Analysis of a multi-numerics/multi-physics problem

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1 Introduction and Model Problem

In this work, a numerical method for solving the coupled problem of Stokes and Darcy equations is formulated and analyzed. The coupled system arises from the study of the interaction between surface and subsurface flow. Discontinuous finite elements and mixed finite elements (MFE) are used in subregions of the subsurface domain while only discontinuous finite elements are used in the surface domain. While mixed elements are popular and efficient on regular grids, discontinuous Galerkin (DG) methods are accurate and easily implementable on highly unstructured meshes. The proposed method enables to take advantage of one of these locally conservative methods in a particular subregion of the subsurface. This work is an extension of the coupling of DG for Stokes and MFE for Darcy [8], and DG for Stokes and Darcy [7]. Similar couplings are studied in the work of Layton, Schieweck and Yotov [5] and Discacciati and Quarteroni [2, 3]. Let Ω be a domain in $\mathbb{R}^2$, subdivided into three subdomains Ω_1, Ω_2, and Ω_3. Denote by

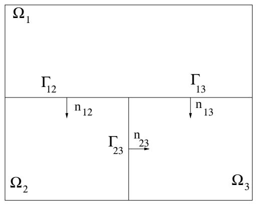

Fig. 1. Computational domain

Γ_{ij} the interface between Ω_i and Ω_j for $i < j$ (see figure 1). Define also the boundary $\Gamma_i = \partial\Omega_i \cap \partial\Omega$, for $i = 1, 2, 3$. The velocity $\boldsymbol{u}_i$ (resp. pressure p_i) denotes the restriction of the fluid velocity $\boldsymbol{u}$ (resp. pressure p) to the subdomain Ω_i. We assume that the fluid satisfies the Stokes equations in Ω_1.

$$-\nabla \cdot (2\mu \boldsymbol{D}(\boldsymbol{u}_1) - p_1 \boldsymbol{I}) = \boldsymbol{f}_1, \quad \text{in} \quad \Omega_1, \tag{1}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \text{in} \quad \Omega_1. \tag{2}$$

Here, $\boldsymbol{f}_1$ is an external force acting on the fluid, $\mu > 0$ is the constant fluid viscosity and $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the strain tensor. Let $\boldsymbol{n}$ denote the unit outward normal vector to the boundary $\partial\Omega$. The single phase flow problem is solved on the region $\Omega_2 \cup \Omega_3$.

$$\nabla \cdot \boldsymbol{u}_i = f_i, \quad \boldsymbol{u}_i = -\boldsymbol{K}\nabla p_i, \quad \text{in} \quad \Omega_i, \, i = 2, 3. \tag{3}$$

The permeability tensor $\boldsymbol{K}$ is symmetric, positive definite, bounded below and above uniformly. As boundary conditions, we consider $\boldsymbol{u} = 0$ on the Stokes boundary Γ_1, and

$\boldsymbol{K}\nabla p \cdot \boldsymbol{n} = 0$ on the Darcy boundary $\Gamma_2 \cup \Gamma_3$. At the interface, the transmissibility conditions arise from the mass conservation, the balance of forces across each interface, and the Beaver-Joseph-Saffman law [1, 9]. Denoting by $\boldsymbol{n}_{ij}$ (resp. $\boldsymbol{\tau}_{ij}$) the normal (resp. tangential) unit vector to the interface Γ_{ij}, for $i < j$, the conditions are written as:

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} = -\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, \quad \text{on} \quad \Gamma_{12}, \tag{4}$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{13} = \boldsymbol{u}_3 \cdot \boldsymbol{n}_{13}, \quad \text{on} \quad \Gamma_{13}, \tag{5}$$

$$\boldsymbol{u}_3 \cdot \boldsymbol{n}_{23} = -\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{23}, \quad p_2 = p_3, \quad \text{on} \quad \Gamma_{23}, \tag{6}$$

$$p_1 - 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{1i}) \cdot \boldsymbol{n}_{1i}) = p_i, \quad \text{on} \quad \Gamma_{1i}, \quad i = 2, 3, \tag{7}$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{1i} = -2G(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{1i}) \cdot \boldsymbol{\tau}_{1i}, \quad \text{on} \quad \Gamma_{1i}, \quad i = 2, 3. \tag{8}$$

As the pressure is unique up to an additive constant, we assume that $\int_\Omega p = 0$. A weak solution $(\boldsymbol{u}, p)$ of the coupled Stokes-Darcy equations exists (see proof in [5]). We assume that it is also a strong solution with enough regularity. Section 2 defines the discrete spaces and the numerical method. Section 3 contains the error analysis.

