)$, the solution ϕ of the dual problem

$$-\nabla\cdot K\nabla\phi + \alpha\phi = f, \quad \text{in} \quad \Omega,$$
$$K\nabla\phi\cdot\nu = 0, \quad \text{on} \quad \partial\Omega,$$

belongs to $H^2(\Omega)$, with continuous dependence on f. Then,

$$\|P^{NIPG} - p\|_{0,\Omega} \leq C\frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}(n-1)}}{r^{s-3}}|||p|||_s,$$

for $r \geq 1, s \geq 2$ and C independent of h, r, p. In particular, optimal L^2 rates of convergence are obtained if $\beta \geq 3$ for $n = 2$ and if $\beta \geq \frac{3}{2}$ for $n = 3$.

Proof. Consider the dual problem

$$-\nabla \cdot K\nabla\phi + \alpha\phi = P^{NIPG} - p, \quad \text{in} \quad \Omega,$$
$$K\nabla\phi \cdot \nu = 0, \quad \text{on} \quad \partial\Omega.$$

By assumption $\phi \in H^2(\Omega)$ and there is a constant C that depends on Ω such that

$$\|\phi\|_{2,\Omega} \leq C\|P^{NIPG} - p\|_{0,\Omega}.$$

Denote $\chi = P^{NIPG} - p$. Then

$$\|\chi\|_{0,\Omega}^2 = (-\nabla \cdot K\nabla\phi + \alpha\phi, \chi).$$

Integrating by parts on each element yields:

$$\begin{aligned}\|\chi\|_{0,\Omega}^2 &= \sum_{j=1}^{N_h}\int_{E_j} K\nabla\phi\nabla\chi + \alpha\phi\chi - \sum_{j=1}^{N_h}\int_{\partial E_j}(K\nabla\phi \cdot \nu)\chi,\\ &= \sum_{j=1}^{N_h}\int_{E_j} K\nabla\phi\nabla\chi + \alpha\phi\chi - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi \cdot \nu_k\}[\chi] - \sum_{k=1}^{P_h}\int_{e_k}[K\nabla\phi \cdot \nu_k]\{\chi\}.\end{aligned}$$

By subtracting the orthogonality equation for any $\phi^* \in \mathcal{D}_r(\mathcal{E}_h)$:

$$a_{NS}(\chi, \phi^*) + J_0^{\sigma,\beta}(\chi, \phi^*) = 0,$$

and using the symmetry of K, we obtain:

$$\begin{aligned}\|\chi\|_{0,\Omega}^2 &= \sum_{j=1}^{N_h}\int_{E_j} K\nabla(\phi - \phi^*)\nabla\chi + \alpha(\phi - \phi^*)\chi\\ &\quad - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi \cdot \nu_k\}[\chi] - \sum_{k=1}^{P_h}\int_{e_k}[K\nabla\phi \cdot \nu_k]\{\chi\}\\ &\quad + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi \cdot \nu_k\}[\phi^*] - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi^* \cdot \nu_k\}[\chi] - J_0^{\sigma,\beta}(\chi, \phi^*).\end{aligned}$$

The regularity of ϕ and K imply that the jumps $[K\nabla\phi\cdot\nu_k]|_{e_k}=0$; thus, by choosing $\phi^*\in C^0(\bar{\Omega})$, we obtain

$$\|\chi\|_{0,\Omega}^2 = \sum_{j=1}^{N_h}\int_{E_j} K\nabla(\phi-\phi^*)\nabla\chi + \alpha(\phi-\phi^*)\chi + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\phi-\phi^*)\cdot\nu_k\}[\chi] - 2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\nu_k\}[\chi].$$

The first two terms are bounded in the following fashion

$$\begin{aligned}|\sum_{j=1}^{N_h}\int_{E_j} K\nabla(\phi-\phi^*)\nabla\chi| &\leq C\sum_{j=1}^{N_h}\|\phi-\phi^*\|_{1,E_j}\|K^{\frac{1}{2}}\nabla\chi\|_{0,E_j},\\ &\leq C\frac{h}{r}\|\phi\|_{2,\Omega}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0,\\ &\leq C\frac{h}{r}\|\chi\|_{0,\Omega}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0.\end{aligned}$$

$$\begin{aligned}|\sum_{j=1}^{N_h}\int_{E_j}\alpha(\phi-\phi^*)\chi| &\leq C\|\alpha\|_\infty\frac{h^2}{r^2}\|\phi\|_{2,\Omega}\|\chi\|_{0,\Omega},\\ &\leq C\|\alpha\|_\infty\frac{h^2}{r^2}\|\chi\|_{0,\Omega}^2.\end{aligned}$$

By Cauchy-Schwarz inequality and Theorem 3.1, we have

$$\begin{aligned}|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\phi-\phi^*)\cdot\nu_k\}[\chi]| &\leq \sum_{k=1}^{P_h}\left(\frac{|e_k|^\beta}{\sigma_k}\right)^{\frac{1}{2}}\|K\nabla(\phi-\phi^*)\cdot\nu_k\|_{0,e_k}\left(\frac{\sigma_k}{|e_k|^\beta}\right)^{\frac{1}{2}}\|\chi\|_{0,e_k},\\ &\leq J_0^{\sigma,\beta}(\chi,\chi)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|K\nabla(\phi-\phi^*)\|_{0,e_k}^2\right)^{\frac{1}{2}},\\ &\leq C\frac{h^{\mu-\frac{1}{2}}|e_k|^{\frac{\beta}{2}}}{r^{s-3}}|\!|\!|p|\!|\!|_s\|\chi\|_{0,\Omega}.\end{aligned}$$

Similarly,

$$\begin{aligned}|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\nu_k\}[\chi]| &\leq \sum_{k=1}^{P_h}\left(\frac{|e_k|^\beta}{\sigma_k}\right)^{\frac{1}{2}}\|K\nabla\phi\cdot\nu_k\|_{0,e_k}\left(\frac{\sigma_k}{|e_k|^\beta}\right)^{\frac{1}{2}}\|\chi\|_{0,e_k},\\ &\leq J_0^{\sigma,\beta}(\chi,\chi)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{\sigma_k}\|\nabla\phi\|_{0,e_k}^2\right)^{\frac{1}{2}},\\ &\leq C\frac{h^{\mu-\frac{3}{2}}|e_k|^{\frac{\beta}{2}}}{r^{s-3}}|\!|\!|p|\!|\!|_s\|\chi\|_{0,\Omega}.\end{aligned}$$

Combining the previous bounds, we have

$$\|\chi\|_{0,\Omega} \leq C\frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}(n-1)}}{r^{s-3}}|||p|||_s + C\|\alpha\|_\infty \frac{h^{\mu+1}}{r^s}|||p|||_s.$$

□

4. A priori error estimates for the NCG method

In this section, we derive an error estimate for the H^1 norm and the L^2 norm that are both h-optimal for the constrained Galerkin method applied to the pure Neumann problem.

Theorem 4.1. Under the assumptions of Lemma 2.1 and if $p \in H^s(\mathcal{E}_h)$, we have

$$|||P^{NCG} - p|||_1 \leq C(K, \|\alpha\|_\infty)\frac{h^{\mu-1}}{r^{s-2}}|||p|||_s$$

where $\mu = \min(r+1, s)$, C independent of h, r, p and $r \geq 1, s \geq 2$.