2 Numerical Method

The discontinuous Galerkin method is used in the region $\Omega_1 \cup \Omega_2$, while the mixed finite element method is used in the region Ω_3. For $i = 1, 2, 3$, let $\mathcal{E}_h^i$ be a non-degenerate quasi-uniform subdivision of Ω_i, let Γ_h^i be the set of interior edges and let h denote the maximum diameter of elements. We assume that the meshes match at the interface Γ_{13} match, but they may not match at the interfaces Γ_{12} and Γ_{23}. Given a fixed normal vector $\boldsymbol{n}_e$, on each interior edge, pointing from E_e^1 to E_e^2, the average $\{w\}$ and jump $[w]$ of function w is defined:

$$\{w\} = \frac{1}{2}(w|_{E_e^1}) + \frac{1}{2}(w|_{E_e^2}), \quad [w] = (w|_{E_e^1}) - (w|_{E_e^2}), \quad \forall e = \partial E_e^1 \cap \partial E_e^2.$$

On a boundary edge, the jump and average coincide with the trace of the function. For any integer $k \geq 0$, the Sobolev space on a domain $\mathcal{O}$ is denoted by $H^k(\mathcal{O}) = \{v \in L^2(\mathcal{O}) : D^m v \in L^2(\mathcal{O}), \quad \forall |m| \leq k\}$, with norm $\|\cdot\|_{k,\mathcal{O}}$. We denote by $L_0^2(\mathcal{O})$ the space of square-integrable functions with zero average, with L^2 inner-product $(\cdot,\cdot)_{\mathcal{O}}$.

Let k_1, k_2 and k_3 be positive integers. The DG discrete spaces are

$$\begin{aligned}
\boldsymbol{X}_h^1 &= \{\boldsymbol{v}_1 \in (L^2(\Omega_1))^2 : \quad \forall E \in \mathcal{E}_h^1, \quad \boldsymbol{v}_1 \in (I\!P_{k_1}(E))^2\}, \\
M_h^1 &= \{q_1 \in L^2(\Omega_2) : \quad \forall E \in \mathcal{E}_h^1, \quad q_1 \in I\!P_{k_1-1}(E)\}, \\
M_h^2 &= \{q_2 \in L^2(\Omega_2) : \quad \forall E \in \mathcal{E}_h^2, \quad q_2 \in I\!P_{k_2}(E)\}.
\end{aligned}$$

The MFE discrete spaces are the classical ones, such as the Raviart-Thomas spaces [6]. We assume that the mixed velocity spaces $\boldsymbol{X}_h^3 \subset \boldsymbol{H}(div; \Omega_3)$ contains polynomials of degree k_3 and the pressure spaces $M_h^3 \subset L^2(\Omega_3)$ polynomials of degree $k_3 - 1$. We associate to these spaces the following norms:

$$\|\boldsymbol{v}_1\|_{X_h^1}^2 = \sum_{E \in \mathcal{E}_h^1} \|\nabla \boldsymbol{v}_1\|_{0,E}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\boldsymbol{v}_1]\|_{0,e}^2 + \frac{1}{G} \sum_{e \in \Gamma_{12}} \|\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{0,e}^2,$$

$$\|q_2\|_{M_h^2}^2 = \sum_{E \in \mathcal{E}_h^2} \|\nabla q_2\|_{0,E}^2 + \sum_{e \in \Gamma_h^2} \frac{\sigma_{2,e}}{|e|} \|[q_2]\|_{0,e}^2,$$

$$\|\boldsymbol{v}_3\|_{X_h^3}^2 = \|\boldsymbol{v}_3\|_{0,\Omega_3}^2, \quad \|q_1\|_{M_h^1}^2 = \|q_1\|_{0,\Omega_1}^2, \quad \|q_3\|_{M_h^3}^2 = \|q_3\|_{0,\Omega_3}^2.$$

Here, $|e|$ denotes the measure of each edge e, the parameters $\sigma_{1,e}$ and $\sigma_{2,e}$ are positive constants defined later. Throughout the paper, C denotes a generic positive constant whose is independent of the mesh size h. We now define the global finite element spaces.