Proof. We have the following orthogonality equation:

$$a_{NS}(P^{NCG} - p, v) = 0, \quad \forall v \in \mathcal{D}_r^*(\mathcal{E}_h).$$

We can show [10] that there is an interpolant $\tilde{p} \in \mathcal{D}_r^*(\mathcal{E}_h)$ that is optimally close to p in the H^m Sobolev norms. Let $\chi = P^{NCG} - \tilde{p}$; then

$$\begin{aligned} a_{NS}(\chi,\chi) &= a_{NS}(p-\tilde{p},\chi), \\ &= \sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p})\nabla\chi + \alpha\chi(p-\tilde{p})) \\ &\quad - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]. \end{aligned}$$

The first two terms are easily bounded by Cauchy-Schwarz and an approximation result:

$$\begin{aligned} \left|\sum_{j=1}^{N_h}\int_{E_j}K\nabla(p-\tilde{p})\nabla\chi\right| &\leq |||K^{\frac{1}{2}}\nabla\chi|||_0 |||K^{\frac{1}{2}}\nabla(p-\tilde{p})|||_0, \\ &\leq C|||K^{\frac{1}{2}}\nabla\chi|||_0\left(\sum_{j=1}^{N_h}\frac{h_j^{2\mu-2}}{r^{2s-2}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

In a similar manner, we have:

$$|\sum_{j=1}^{N_h}\int_{E_j}\alpha\chi(p-\tilde{p})| \leq \|\alpha^{\frac{1}{2}}\chi\|_0\|\alpha^{\frac{1}{2}}(p-\tilde{p})\|_0,$$
$$\leq C\|\alpha\|_\infty^{\frac{1}{2}}\|\alpha^{\frac{1}{2}}\chi\|_0\left(\sum_{j=1}^{N_h}\frac{h_j^{2\mu}}{r^{2s}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Now we try to estimate the third term. Let c_k be any constant:

$$A\equiv\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi]=\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\nu_k\}[\chi-c_k].$$

We have by Cauchy-Schwarz and Holder inequality:

$$|A|\leq\left(\sum_{k=1}^{P_h}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\|[\chi-c_k]\|_{0,e_k}^2\right)^{\frac{1}{2}}. \tag{4.3}$$

We look at one edge or face e_k, assuming that it is the interface between two elements E^1 and E^2 of $\mathcal{E}_h$. Let B_1 and B_2 be the matrices of the mappings from the reference element $\hat{E}$ onto E^1 and E^2 respectively. It is known that

$$\|B_i^{-1}\|\leq C\frac{1}{h},\quad \|B_i\|\leq Ch,\quad |\det B_i|\leq Ch^n,\quad i=1,2,$$

where $\|\cdot\|$ denotes the matrix norm subordinated to the Euclidean norm ([4]). Then, by the trace theorem,

$$\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}\leq\hat{C}|e_k|^{\frac{1}{2}}(\|B_1^{-1}\|\|\nabla(p-\tilde{p})\|_{0,E^1}+\|\nabla^2(p-\tilde{p})\|_{0,E^1}$$
$$+\|B_2^{-1}\|\|\nabla(p-\tilde{p})\|_{0,E^2}+\|\nabla^2(p-\tilde{p})\|_{0,E^2}).$$

Summing on k,

$$\left(\sum_{k=1}^{P_h}\|\{K\nabla(p-\tilde{p})\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\leq C\frac{h^{\mu-\frac{3}{2}}}{r^{s-2}}\|p\|_s. \tag{4.4}$$

The other factor is bounded in the following way.

$$\|[\chi-c_k]\|_{0,e_k}\leq\|(\chi-c_k)_1\|_{0,e_k}+\|(\chi-c_k)_2\|_{0,e_k}.$$

Since $P^{NCG}\in\mathcal{D}_r^*(\mathcal{E}_h)$, we have

$$\int_{e_k}(\chi)_1 d\sigma=\int_{e_k}(\chi)_2 d\sigma.$$

Therefore, it suffices to estimate $\|(\chi - c_k)_1\|_{0,e_k}$.

$$\|(\chi - c_k)_1\|_{0,e_k} \le |e_k|^{\frac{1}{2}} \|\widehat{\chi - c_k}\|_{L^2(\hat{e})}.$$

Choose

$$c_k = \frac{1}{|e_k|} \int_{e_k} (\chi)_1 d\sigma.$$

We note that $c_k = \frac{1}{|\hat{e}|} \int_{\hat{e}} \hat{\chi} d\hat{\sigma}$ and that the mapping $\hat{f} \mapsto \hat{f} - \frac{1}{|\hat{e}|} \int_{\hat{e}} \hat{f} d\hat{\sigma}$ is continuous on $H^1(\hat{e})$ and vanishes on constant functions. Thus,

$$\|\hat{\chi} - c_k\|_{0,\hat{e}} \le \hat{C} \|\hat{\nabla}_{\hat{e}} \hat{\chi}\|_{0,\hat{e}},$$

where $\hat{\nabla}_{\hat{e}}$ denotes the tangential gradient on $\hat{e}$. But since $\hat{\nabla}_{\hat{e}} \hat{\chi}$ belongs to a finite-dimensional space, on which all norms are equivalent and since the subdivision of Ω is regular, we get:

$$\|\hat{\chi} - c_k\|_{0,\hat{e}} \le \hat{C} \|\hat{\nabla} \hat{\chi}\|_{0,\hat{E}} \le \hat{C} \|\nabla \chi\|_{0,E^1}.$$

Therefore,

$$\|(\chi - c_k)_{E^1}\|_{0,e_k} \le C |e_k|^{\frac{1}{2}} \|\nabla \chi\|_{0,E^1}.$$

Thus, summing on k, we have

$$\sum_{k=1}^{P_h} \|[\chi - c_k]\|_{0,e_k}^2 \le C \sum_{j=1}^{N_h} h_j \|\nabla \chi\|_{0,E_j}^2. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we obtain a bound for A:

$$|A| \le C \frac{h^{\mu-1}}{r^{s-2}} \|\!|p|\!\|_s \|\!|K^{\frac{1}{2}} \nabla \chi|\!\|_0. \tag{4.6}$$

The last term is bounded in the following way:

$$\left| \int_{e_k} \{K \nabla \chi \cdot \nu_k\} [p - \tilde{p}] \right| \le C \|\{\nabla \chi \cdot \nu_k\}\|_{0,e_k} \|[p - \tilde{p}]\|_{0,e_k}.$$

Since $\hat{\nabla} \hat{\chi}$ belongs to a finite-dimensional space, we have

$$\|\{\nabla \chi \cdot \nu_k\}\|_{0,e_k} \le \hat{C} |e_k|^{\frac{1}{2}} \left(\|B_1^{-1}\| \|\nabla \chi\|_{0,E^1} + \|B_2^{-1}\| \|\nabla \chi\|_{0,E^2} \right).$$

The other factor is bounded by

$$\begin{aligned} \|[p - \tilde{p}]\|_{0,e_k} \le \hat{C} |e_k|^{\frac{1}{2}} (&|\det B_1|^{-\frac{1}{2}} \|p - \tilde{p}\|_{0,E^1} + \|\nabla(p - \tilde{p})\|_{0,E^1} \\ &+ |\det B_2|^{-\frac{1}{2}} \|p - \tilde{p}\|_{0,E^2} + \|\nabla(p - \tilde{p})\|_{0,E^2}). \end{aligned}$$

Thus,

$$|\int_{e_k} \{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]| \leq \hat{C}|e_k| \left(\|B_1^{-1}\|\|\nabla\chi\|_{0,E^1} + \|B_2^{-1}\|\|\nabla\chi\|_{0,E^2}\right)$$
$$(|\det B_1|^{-\frac{1}{2}}\|p-\tilde{p}\|_{0,E^1} + \|\nabla(p-\tilde{p})\|_{0,E^1}$$
$$+|\det B_2|^{-\frac{1}{2}}\|p-\tilde{p}\|_{0,E^2} + \|\nabla(p-\tilde{p})\|_{0,E^2}).$$

Summing on k,

$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}]| \leq C\frac{h^{\mu-1}}{r^{s-1}}\|\!|p|\!\|_s \|\!|K^{\frac{1}{2}}\nabla\chi|\!\|_0. \tag{4.7}$$

Combining (4.1) , (4.2), (4.6) and (4.7), we obtain:

$$a_{NS}(\chi,\chi) \leq C\frac{h^{\mu-1}}{r^{s-1}}\|\!|p|\!\|_s\|\!|K^{\frac{1}{2}}\nabla\chi|\!\|_0 + C\|\alpha\|_\infty^{\frac{1}{2}}\frac{h^\mu}{r^s}\|\alpha^{\frac{1}{2}}\chi\|_0\|\!|p|\!\|_s.$$

□

Theorem 4.2. Under the assumptions of Theorem 3.2, we have

$$\|P^{NCG}-p\|_{0,\Omega} \leq C\frac{h^\mu}{r^{s-2}}\|\!|p|\!\|_s,$$

for $r\geq 1, s\geq 2$ and C independent of h, r, p.

Proof. As in the proof of Theorem 3.2, we consider the dual problem:

$$\begin{cases} -\nabla\cdot(K\nabla\psi)+\alpha\psi = P^{NCG}-p & \text{in} \quad \Omega, \\ K\nabla\psi\cdot\nu = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Denote $\chi = P^{NCG}-p$. Thus, we have:

$$\|\chi\|_{0,\Omega}^2 = (-(\nabla\cdot K\nabla\psi)+\alpha\psi,\chi),$$
$$= \sum_{j=1}^{N_h}\int_{E_j} K\nabla\psi\nabla\chi + \alpha\psi\chi - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}[\chi]. \tag{4.8}$$

Let ψ^* be in $\mathcal{D}_r^*(\mathcal{E}_h)\cap C^0(\Omega)$. The orthogonality condition implies that:

$$0 = \sum_{j=1}^{N_h}\int_{E_j} K\nabla\chi\nabla\psi^* + \alpha\chi\psi^* + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi^*\cdot\nu_k\}[\chi]. \tag{4.9}$$

Now, we subtract (4.9) from (4.8):

$$\|\chi\|_{0,\Omega}^2 = \sum_{j=1}^{N_h}\int_{E_j} K\nabla(\psi-\psi^*)\nabla\chi + \alpha(\psi-\psi^*)\chi$$

$$-2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}[\chi]+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\psi-\psi^*)\cdot\nu_k\}[\chi]. \quad (4.10)$$

The first two terms are easily bounded by using Cauchy-Schwarz, Holder inequalities and the approximation property:

$$\begin{aligned}|\sum_{j=1}^{N_h}\int_{E_j}K\nabla(\psi-\psi^*)\nabla\chi| &\leq C\left(\sum_{j=1}^{N_h}\|\nabla(\psi-\psi^*)\|_{0,E_j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N_h}\|K^{\frac{1}{2}}\nabla\chi\|_{0,E_j}^2\right)^{\frac{1}{2}},\\ &\leq C\left(\sum_{j=1}^{N_h}\frac{h_j^2}{r^2}\|\psi\|_{2,E_j}^2\right)^{\frac{1}{2}}\|\!|K^{\frac{1}{2}}\nabla\chi\|\!|_0,\\ &\leq C\frac{h}{r}\|\chi\|_{0,\Omega}\|\!|K^{\frac{1}{2}}\nabla\chi\|\!|_0.\end{aligned}$$

In a similar manner, we have:

$$\begin{aligned}\left|\sum_{j=1}^{N_h}\int_{E_j}\alpha(\psi-\psi^*)\chi\right| &\leq C\sum_{j=1}^{N_h}\|(\psi-\psi^*)\|_{0,E_j}\|\alpha^{\frac{1}{2}}\chi\|_{0,E_j},\\ &\leq C\left(\sum_{j=1}^{N_h}\frac{h_j^4}{r^4}\|\psi\|_{2,E_j}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N_h}\|\alpha^{\frac{1}{2}}\chi\|_{0,E_j}^2\right)^{\frac{1}{2}},\\ &\leq C\frac{h^2}{r^2}\|\chi\|_{0,\Omega}\|\!|K^{\frac{1}{2}}\nabla\chi\|\!|_0.\end{aligned}$$

Then, we will bound the third term in (4.10). Let $\tilde{p}$ be an element of $\mathcal{D}_r^*(\mathcal{E}_h)\cap C^0(\bar{\Omega})$ and $\vec{c}$ be any constant vector.

$$\begin{aligned}2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}[\chi] &= 2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}\left[P^{NCG}-\tilde{p}\right],\\ &= 2\sum_{k=1}^{P_h}\int_{e_k}\{(K\nabla\psi-\vec{c})\cdot\nu_k\}[P^{NCG}-\tilde{p}].\end{aligned}$$

As it was proved in Theorem 4.1, we have:

$$\sum_{k=1}^{P_h}\|[P^{NCG}-\tilde{p}]\|_{0,e_k}^2\leq C\sum_{j=1}^{N_h}h_j\|\nabla(P^{NCG}-\tilde{p})\|_{0,E_j}^2.$$

By the triangle inequality, Theorem 4.1 and choosing for $\tilde{p}$ an interpolant of p, we have

$$\sum_{j=1}^{N_h}\|\nabla(P^{NCG}-\tilde{p})\|_{0,E_j}^2\leq 2\sum_{j=1}^{N_h}\|\nabla(P^{NCG}-p)\|_{0,E_j}^2+2\sum_{j=1}^{N_h}\|\nabla(p-\tilde{p})\|_{0,E_j}^2,$$

$$\leq C\frac{h^{2\mu-2}}{r^{2s-4}}\sum_{j=1}^{N_h}\|p\|_{s,E_j}^2.$$

On the other hand, we have

$$\|\{(K\nabla\psi-\vec{c})\cdot\nu_k\}\|_{0,e_k}\leq\|\{K\nabla\psi-\vec{c}\}\|_{0,e_k}\leq Ch^{\frac{1}{2}}\|\nabla(K\nabla\psi)\|_{0,E^1\cup E^2}.$$

If we assume that $K\nabla\psi\in H^1(E_j)$ with

$$\sum_{j=1}^{N_h}\|\nabla(K\nabla\psi)\|_{0,E_j}^2\leq\|\chi\|_{0,\Omega}^2,$$

then

$$|2\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\nu_k\}\,[\chi]|\leq C\frac{h^{\mu}}{r^{s-2}}\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s.$$

Let A denote the last term in (4.10). We have by Cauchy-Schwarz and Holder inequality:

$$|A|\leq C\left(\sum_{k=1}^{P_h}\|\{\nabla(\psi-\psi^*)\cdot\nu_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\|[\chi]\|_{0,e_k}^2\right)^{\frac{1}{2}}.$$

By the approximation property:

$$\sum_{k=1}^{P_h}\|\{\nabla(\psi-\psi^*)\cdot\nu_k\}\|_{0,e_k}^2\leq C\sum_{j=1}^{N_h}\frac{h_j}{r}\|\psi\|_{2,E_j}^2\leq C\frac{h}{r}\|\chi\|_{0,\Omega}.$$

We also have

$$\sum_{k=1}^{P_h}\|[\chi]\|_{0,e_k}^2\leq C\frac{h}{r}|\!|\!|\nabla\chi|\!|\!|_0\leq C\frac{h^{2\mu-1}}{r^{2s-1}}|\!|\!|p|\!|\!|_s.$$

Thus, we obtain:

$$|A|\leq\frac{h^{\mu}}{r^{s}}\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s.$$

The theorem is obtained by combining all the above results. □

5. A priori error estimate for the DG method

In this section, we derive an *a priori* optimal error estimate in two dimensions for the problem with Neumann, or Dirichlet, or mixed boundary conditions. We make several additional assumptions:

- $K \in [W^{1,\infty}(E_j)]^{2\times 2}$, $\forall j = 1,\dots,N_h$, uniformly in E_j, i.e. $\exists \gamma_2 > 0$, $\max_{E_j}\left(\sum_{i,m=1}^{2} \|\nabla k_{im}\|^2_{\infty,E_j}\right)^{\frac{1}{2}} \leq \gamma_2$.
- We denote $\bar{K} = (\bar{k}_{ij})$, where

$$\bar{k}_{ij} = \frac{1}{|E|}\int_E k_{ij}, \quad \forall E \in \mathcal{E}_h \tag{5.1}$$

  It is easy to prove that $\bar{K}$ is also symmetric, positive definite and that the largest eigenvalue of $\bar{K}$ is $\leq \gamma_1$ and its smallest eigenvalue is $\geq \gamma_0$, where γ_1 and γ_0 are the constants of (2.1).

We first prove an approximation result that holds for $n = 2$ or 3.

Lemma 5.1. Let $p \in H^s(\mathcal{E}_h)$, for $s \geq 2$ and let $r \geq 2$. Let $\bar{K}$ be defined by (5.1). There exists an interpolant of p, $\tilde{p}^I \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying

$$\int_{e_k} \{\bar{K}\nabla(\tilde{p}^I - p)\cdot \nu_k\} = 0, \quad \forall k = 1,\dots,P_h,$$

$$\int_{e_k} \bar{K}\nabla(\tilde{p}^I - p)\cdot \nu_k = 0, \quad \forall e_k \in \Gamma_D,$$

$$\|\tilde{p}^I - p\|_{0,\Omega} \leq C\frac{h^\mu}{r^{s-2}}|\!|\!| p |\!|\!|_s,$$

$$|\!|\!| \nabla(\tilde{p}^I - p) |\!|\!|_0 \leq C\frac{h^{\mu-1}}{r^{s-2}}|\!|\!| p |\!|\!|_s,$$

$$|\!|\!| \nabla^2(\tilde{p}^I - p) |\!|\!|_0 \leq C\frac{h^{\mu-1}}{r^{s-2}}|\!|\!| p |\!|\!|_s,$$

where $\mu = \min(r+1, s)$ and C is independent of h and r.

Proof. We prove the Lemma for a triangular element and for a parallelogram in $2D$ and for a tetrahedra in $3D$. The proof can be extended to an arbitrary quadrilateral.