$$\boldsymbol{X}_h = \{\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_3) : \boldsymbol{v}_i \in \boldsymbol{X}_h^i, \quad \forall \eta \in \boldsymbol{X}_h^3 \cdot \boldsymbol{n}_{13} \quad \sum_{e \in \Gamma_{13}} \int_e \eta (\boldsymbol{v}_1 - \boldsymbol{v}_3) \cdot \boldsymbol{n}_{13} = 0\}, \tag{9}$$

$$M_h = \{q = (q_1, q_2, q_3) : q_i \in M_h^i, \quad q \in L_0^2(\Omega)\}. \tag{10}$$

We recall a result proved in [4] that generalize a Sobolev imbedding. There exists a constant C independent of h such that

$$\forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1, \quad \forall q_2 \in M_h^2, \quad \|\boldsymbol{v}_1\|_{0,\Omega_1} \leq C \|\boldsymbol{v}_1\|_{X_h^1}, \quad \|q_2\|_{0,\Omega_2} \leq C \|q_2\|_{M_h^2}. \tag{11}$$

We introduce the following bilinear forms $a_1 : \boldsymbol{X}_h^1 \times \boldsymbol{X}_h^1 \to \mathbb{R}$, $b_1 : \boldsymbol{X}_h^1 \times M_h^1 \to \mathbb{R}$, $a_2 : M_h^2 \times M_h^2 \to \mathbb{R}$, $a_3 : \boldsymbol{X}_h^3 \times \boldsymbol{X}_h^3 \to \mathbb{R}$ and $b_3 : \boldsymbol{X}_h^3 \times M_h^3 \to \mathbb{R}$:

$$\begin{aligned}
a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) = {} & 2\mu \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \int_e [\boldsymbol{u}_1] \cdot [\boldsymbol{v}_1] \\
& -2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\nabla \boldsymbol{u}_1 \boldsymbol{n}_e\} \cdot [\boldsymbol{v}_1] + 2\mu\epsilon_1 \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\nabla \boldsymbol{v}_1 \boldsymbol{n}_e\} \cdot [\boldsymbol{u}_1] \\
& + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12})(\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}) + \frac{\mu}{G} \sum_{e \in \Gamma_{13}} \int_e (\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{13})(\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{13}), \\
b_1(\boldsymbol{v}_1, p_1) = {} & - \sum_{E \in \mathcal{E}_h} \int_E p_1 \nabla \cdot \boldsymbol{v}_1 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{p_1\}[\boldsymbol{v}_1] \cdot \boldsymbol{n}_e, \\
a_2(p_2, q_2) = {} & \sum_{E \in \mathcal{E}_h^2} \int_E \boldsymbol{K} \nabla p_2 \cdot \nabla q_2 + \sum_{e \in \Gamma_h^2} \frac{\sigma_{2,e}}{|e|} \int_e [p_2][q_2] \\
& - \sum_{e \in \Gamma_h^2} \int_e \{\boldsymbol{K} \nabla p_2 \cdot \boldsymbol{n}_e\}[q_2] + \epsilon_2 \sum_{e \in \Gamma_h^2} \int_e \{\boldsymbol{K} \nabla q_2 \cdot \boldsymbol{n}_e\}[p_2]. \\
a_3(\boldsymbol{u}_3, \boldsymbol{v}_3) = {} & \int_{\Omega_3} \boldsymbol{K}^{-1} \boldsymbol{u}_3 \cdot \boldsymbol{v}_3, \quad b_3(\boldsymbol{v}_2, q_2) = \int_{\Omega_3} q_3 \nabla \cdot \boldsymbol{v}_3.
\end{aligned}$$

By introducing the parameters ϵ_1, ϵ_2 that take the value 1 or -1, we allow for non-symmetric or symmetric bilinear forms a_1 and a_2. We assume that in the non-symmetric case ($\epsilon_1 = 1$), the parameter $\sigma_{1,e}$ is equal to 1 and in the symmetric case ($\epsilon_1 = -1$), the parameter $\sigma_{1,e}$ is bounded below by a sufficiently large positive value. The same assumptions hold true for ϵ_2 and $\sigma_{2,e}$. Combining the subdomains bilinear forms, we define

$$A(\boldsymbol{u}, p; \boldsymbol{v}, q) = a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) + a_2(p_2, q_2) + a_3(\boldsymbol{u}_3, \boldsymbol{v}_3), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}_h, \quad \forall p, q \in M_h, \tag{12}$$

$$B(\boldsymbol{v}, q) = b_1(\boldsymbol{v}_1, q_1) - b_3(\boldsymbol{v}_3, q_3), \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \forall q \in M_h. \tag{13}$$