The case of triangles:

Let E be a triangle with sides e_i, e_j and e_k and corresponding unit normal vectors $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j$ and $\boldsymbol{\nu}_k$. We will show that given f in $H^s(E)$ with $s \geq 2$, there is a polynomial q in $I\!P_2(E)$ such that $\int_{e_k} \bar{K}\nabla(q-f)\cdot\boldsymbol{\nu}_k = 0$, $\int_{e_i} \bar{K}\nabla q\cdot\boldsymbol{\nu}_i = 0$ and $\int_{e_j} \bar{K}\nabla q\cdot\boldsymbol{\nu}_j = 0$.

We denote a_1, a_2, a_3 the vertices of E so that $e_i = [a_1, a_2]$, $e_j = [a_1, a_3]$ and $e_k = [a_2, a_3]$ (see Fig. 1). There is an invertible affine transformation

$$x = F(\hat{x}) = B\hat{x} + b$$

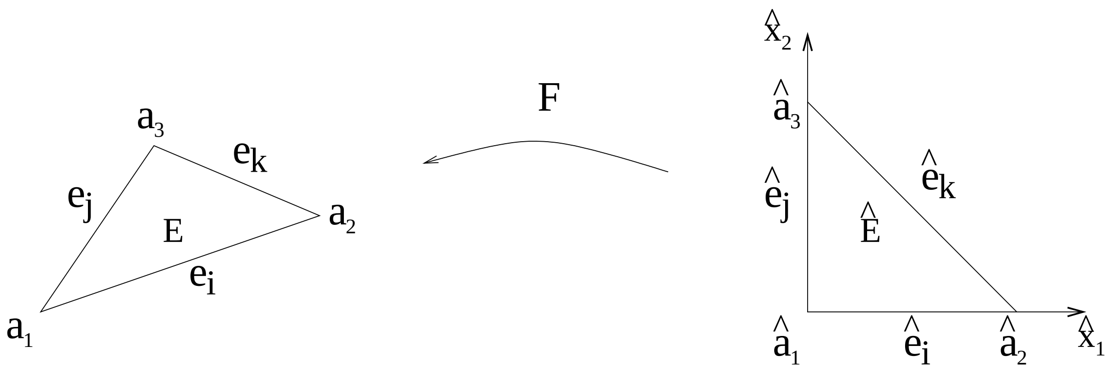

Figure 1. Mapping between an arbitrary triangular element and the reference element.

that maps the reference element $\hat{E}$ onto E such that e_k is mapped onto the line $\hat{x}_1 + \hat{x}_2 = 1$, where $\hat{x}$ denotes the coordinates of a point of $\hat{E}$. Let λ_1, λ_2 and λ_3 be the barycentric coordinates of a_1, a_2 and a_3 in E, and let us consider a polynomial q of the form:

$$q = 4q(a_{12})\lambda_1\lambda_2 + 4q(a_{13})\lambda_1\lambda_3,$$

where a_{12} and a_{13} are the midpoints of e_i and e_j respectively. To simplify, we denote $q_{12} = q(a_{12})$ and $q_{13} = q(a_{13})$. The following relations hold:

$$\begin{aligned} \hat{q}(\hat{x}) &= 4q_{12}(1-\hat{x}_1-\hat{x}_2)\hat{x}_1 + 4q_{13}(1-\hat{x}_1-\hat{x}_2)\hat{x}_2, \\ \hat{\nabla}\hat{q} &= (4q_{12}(1-2\hat{x}_1-\hat{x}_2) - 4q_{13}\hat{x}_2, -4q_{12}\hat{x}_1 + 4q_{13}(1-\hat{x}_1-2\hat{x}_2)), \end{aligned}$$

and $\hat{\nabla}\hat{q}$ on selected lines is given by

$$\begin{aligned} \text{on} \quad \hat{x}_2 = 0, &\quad \hat{\nabla}\hat{q} = (4q_{12}(1-2\hat{x}_1), -4q_{12}\hat{x}_1 + 4q_{13}(1-\hat{x}_1)), \\ \text{on} \quad \hat{x}_2 = 0, &\quad \hat{\nabla}\hat{q} = (4q_{12}(1-\hat{x}_2) - 4q_{13}\hat{x}_2, 4q_{13}(1-2\hat{x}_2)), \\ \text{on} \quad \hat{x}_1 + \hat{x}_2 = 1, &\quad \hat{\nabla}\hat{q} = (-4q_{12}\hat{x}_1 - 4q_{13}(1-\hat{x}_1), -4q_{12}\hat{x}_1 - 4q_{13}(1-\hat{x}_1)). \end{aligned}$$

Let A denote the matrix $(B^t\bar{K}^{-1}B)^{-1}$.

$$\int_{e_i} \bar{K}\nabla q \cdot \boldsymbol{\nu}_i = |e_i| \int_0^1 \|B^t\boldsymbol{\nu}_i\| A\hat{\nabla}\hat{q} \cdot \hat{\boldsymbol{\nu}}.$$

But $\hat{\boldsymbol{\nu}} = (0, -1)$, so denoting by A_{lm} the coefficients of A, we have

$$\begin{aligned} \int_0^1 A\hat{\nabla}\hat{q} \cdot \hat{\boldsymbol{\nu}} &= -\int_0^1 (A_{12}4q_{12}(1-2\hat{x}_1) + A_{22}(-4q_{12}\hat{x}_1 + 4q_{13}(1-\hat{x}_1)) \\ &= -2A_{22}(-q_{12} + q_{13}). \end{aligned}$$

Thus, this integral vanishes if we choose $q_{12} = q_{13}$. Then, on e_j, considering that $\hat{\boldsymbol{\nu}} = (-1, 0)$, we have

$$\int_0^1 A\hat{\nabla}\hat{q} \cdot \hat{\boldsymbol{\nu}} = -\int_0^1 (A_{11}4q_{12}(1-2\hat{x}_2) + A_{12}4q_{12}(1-2\hat{x}_2)) \\ = -2A_{11}(q_{12} - q_{13}) = 0.$$

Now, on e_k, $\hat{\boldsymbol{\nu}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\sqrt{2}\int_0^1 A\hat{\nabla}\hat{q} \cdot \hat{\boldsymbol{\nu}} = -4q_{12}(A_{11} + 2A_{12} + A_{22}) = -4q_{12}A\mathbf{1}\cdot\mathbf{1}.$$

And since the matrix A is positive-definite, $A\mathbf{1}\cdot\mathbf{1} \neq 0$, so we can always determine q_{12}. Now, let us compute the factor of q_{12} in

$$\int_{e_k} \bar{K}\nabla q \cdot \boldsymbol{\nu}_k = |e_k|\|B^t\boldsymbol{\nu}_k\| \int_0^1 A\hat{\nabla}\hat{q} \cdot \hat{\boldsymbol{\nu}} = \frac{|e_k|}{\sqrt{2}}\|B^t\boldsymbol{\nu}_k\|(-4q_{12})A\mathbf{1}\cdot\mathbf{1}.$$

We want the following equation to hold

$$\int_{e_k} \bar{K}\nabla q \cdot \boldsymbol{\nu}_k = \int_{e_k} \bar{K}\nabla f \cdot \boldsymbol{\nu}_k$$

$$\frac{|e_k|}{\sqrt{2}}\|B^t\boldsymbol{\nu}_k\|(-4q_{12})A\mathbf{1}\cdot\mathbf{1} = |e_k|\|B^t\boldsymbol{\nu}_k\| \int_0^1 A\hat{\nabla}\hat{f} \cdot \hat{\boldsymbol{\nu}}$$

But

$$\left|\int_0^1 A\hat{\nabla}\hat{f} \cdot \hat{\boldsymbol{\nu}}\right| \leq \int_0^1 \|A\|\|\hat{\nabla}\hat{f}\| \leq \|A\|\|\hat{\nabla}\hat{f}\|_{0,\hat{e}}.$$

Since $\|B^t\boldsymbol{\nu}_k\|$ is different from zero, we are left with

$$|q_{12}|A\mathbf{1}\cdot\mathbf{1} \leq \frac{1}{2\sqrt{2}}\|A\|\|\hat{\nabla}\hat{f}\|_{0,\hat{e}}$$

But, we have

$$A\mathbf{1}\cdot\mathbf{1} \geq \frac{1}{\lambda_{\max}}\|\mathbf{1}\|^2 = \frac{2}{\lambda_{\max}} \quad \text{and} \quad \|A\| \leq \gamma_{\max}\|B^{-1}\|^2,$$

where $\lambda_{\max}$ is the largest eigenvalue of $B^t\bar{K}^{-1}B$, i.e.

$$\lambda_{\max} = \|B^t\bar{K}^{-1}B\| \leq \|B\|^2\|\bar{K}^{-1}\| \leq \|B\|^2\frac{1}{\gamma_{\min}}.$$

Thus,

$$|q_{12}| \leq \frac{\gamma_{\max}}{4\gamma_{\min}}\sigma_E^2\|\hat{\nabla}\hat{f}\|_{0,\hat{e}}.$$

Let $f = p - p^I$, where $p^I \in I\!P_r(E)$ is the interpolant ot p satisfying (**??**)-(**??**). From the above construction, it follows that there exists a unique polynomial q of the form:

$$q = 4q(a_{12})\lambda_1(1-\lambda_1)$$

that satisfies

$$\int_{e_k} \bar{K}\nabla q \cdot \boldsymbol{\nu}_k = \int_{e_k} \bar{K}\nabla(p - p^I)\cdot \boldsymbol{\nu}_k$$

and

$$\int_{e_i} \bar{K}\nabla q \cdot \boldsymbol{\nu}_i = \int_{e_j} \bar{K}\nabla q \cdot \boldsymbol{\nu}_j = 0.$$

Moreover

$$\begin{aligned} |q(a_{12})| &\leq \frac{\gamma\mathrm{max}}{4\gamma_{\mathrm{min}}}\sigma_E^2 \|\hat{\nabla}(\hat{p} - \hat{p}^I)\|_{0,\hat{e}}. \\ &\leq C(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}) \end{aligned}$$

Therefore

$$\begin{aligned} \|q\|_{0,E} &\leq C|\mathrm{det}B|^{\frac{1}{2}}|q(a_{12})| \\ &\leq Ch(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}), \end{aligned}$$

and

$$\begin{aligned} \|\nabla q\|_{0,E} &\leq C|\mathrm{det}B|^{\frac{1}{2}}\|B^{-1}\||q(a_{12})| \\ &\leq C(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}), \end{aligned}$$

and

$$\begin{aligned} \|\nabla^2 q\|_{0,E} &\leq C|\mathrm{det}B|^{\frac{1}{2}}\|B^{-1}\|^2|q(a_{12})| \\ &\leq Ch^1(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}). \end{aligned}$$

So, if we define $\tilde{p}^I = q + p^I$, then

$$\begin{aligned} \|\tilde{p}^I - p\|_{0,\Omega} &\leq \|q\|_{0,\Omega} + \|p^I - p\|_{0,\Omega} \\ &\leq Ch(\|\nabla(p^I-p)\|_{0,\Omega} + h\|\nabla^2(p^I-p)\|_{0,\Omega}) + \|p^I - p\|_{0,\Omega}, \end{aligned}$$

which has the same order of approximation as $\|p^I - p\|_{0,\Omega}$ and

$$\begin{aligned} \|\nabla(\tilde{p}^I - p)\|_{0,\Omega} &\leq \|\nabla q\|_{0,\Omega} + \|\nabla(p^I - p)\|_{0,\Omega} \\ &\leq C(\|\nabla(p^I-p)\|_{0,\Omega} + h\|\nabla^2(p^I-p)\|_{0,\Omega}) + \|\nabla(p^I - p)\|_{0,\Omega}, \end{aligned}$$

which is also of the same order as $\|\nabla(p^I - p)\|_{0,\Omega}$. Similarly, the order of $\|\nabla^2(\tilde{p}^I - p)\|_{0,\Omega}$ is optimal in h.

The case of parallelograms:
Let E be a parallelogram with vertices $a_i, 1 \le i \le 4$, let $e_k = [a_1, a_2]$, $\hat{E}$ be the reference unit square and let E be mapped onto $\hat{E}$ so that e_k is mapped onto $\hat{e}$ (see Fig. 2). Let b_1 be the midpoint of $[a_1, a_2]$, b_2 the midpoint of $[a_1, a_3]$, b_3 of $[a_3, a_4]$ and b_4 of $[a_2, a_4]$, and $\hat{b}_i$ the corresponding image of b_i. We will show that given $\alpha \in I\!R$, there is a polynomial $q \in I\!P_2(E)$ such that $\int_{e_k} \bar{K}\nabla q \cdot \boldsymbol{\nu}_k = \alpha$ and $\int_e \bar{K}\nabla q \cdot \boldsymbol{\nu} = 0$ on the other edges. We look at a polynomial of the form

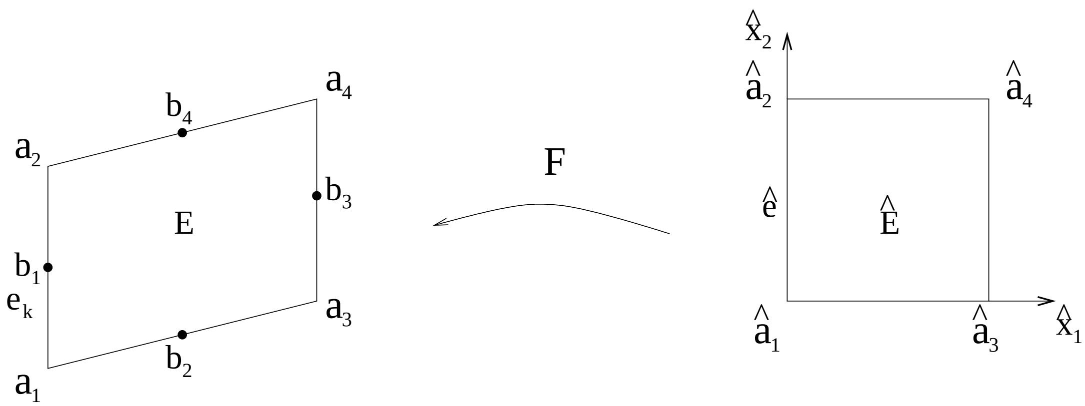

Figure 2. Mapping between an arbitrary parallelogram element and the reference element.

$$
\begin{aligned}
\hat{q} = {} & 2\hat{q}_1(\frac{1}{2} - \hat{x}_1)(1 - \hat{x}_1) + 2\hat{q}_2(\frac{1}{2} - \hat{x}_2)(1 - \hat{x}_2) \\
& -2\hat{q}_4(\frac{1}{2} - \hat{x}_2)\hat{x}_2 - 2\hat{q}_3\hat{x}_1(\frac{1}{2} - \hat{x}_1),
\end{aligned}
$$

where $\hat{q}_i = \hat{q}(\hat{b}_i)$. Then, we have

$$
\hat{\nabla}\hat{q} = (2\hat{q}_1(2\hat{x}_1 - \frac{3}{2}) - 2\hat{q}_3(\frac{1}{2} - 2\hat{x}_1), 2\hat{q}_2(2\hat{x}_2 - \frac{3}{2}) - 2\hat{q}_4(\frac{1}{2} - 2\hat{x}_2))
$$

On the edges, we have

$$
\begin{aligned}
&\text{on} \quad [\hat{a}_1, \hat{a}_3], \quad \hat{x}_2 = 0, \quad \hat{\nabla}\hat{q} = (2\hat{q}_1(2\hat{x}_1 - \frac{3}{2}) - 2\hat{q}_3(\frac{1}{2} - 2\hat{x}_1), -3\hat{q}_2 - \hat{q}_4), \\
&\text{on} \quad [\hat{a}_3, \hat{a}_4], \quad \hat{x}_1 = 1, \quad \hat{\nabla}\hat{q} = (\hat{q}_1 + 3\hat{q}_3, 2\hat{q}_2(2\hat{x}_2 - \frac{3}{2}) - 2\hat{q}_4(\frac{1}{2} - 2\hat{x}_2)), \\
&\text{on} \quad [\hat{a}_2, \hat{a}_4], \quad \hat{x}_2 = 1, \quad \hat{\nabla}\hat{q} = (2\hat{q}_1(2\hat{x}_1 - \frac{3}{2}) - 2\hat{q}_3(\frac{1}{2} - 2\hat{x}_1), \hat{q}_2 + 3\hat{q}_4),
\end{aligned}
$$

$$\text{on}\quad [\hat a_1,\hat a_2]=\hat e,\quad \hat x_1=0,\quad \hat\nabla\hat q=(-3\hat q_1-\hat q_3, 2\hat q_1(2\hat x_2-\frac{3}{2})-2\hat q_4(\frac{1}{2}-2\hat x_2)).$$