It is easy to show that there is a coercivity constant $\kappa > 0$ such that

$$A(\boldsymbol{v}, q; \boldsymbol{v}, q) \geq \kappa (\|\boldsymbol{v}_1\|_{X_h^1}^2 + \|\boldsymbol{v}_3\|_{X_h^3}^2 + \|q_2\|_{M_h^2}^2), \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \forall q \in M_h. \tag{14}$$

Finally, we define a bilinear form $\Lambda : (\boldsymbol{X}_h \times M_h) \times (\boldsymbol{X}_h \times M_h) \to \mathbb{R}$ acting on the interfaces Γ_{12} and Γ_{23}.

$$\Lambda(\boldsymbol{u},p;\boldsymbol{v},q) = \int_{\Gamma_{12}} (p_2 \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} - q_2 \boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}) + \int_{\Gamma_{23}} (q_2 \boldsymbol{u}_3 \cdot \boldsymbol{n}_{23} - p_2 \boldsymbol{v}_3 \cdot \boldsymbol{n}_{23}).$$

With these forms, the numerical method is: find $(\boldsymbol{U},P) \in \boldsymbol{X}_h \times M_h$ such that

$$\forall (\boldsymbol{v},q) \in \boldsymbol{X}_h \times M_h,\ A(\boldsymbol{U},P;\boldsymbol{v},q) + B(\boldsymbol{v},P) + \Lambda(\boldsymbol{U},P;\boldsymbol{v},q) = (\boldsymbol{f}_1,\boldsymbol{v}_1)_{\Omega_1} + (f_2,q_2)_{\Omega_2}, \tag{15}$$

$$\forall q \in M_h, \quad B(\boldsymbol{U},q) = -(f_3,q_3)_{\Omega_3}. \tag{16}$$

Lemma 1. *If $(\boldsymbol{u},p)$ is the strong solution of the coupled Stokes-Darcy flow problem (1)-(8), then $(\boldsymbol{u},p)$ satisfy the variational equations: for all $(\boldsymbol{v},q) \in \boldsymbol{X}_h \times M_h$*

$$A(\boldsymbol{u},p;\boldsymbol{v},q) + B(\boldsymbol{v},p) + \Lambda(\boldsymbol{u},p;\boldsymbol{v},q) = (\boldsymbol{f}_1,\boldsymbol{v}_1)_{\Omega_1} + (f_2,q_2)_{\Omega_2} - \int_{\Gamma_{13}} p_3(\boldsymbol{v}_1 - \boldsymbol{v}_3)\cdot \boldsymbol{n}_{13}, \tag{17}$$

$$B(\boldsymbol{u},q) = -(f_3,q_3)_{\Omega_3}. \tag{18}$$

Proof. Multiplying the Stokes equation (1) by $\boldsymbol{v}_1 \in \boldsymbol{X}_h^1$, integrating by parts over one element E, summing over all elements in $\mathcal{E}_h^1$ and using the regularity of the strong solution yields:

$$\begin{aligned}
&\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla \cdot \boldsymbol{v}_1) \\
&- \sum_{e\in\Gamma_h^1} \int_e \{-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + \epsilon \sum_{e\in\Gamma_h^1} \int_e \{2\mu \boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_1] \\
&- \sum_{e\in\Gamma_{12}\cup\Gamma_{13}} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 \\
&- \sum_{e\in\Gamma_1} \int_e (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n} \cdot \boldsymbol{v}_1 + \epsilon \sum_{e\in\Gamma_1} \int_e 2\mu \boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n} \cdot \boldsymbol{u}_1 = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1.
\end{aligned}$$

Using the interface conditions (7), (8), we obtain

$$a_1(\boldsymbol{u}_1,\boldsymbol{v}_1) + b_1(\boldsymbol{v}_1,p_1) + \sum_{e\in\Gamma_{12}} \int_e p_2 \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} + \sum_{e\in\Gamma_{13}} \int_e p_3 \boldsymbol{v}_1 \cdot \boldsymbol{n}_{13} = (\boldsymbol{f}_1,\boldsymbol{v}_1)_{\Omega_1}.$$