In the case of parallelograms, the transformation that maps $\hat E$ onto E is affine, so if we denote $A=(B^t\bar K^{-1}B)^{-1}$:

$$\int_{e_k}\bar K\nabla q\cdot\boldsymbol{\nu}_k=|e_k|\int_0^1 A\hat\nabla\hat q\cdot\hat{\boldsymbol{\nu}}\|B^t\boldsymbol{\nu}_k\|$$

Since $\boldsymbol{\nu}=(-1,0)$, we have

$$\int_0^1 A\hat\nabla\hat q\cdot\hat{\boldsymbol{\nu}}=A_{11}(3\hat q_1+\hat q_3)+A_{12}(\hat q_2-\hat q_4).$$

Since A is positive-definite, $A_{11}>0$, hence

$$3\hat q_1+\hat q_3=-\frac{A_{12}}{A_{11}}(\hat q_2-\hat q_4)+\frac{\alpha}{A_{11}}.$$

Now, the condition $\int_{[a_1,a_3]}\bar K\nabla q\cdot\boldsymbol{\nu}=0$ can be written as

$$\int_0^1 A\hat\nabla\hat q\cdot\hat{\boldsymbol{\nu}}=0,$$

with $\hat{\boldsymbol{\nu}}=(0,-1)$. This leads to the equation

$$3\hat q_2+\hat q_4=\frac{A_{12}}{A_{22}}(\hat q_3-\hat q_1).$$

On the other edge, the condition $\int_{[a_3,a_4]}\bar K\nabla q\cdot\boldsymbol{\nu}=0$ can be written as

$$\int_0^1 A\hat\nabla\hat q\cdot\hat{\boldsymbol{\nu}}=0,$$

with $\hat{\boldsymbol{\nu}}=(1,0)$. This leads to the equation

$$\hat q_1+3\hat q_3=\frac{A_{12}}{A_{11}}(\hat q_2-\hat q_4).$$

Finally, the condition $\int_{[a_2,a_4]}\bar K\nabla q\cdot\boldsymbol{\nu}=0$ can be written as

$$\int_0^1 A\hat\nabla\hat q\cdot\hat{\boldsymbol{\nu}}=0,$$

with $\hat{\boldsymbol{\nu}}=(0,1)$. This leads to the equation

$$\hat q_2+3\hat q_4=\frac{A_{12}}{A_{22}}(\hat q_1-\hat q_3).$$

Therefore,

$$\hat q_1=\frac{\alpha}{4A_{11}}(\frac{A_{12}^2}{\det A}+\frac{3}{2}),$$

$$\hat{q}_2 = -\frac{\alpha}{4}\frac{A_{12}}{\det A},$$
$$\hat{q}_3 = -\frac{\alpha}{4A_{11}}(\frac{A_{12}^2}{\det A} + \frac{1}{2}),$$
$$\hat{q}_4 = \frac{\alpha}{4}\frac{A_{12}}{\det A}.$$

But,

$$\det A = \frac{\det \bar{K}}{(\det B)^2}.$$

So, we obtain

$$A_{11} = \frac{\det \bar{K}}{(\det B)^2}\bar{K}^{-1}\vec{a}_{12}\cdot\vec{a}_{12} = \det A(\bar{K}^{-1}\vec{a}_{12})\cdot\vec{a}_{12}$$
$$A_{22} = \frac{\det \bar{K}}{(\det B)^2}\bar{K}^{-1}\vec{a}_{13}\cdot\vec{a}_{13} = \det A(\bar{K}^{-1}\vec{a}_{13})\cdot\vec{a}_{13}$$
$$A_{12} = -\frac{\det \bar{K}}{(\det B)^2}\bar{K}^{-1}\vec{a}_{12}\cdot\vec{a}_{13} = -\det A(\bar{K}^{-1}\vec{a}_{12})\cdot\vec{a}_{13}$$

Thus, we can bound $\hat{q}_2$

$$|\hat{q}_2| \leq \frac{|\alpha|}{4\gamma_0}\|\vec{a}_{12}\|\|\vec{a}_{13}\| \leq \frac{|\alpha|h_E^2}{4\gamma_0}$$

We also have

$$\frac{A_{12}^2}{\det A} \leq \det A\frac{h_E^4}{\gamma_{\min}^2} \leq \hat{C}\frac{\gamma\max}{\gamma_{\min}^2}\frac{h_E^4}{\rho_E^4} \leq \hat{C}\sigma^4\frac{\gamma_1}{\gamma_0}$$
$$A_{11} \geq \det A\frac{\|\vec{a}_{12}\|^2}{\gamma\max} \geq \hat{C}\frac{\gamma\min}{h_E^4}\|\vec{a}_{12}\|^2 \geq \hat{C}\gamma_0\frac{\|\vec{a}_{12}\|^2}{h_E^4}$$

Thus, we have

$$|\hat{q}_3| \leq \hat{C}\frac{|\alpha|}{4\gamma_0}\frac{h_E^4}{|e_k|^2},$$

and similarly for $|\hat{q}_1|$. It remains to substitute the expression of α. We want

$$\int_{e_k}\bar{K}\nabla q\cdot\boldsymbol{\nu}_k = \int_{e_k}\bar{K}\nabla(p-p^I)\cdot\boldsymbol{\nu}_k$$

and

$$\int_{e_k}\bar{K}\nabla q\cdot\boldsymbol{\nu}_k = |e_k|\|B^t\boldsymbol{\nu}_k\|\alpha$$

Thus,

$$\alpha = \frac{1}{|e_k| \|B^t \boldsymbol{\nu}_k\|} \int_{e_k} \bar{K} \nabla (p - p^I) \cdot \boldsymbol{\nu}_k$$

But

$$|e_k| \|B^t \boldsymbol{\nu}_k\| = \sqrt{2} |E|$$

So, when we substitute this expression for α in the above estimates for $|\hat{q}_i|$, the factor multiplying $\int_{e_k}$ does not depend on h. Note that one can obtain shaper bounds for $|\hat{q}_3|$ and $|\hat{q}_1|$ by not expanding $\det B = |E|$ and canceling at the last moment. The construction is the same for genuine quadrilaterals but the difficulty lies in solving $\int_{e_i} \bar{K} \nabla q \cdot \boldsymbol{\nu}_i = 0$.

The case of tetrahedra:
Let E be a tetrahedra with faces e_i, e_j, e_k, e_l and corresponding unit normal vectors $\boldsymbol{\nu}_i, \boldsymbol{\nu}_j, \boldsymbol{\nu}_k$ and $\boldsymbol{\nu}_l$. We will show that given f in $H^s(E)$ with $s \geq 2$, there is a polynomial q in $I\!P_2(E)$ such that $\int_{e_k} K \nabla (q - f) \cdot \boldsymbol{\nu}_k = 0$, $\int_{e_i} K \nabla q \cdot \boldsymbol{\nu}_i = 0$, $\int_{e_j} K \nabla q \cdot \boldsymbol{\nu}_j = 0$ and $\int_{e_l} K \nabla q \cdot \boldsymbol{\nu}_l = 0$.
We denote a_1, a_2, a_3 and a_4 the vertices of E so that $e_i = [a_1, a_2, a_4]$, $e_j = [a_1, a_2, a_3]$, $e_l = [a_1, a_3, a_4]$ and $e_k = [a_2, a_3, a_4]$ (see Fig. 3). There is an invertible

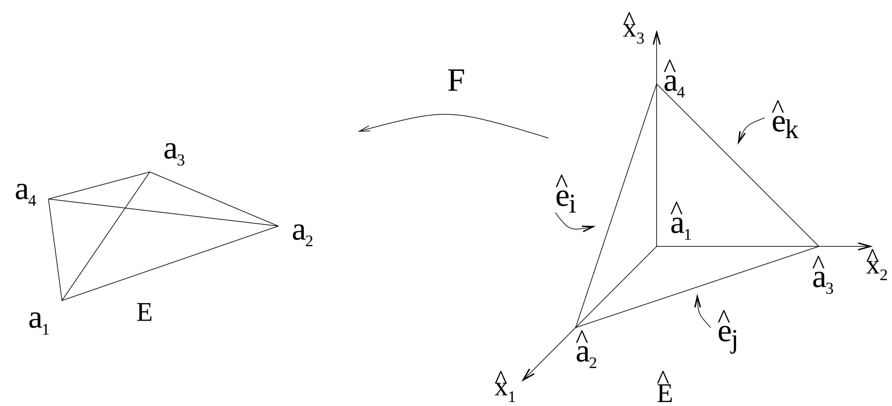

Figure 3. Mapping between an arbitrary tetrahedra element and the reference element.

affine transformation

$$x = F(\hat{x}) = B\hat{x} + b$$

that maps the reference element $\hat{E}$ onto E such that e_k is mapped onto the plane $\hat{x}_1 + \hat{x}_2 + \hat{x}_3 = 1$, where $\hat{x}$ denotes the coordinates of a point of $\hat{E}$. Let $\lambda_1, \lambda_2, \lambda_3$

and λ_4 be the barycentric coordinates of a_1, a_2, a_3 and a_4 in E. We are looking at a polynomial q of the form:

$$q = q(a_1)\lambda_1\lambda_2(3\lambda_1 - 2).$$

We have

$$\hat{\nabla}\hat{q}(\hat{x}) = q(a_1)(-4 + 6(\hat{x}_1 + \hat{x}_2 + \hat{x}_3))\mathbf{1},$$

where $\mathbf{1} = (1,1,1)$. It is easy to check that each component of $\hat{\nabla}\hat{q}$ has zero mean value on $\hat{e}_i, \hat{e}_j$ and $\hat{e}_k$. On $\hat{e}_k$, $\hat{x}_1 + \hat{x}_2 + \hat{x}_3 = 1$, thus

$$\hat{\nabla}\hat{q}(\hat{x}) = 2q(a_1)\mathbf{1}.$$

We want to satisfy

$$\int_{e_k} \bar{K}\nabla q \cdot \boldsymbol{\nu}_k = \int_{e_k} \bar{K}\nabla f \cdot \boldsymbol{\nu}_k.$$

This is equivalent to

$$\frac{2}{\sqrt{3}} q(a_1)|e_k| \|B^t \boldsymbol{\nu}_k\| A\mathbf{1} \cdot \mathbf{1} = |e_k| \|B^t \boldsymbol{\nu}_k\| \int_{\hat{e}_k} A\hat{\nabla}\hat{f} \cdot \hat{\boldsymbol{\nu}}_k.$$

Since $\|B^t \boldsymbol{\nu}_k\|$ is different from zero, we obtain

$$|q(a_1)| A\mathbf{1} \cdot \mathbf{1} \leq \|A\| \frac{\sqrt{3}}{2\sqrt{2}} \|\hat{\nabla}\hat{f}\|_{0,\hat{e}_k}.$$

But, we have

$$A\mathbf{1} \cdot \mathbf{1} \geq \frac{\|\mathbf{1}\|^2}{\lambda_{\max}} \geq \frac{3}{\lambda_{\max}},$$

where

$$\lambda_{\max} \leq \|B\|^2 \|\bar{K}^{-1}\| \leq \frac{1}{\gamma_{\min}} \|B\|^2,$$

and

$$\|A\| \leq \gamma_{\max} \|B^{-1}\|^2.$$

Therefore, after some simplifications,

$$|q(a_1)| \leq \hat{C} \frac{1}{2\sqrt{6}} \frac{\gamma_{\max}}{\gamma_{\min}} \sigma_E^2 \|\hat{\nabla}\hat{f}\|_{0,\hat{e}_k},$$

where

$$\hat{C} = (\frac{\hat{h}}{\hat{\rho}})^2.$$

Let $f = p - p^I$, where p^I is an element of $I\!P_r(E)$ that satisfies the approximation properties (**??**)-(**??**). From the construction described above, we have

$$|q(a_1)| \leq \hat{C}\frac{1}{2\sqrt{6}}\frac{\gamma\max}{\gamma_{\min}}\sigma_E^2\|\hat{\nabla}(\hat{p}-\hat{p}^I)\|_{0,\hat{e}_k},$$

which yields

$$\begin{aligned}\|q\|_0, E &\leq C|\det B|^{\frac{1}{2}}|q(a_1)|\\ &\leq Ch(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}),\end{aligned}$$

and

$$\begin{aligned}\|\nabla q\|_0, E &\leq C|\det B|^{\frac{1}{2}}\|B^{-1}\||q(a_1)|\\ &\leq C(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E})\end{aligned}$$

and

$$\begin{aligned}\|\nabla^2 q\|_0, E &\leq C|\det B|^{\frac{1}{2}}\|B^{-1}\|^2|q(a_1)|\\ &\leq Ch^{-1}(\|\nabla(p-p^I)\|_{0,E} + h\|\nabla^2(p-p^I)\|_{0,E}).\end{aligned}$$

We conclude in the same way as in the triangular case that if $\tilde{p}^I$ denotes $(q+p^I)$, then $\tilde{p}^I$ satisfies (5.2)-(5.2).

□

Theorem 5.2. Assume $\Omega \subset I\!R^n$ for $n = 2,3$. If $\alpha \equiv 0$, then there is a constant C independent of h, r, p such that for $s \geq 2$ and $r \geq 2$

$$\|K^{\frac{1}{2}}\nabla(p - P^{DG})\|_0 \leq C(K)\frac{h^{\mu-1}}{r^{s-4}}\|p\|_s.$$

If $\alpha \geq \alpha_0 > 0$, then the following inequality holds

$$\|p - P^{DG}\|_1 \leq C(K, \|\alpha\|_\infty)\frac{h^{\mu-1}}{r^{s-4}}\|p\|_s,$$

where $\mu = \min(r+1, s)$.

Proof. The difference between P^{DG} and p satisfies:

$$a_{NS}(P^{DG} - p, v) = 0, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h).$$

We take $v = P^{DG} - \tilde{p}^I$, where $\tilde{p}^I$ is the interpolant of p constructed in Lemma 5.1 such that

$$\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\} = 0, \quad \forall k = 1,\ldots,P_h \tag{5.2}$$

The interpolant $\tilde{p}^I$ has approximation errors in L^2 norm and in H^1 and H^2 seminorms that are optimal in h. Denote $\chi = P^{DG} - \tilde{p}^I$.

$$\begin{aligned}a_{NS}(\chi,\chi) = &\sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p}^I)\cdot\nabla\chi + \alpha(p-\tilde{p}^I)\chi)\\&-\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]\\&-\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla(p-\tilde{p}^I)\cdot\nu_D)\chi + \sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\chi\cdot\nu_D)(p-\tilde{p}^I).\end{aligned}$$

In view of (5.2), we can write:

$$\begin{aligned}a_{NS}(\chi,\chi) = &\sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p}^I)\cdot\nabla\chi + \alpha(p-\tilde{p}^I)\chi)\\&-\sum_{k=1}^{P_h}\int_{e_k}\{(K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi] - \sum_{k=1}^{P_h}\int_{e_k}\{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}([\chi]-c_k)\\&+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I] - \sum_{e_k\in\Gamma_D}\int_{e_k}((K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k)\chi\\&-\sum_{e_k\in\Gamma_D}\int_{e_k}(\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k)(\chi-c_k) + \sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\chi\cdot\nu_D)(p-\tilde{p}^I)\end{aligned} \tag{5.3}$$

where c_k is any constant depending on e_k and $\bar{K}$ is defined by (5.1). The first two terms in (5.3) can be bounded in the following way:

$$|\sum_{j=1}^{N_h}\int_{E_j}K\nabla(p-\tilde{p}^I)\cdot\nabla\chi| \le C|\!|\!|\nabla(p-\tilde{p}^I)|\!|\!|_0 |\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0,$$

$$|\sum_{j=1}^{N_h}\int_{E_j}\alpha(p-\tilde{p}^I)\chi| \le C\|\alpha\|_\infty\|p-\tilde{p}^I\|_0\|\chi\|_0.$$

To bound the other terms, we consider the contribution from each interior edge. We assume that $e_k = \partial E^1 \cap \partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$ and we denote by B_1 and B_2 the matrices of the mappings from the reference element $\hat{E}$ onto E^1 and E^2 respectively. The third term in (5.3) is bounded by

$$\begin{aligned}|\int_{e_k}\{(K-\bar{K})\nabla(p-\tilde{p}^I)\cdot\nu_k\}[\chi]| \le &\frac{1}{2}\|K-\bar{K}\|_{\infty,E^1}\|\nabla(p-\tilde{p}^I)\cdot\nu_k\|_{0,e_k}\|[\chi]\|_{0,e_k}\\&+\frac{1}{2}\|K-\bar{K}\|_{\infty,E^2}\|\nabla(p-\tilde{p}^I)\cdot\nu_k\|_{0,e_k}\|[\chi]\|_{0,e_k}.\end{aligned}$$