Similarly, multiplying (3) by a test function $q_2 \in M_h^2$, integrating by parts on one element, summing over all elements in $\mathcal{E}_h^2$ and using the boundary condition yields:

$$a_2(p_2,q_2) - \int_{\Gamma_{23}} \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{23} q_2 + \int_{\Gamma_{12}} \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12} q_2 = (f_2,q_2)_{\Omega_2}.$$

With the conditions (4) and (6), we obtain

$$a_2(p_2,q_2) - \sum_{e\in\Gamma_{12}} \int_e \boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} q_2 + \sum_{e\in\Gamma_{23}} \int_e \boldsymbol{u}_3 \cdot \boldsymbol{n}_{23} q_2 = (f_2,q_2)_{\Omega_2}.$$

Rewriting (3) as $\boldsymbol{K}^{-1}\boldsymbol{u}_3 = -\nabla p_3$, we easily obtain by multiplying by $\boldsymbol{v}_3 \in \boldsymbol{X}_h^3$, integrating by parts and using (6):

$$a_3(\boldsymbol{u}_3, \boldsymbol{v}_3) - b_3(\boldsymbol{v}_3, p_3) - \sum_{e \in \Gamma_{13}} \int_e p_3 \boldsymbol{v}_3 \cdot \boldsymbol{n}_{13} - \sum_{e \in \Gamma_{23}} \int_e p_2 \boldsymbol{v}_3 \cdot \boldsymbol{n}_{23} = 0.$$

Finally, the regularity of $\boldsymbol{u}$ and a simple integration of (3) give:

$$b_1(\boldsymbol{u}_1, q_1) = 0, \quad b_3(\boldsymbol{u}_3, q_3) = (f_3, q_3)_{\Omega_3}.$$

The final result is obtained by adding the previous variational equations.

We now recall some approximation properties satisfied by the spaces $\boldsymbol{X}_h$ and M_h [4, 8]. Given $\boldsymbol{v} \in (H^1(\Omega_1 \cup \Omega_3))^2$, there exists $\tilde{\boldsymbol{v}} \in \boldsymbol{X}_h$ such that

$$\forall q \in M_h, \quad B(\boldsymbol{v} - \tilde{\boldsymbol{v}}, q) = 0, \tag{19}$$

$$\forall e \in \Gamma_h^1 \cup \Gamma_1, \quad \forall \boldsymbol{q} \in (\mathbb{P}_{k_1-1}(e))^2, \quad \int_e [\tilde{\boldsymbol{v}}_1] \cdot \boldsymbol{q} = 0, \tag{20}$$

$$\forall e \in \Gamma_{12} \cup \Gamma_{13}, \quad \forall \boldsymbol{q} \in (\mathbb{P}_{k_1-1}(e))^2, \quad \int_e (\boldsymbol{v}_1 - \tilde{\boldsymbol{v}}_1) \cdot \boldsymbol{q} = 0, \tag{21}$$

$$\forall e \in \Gamma_{23}, \quad \forall \eta \in \boldsymbol{X}_h^3 \cdot \boldsymbol{n}_{23}, \quad \int_e (\boldsymbol{v}_3 - \tilde{\boldsymbol{v}}_3) \cdot \boldsymbol{n}_{23} \eta = 0. \tag{22}$$

If in addition, $\boldsymbol{v}_1 \in (H^{k_1+1}(\Omega_1))^2$, $\boldsymbol{v}_3 \in (H^{k_3+1}(\Omega_3))^2$, the approximation $\tilde{\boldsymbol{v}}$ satisfies

$$\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{0,\Omega_1} \leq C h^{k_1+1} |\boldsymbol{v}|_{k_1+1,\Omega_1}, \quad \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{X_h^1} \leq C h^{k_1} |\boldsymbol{v}|_{k_1+1,\Omega_1}, \quad \|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_{X_h^3} \leq C h^{k_3+1} |\boldsymbol{v}|_{k_3+1,\Omega_3}. \tag{23}$$