It is easy to check that

$$\sup_{x\in E}\|K(x)-\bar K\| \le \sup_{x\in E}\|K(x)-\bar K\|_S,$$

where $\|K(x)\|_S^2=\sum_{i,j=1}^2 k_{ij}^2(x)$. Therefore, it suffices to look at $\|k_{ij}-\bar k_{ij}\|_{\infty,E}=\|\hat k_{ij}-\bar k_{ij}\|_{\infty,\hat E}$. But $\bar k_{ij}$ is also the average of $\hat k_{ij}$, so the mapping $\hat k_{ij}\mapsto \hat k_{ij}-\bar k_{ij}$ vanishes if $\hat k_{ij}$ is a constant function. Thus,

$$\|k_{ij}-\bar k_{ij}\|_{\infty,E}\le C\|\hat\nabla \hat k_{ij}\|_{\infty,\hat E}\le Ch_E\|\nabla k_{ij}\|_{\infty,E},\quad \forall E\in\mathcal{E}_h.$$

By a trace theorem, we have:

$$\|\nabla(p-\tilde p^I)\cdot\nu_k\|_{0,e_k}\le |e_k|^{\frac12}(\|B_1^{-1}\|\|\nabla(p-\tilde p^I)\|_{0,E^1}+\|B_1^{-1}\|\|B_1\|\|\nabla^2(p-\tilde p^I)\|_{0,E^1}),$$
$$\|[\chi]\|_{0,e_k}\le \hat C|e_k|^{\frac12}r^2(\|B_1^{-1}\|\|\chi\|_{0,E^1}+\|B_2^{-1}\|\|\chi\|_{E^2}).$$

Thus, by the approximation result (5.2) and (5.2)

$$|\sum_{k=1}^{P_h}\int_{e_k}\{(K-\bar K)\nabla(p-\tilde p^I)\cdot\nu_k\}[\chi]|\le C\frac{h^{\mu-1}}{r^{s-4}}|\!|\!|p|\!|\!|_s\|\chi\|_0.$$

The fourth term can also be bounded as follows:

$$\left|\int_{e_k}\{\bar K\nabla(p-\tilde p^I)\cdot\nu_k\}([\chi]-c_k)\right|\le\|\{\bar K\nabla(p-\tilde p^I)\cdot\nu_k\}\|_{0,e_k}\|[\chi]-c_k\|_{0,e_k}.$$

By the trace theorem,

$$\|\{\bar K\nabla(p-\tilde p^I)\cdot\nu_k\}\|_{0,e_k}\le \hat C|e_k|^{\frac12}(\|B_1^{-1}\|\|\nabla(p-\tilde p^I)\|_{0,E^1}+\|\nabla^2(p-\tilde p^I)\|_{0,E^1}$$
$$+\|B_2^{-1}\|\|\nabla(p-\tilde p^I)\|_{0,E^2}+\|\nabla^2(p-\tilde p^I)\|_{0,E^2}).$$

Take c_k as follows:

$$c_k=\frac{1}{|e_k|}\int_{e_k}[\chi]=\frac{1}{|e_k|}\int_{e_k}[P^{DG}-\tilde p^I].$$

We have

$$\begin{aligned}\|[\chi]-c_k\|_{0,e_k}&\le |e_k|^{\frac12}\|[\hat\chi]-c_k\|_{0,\hat e}\\&\le |e_k|^{\frac12}\|\frac{d}{d\hat\sigma}[\hat\chi]\|_{0,\hat e}\le \hat C|e_k|^{\frac12}r^2\left(\|\hat\nabla\hat\chi\|_{0,\hat E^1}+\|\hat\nabla\hat\chi\|_{0,\hat E^2}\right)\\&\le \hat C|e_k|^{\frac12}r^2\left(\|K^{\frac12}\nabla\chi\|_{0,E^1}+\|K^{\frac12}\nabla\chi\|_{0,E^2}\right).\end{aligned}$$

Thus,

$$|\int_{e_k}\{\bar K\nabla(p-\tilde p^I)\cdot\nu\}[\chi]|\le C(K)\hat C|e_k|r^2(\|B_1^{-1}\|\|\nabla(p-\tilde p^I)\|_{0,E^1}+\|\nabla^2(p-\tilde p^I)\|_{0,E^1\cup E^2}$$
$$+\|B_2^{-1}\|\|\nabla(p-\tilde p^I)\|_{0,E^2})\|K^{\frac12}\nabla\chi\|_{0,E^1\cup E^2}.$$

Combining the contributions from all the interior edges, we can bound the fourth term of (5.3):

$$\sum_{k=1}^{P_h} \int_{e_k} \{\bar{K}\nabla(p-\tilde{p}^I)\cdot\nu_k\}([\chi]-c_k) \leq C(K)\frac{h^{\mu-1}}{r^{s-4}} \| K^{\frac{1}{2}}\nabla\chi\|_0 \|p\|_s.$$

Now, we bound the fifth term in (5.3):

$$|\int_{e_k} \{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]| \leq C\|\{K\nabla\chi\cdot\nu_k\}\|_{0,e_k}\|[p-\tilde{p}^I]\|_{0,e_k}.$$

Since $\hat{\nabla}\hat{\chi}$ belongs to a finite-dimensional space, we have

$$\begin{aligned}\|\{\nabla\chi\cdot\nu_k\}\|_{0,e_k} &\leq \hat{C}|e_k|^{\frac{1}{2}}r^2\left(\|B_1^{-1}\|\|\hat{\nabla}\hat{\chi}\|_{0,\hat{E}^1}+\|B_2^{-1}\|\|\hat{\nabla}\hat{\chi}\|_{0,\hat{E}^2}\right)\\ &\leq \hat{C}|e_k|^{\frac{1}{2}}r^2\left(|\mathrm{det}B_1|^{-\frac{1}{2}}\|\nabla\chi\|_{0,E^1}+|\mathrm{det}B_2|^{-\frac{1}{2}}\|\nabla\chi\|_{0,E^2}\right).\end{aligned}$$

The other factor is bounded by

$$\begin{aligned}\|[p-\tilde{p}^I]\|_{0,e_k} &\leq |e_k|^{\frac{1}{2}}\|[\widehat{p-\tilde{p}^I}]\|_{0,\hat{e}}\\ &\leq \hat{C}|e_k|^{\frac{1}{2}}\left(\|\widehat{p-\tilde{p}^I}\|_{0,\hat{E}_1\cup\hat{E}_2}+\|\hat{\nabla}(\widehat{p-\tilde{p}^I})\|_{0,\hat{E}_1\cup\hat{E}2}\right)\\ &\leq \hat{C}|e_k|^{\frac{1}{2}}(|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E^1}+\|\nabla(p-\tilde{p}^I)\|_{0,E^1}\\ &\quad+|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E^2}+\|\nabla(p-\tilde{p}^I)\|_{0,E^2}).\end{aligned}$$

Thus,

$$\begin{aligned}|\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]| &\leq C|e_k|r^2\left(|\mathrm{det}B_1|^{-\frac{1}{2}}\|\nabla\chi\|_{0,E^1}+|\mathrm{det}B_2|^{-\frac{1}{2}}\|\nabla\chi\|_{0,E^2}\right)\\ &\quad(|\mathrm{det}B_1|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E^1}+\|\nabla(p-\tilde{p}^I)\|_{0,E^1}\\ &\quad+|\mathrm{det}B_2|^{-\frac{1}{2}}\|p-\tilde{p}^I\|_{0,E^2}+\|\nabla(p-\tilde{p}^I)\|_{0,E^2}).\end{aligned}$$

Hence,

$$|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\nu_k\}[p-\tilde{p}^I]| \leq C(K)\frac{h^{\mu-1}}{r^{s-4}}\|p\|_s\| K^{\frac{1}{2}}\nabla\chi\|_0.$$

Finally, the boundary terms are estimated as the interior terms, and the theorem is obtained by combining all the results together.

□

6. Conclusion

In this paper we have presented optimal *hp* convergence results for three methods for modeling elliptic problems with discontinuous spaces. Unlike the interior penalty methods, which were shown to be effective for modeling sharp fronts arising in miscible displacement in porous media, the NIPG schemes do not require the definition of problem-dependent penalties to be defined. Even though we have obtained optimal hp convergence results for the constrained NCG method, this procedure is more complicated to implement than the DG method. The latter is locally conservative. Computational results for the DG method are described in Part II of this paper.

Acknowledgements

The authors wish to thank Professors J. Tinsley Oden, Ivo Babuska and Dr. Carlos Baumann for their valuable suggestions and constructive comments.

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