Define also the L^2 projection $\tilde{p}$ of the pressure p. If $p_i \in H^{s_i}(\Omega_i)$, we have

$$\forall q \in M_h^i, \quad \forall E \in \mathcal{E}_h^i, \quad i = 1, 2, 3, \quad \int_E q(p - \tilde{p}) = 0, \tag{24}$$

$$m = 0, 1, \quad i = 1, 2, 3, \quad \|p_i - \tilde{p}_i\|_{m,\Omega_i} \leq C h^{s_i - m} |p_i|_{s_i,\Omega_i}. \tag{25}$$

Lemma 2. *The discrete solution to (15), (16) exists and is unique.*

Proof. In a finite-dimensional setting, it suffices to show that the solution is unique. Set $f_i = 0$ and $(\boldsymbol{v}, q) = (\boldsymbol{U}, P)$ in (15), (16). This yields $A(\boldsymbol{U}, P; \boldsymbol{U}, P) + \Lambda(\boldsymbol{U}, P; \boldsymbol{U}, P) = 0$. Since $\Lambda(\boldsymbol{v}, q; \boldsymbol{v}, q) = 0$, we are left with $a_1(\boldsymbol{U}_1, \boldsymbol{U}_1) + a_2(P_2, P_2) + a_3(\boldsymbol{U}_3, \boldsymbol{U}_3) = 0$. This clearly implies that $\boldsymbol{U}_1 = \boldsymbol{U}_3 = 0$, and that P_2 is a global constant over Ω_2.

We now define

$$\bar{P} = \frac{1}{|\Omega_1| + |\Omega_3|} (\int_{\Omega_1} P + \int_{\Omega_3} P),$$

and consider $\hat{P} \in L^2(\Omega_1 \cup \Omega_3)$ such that $P = \hat{P} + \bar{P}$. Then the equation (15) becomes

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad B(\boldsymbol{v}, \hat{P}) + (P_2 - \bar{P})(\int_{\Gamma_{12}} \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} - \int_{\Gamma_{23}} \boldsymbol{v}_3 \cdot \boldsymbol{n}_{23}) = 0. \tag{26}$$

Since $\hat{P} \in L_0^2(\Omega_1 \cup \Omega_3)$, there is a function $\hat{\boldsymbol{z}} \in (H_0^1(\Omega_1 \cup \Omega_3))^2$ such that $-\nabla \cdot \hat{\boldsymbol{z}} = \hat{P}$. Denote by $\tilde{\boldsymbol{z}}$ the approximation of $\hat{\boldsymbol{z}}$ satisfying (19)-(22), and choose $\boldsymbol{v} = \tilde{\boldsymbol{z}}$ in (26). With property (19) and the regularity of $\hat{\boldsymbol{z}}$, we obtain

$$\|\hat{P}\|^2_{0,\Omega_1\cup\Omega_3} = B(\hat{\boldsymbol{z}},\hat{P}) = B(\tilde{\boldsymbol{z}},\hat{P}) = 0.$$

The equation (26), with $\hat{P}=0$ becomes:

$$(P_2-\bar{P})\left(\int_{\Gamma_{12}} \boldsymbol{v}_1\cdot\boldsymbol{n}_{12} - \int_{\Gamma_{23}} \boldsymbol{v}_3\cdot\boldsymbol{n}_{23}\right) = 0, \quad \forall \boldsymbol{v}\in \boldsymbol{X}_h.$$

Since P belongs to $L^2_0(\Omega)$, the constants P_2 and $\bar{P}$ are related by

$$|\Omega_2|P_2 + (|\Omega_1|+|\Omega_3|)\bar{P} = 0.$$

These two equations imply that $P_2=\bar{P}=0$, which concludes the proof.

We now finish this section with some trace and inverse inequalities needed for the analysis. Let E be a mesh element with diameter h_E. Let k be a positive integer. Then, there exists a constant C independent of h_E such that

$$\forall \phi\in I\!P_k(E),\quad \forall e\subset\partial E,\quad \|\phi\|_{0,e}\leq Ch_E^{-1/2}\|\phi\|_{0,E}. \tag{27}$$

$$\forall \phi\in I\!P_k(E),\quad \forall e\subset\partial E,\quad \|\nabla\phi\cdot\boldsymbol{n}_e\|_{0,e}\leq Ch_E^{-1/2}\|\nabla\phi\|_{0,E}. \tag{28}$$

3 Error estimates

Theorem 1. *Let $(\boldsymbol{u},p)$ be the solution of the coupled problem (1)-(8) such that $\boldsymbol{u}|_{\Omega_i}\in (H^{k_i+1}(\Omega_i))^2$ for $i=1,3$, $p|_{\Omega_i}\in H^{k_i}(\Omega_i)$ for $i=1,3$, and $p|_{\Omega_2}\in H^{k_2+1}(\Omega_2)$. Then the discrete solution $(\boldsymbol{U},P)$ of (15), (16) satisfies the following estimate*

$$\|\boldsymbol{U}_1-\boldsymbol{u}_1\|_{X_h^1} + \|P_2-p_2\|_{M_h^2} + \|\boldsymbol{U}_3-\boldsymbol{u}_3\|_{X_h^3} \leq Ch^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1}+|p|_{k_1,\Omega_1})$$
$$+Ch^{k_2}|p|_{k_2+1,\Omega_2} + Ch^{k_3}(|\boldsymbol{u}|_{k_3+1,\Omega_3}+|p|_{k_3,\Omega_3}).$$

Proof. Define $\tilde{\boldsymbol{u}}$ the approximation of $\boldsymbol{u}$ satisfying the properties (19)-(23) and $\tilde{p}$ the approximation of p satisfying (24)-(25) Define $\boldsymbol{\chi}=\boldsymbol{U}-\tilde{\boldsymbol{u}}$ and $\xi=P-\tilde{p}$ and $\boldsymbol{\chi}_i,\xi_i$ their restrictions to the subdomains Ω_i. Subtracting (17) and (18) from (15) and (16), choosing $\boldsymbol{v}_1=\boldsymbol{\chi}_1, q_2=\xi_2$ and $\boldsymbol{v}_3=\boldsymbol{\chi}_3$ and using the coercivity result (14), we obtain

$$\kappa(\|\boldsymbol{\chi}_1\|^2_{X_h^1}+\|\xi^2\|^2_{M_h^2}+\|\boldsymbol{\chi}_3\|^2_{X_h^3}) \leq a_1(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1,\boldsymbol{\chi}_1)+a_2(p_2-\tilde{p}_2,\xi_2)+a_3(\boldsymbol{u}_3-\tilde{\boldsymbol{u}}_3,\boldsymbol{\chi}_3)$$
$$+b_1(\boldsymbol{\chi}_1,p_1-\tilde{p}_1)-b_3(\boldsymbol{\chi}_3,p_3-\tilde{p}_3)-b_1(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1,\xi_1)+b_3(\boldsymbol{u}_3-\tilde{\boldsymbol{u}}_3,\xi_3)+\int_{\Gamma_{12}}(p_2-\tilde{p}_2)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12}$$
$$-\int_{\Gamma_{12}}\xi_2(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1)\cdot\boldsymbol{n}_{12}+\int_{\Gamma_{23}}\xi_2(\boldsymbol{u}_3-\tilde{\boldsymbol{u}}_3)\cdot\boldsymbol{n}_{23}-(p_2-\tilde{p}_2)\boldsymbol{\chi}_3\cdot\boldsymbol{n}_{23}+\int_{\Gamma_{13}}p_3(\boldsymbol{\chi}_1-\boldsymbol{\chi}_3)\cdot\boldsymbol{n}_{13}.$$

The pressure term $b_3(\boldsymbol{\chi}_3,p_3-\tilde{p}_3)$ vanishes because of (24). The terms $b_1(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1,\xi_1)$ and $b_3(\boldsymbol{u}_3-\tilde{\boldsymbol{u}}_3,\xi_3)$ vanish because of property (19) of the approximation $\tilde{\boldsymbol{u}}$. We now bound the remaining terms. By using Cauchy-Schwarz, trace inequalities, and the approximation results (23) and (25), we can bound the first three terms

$$a_1(\boldsymbol{u}_1-\tilde{\boldsymbol{u}}_1,\boldsymbol{\chi}_1)+a_2(p_2-\tilde{p}_2,\xi_2)+a_3(\boldsymbol{u}_3-\tilde{\boldsymbol{u}}_3,\boldsymbol{\chi}_3)\leq \frac{1}{8}\|\boldsymbol{\chi}_1\|^2_{X_h^1}+Ch^{2k_1}|\boldsymbol{u}|^2_{k_1+1,\Omega_1}$$
$$+\frac{1}{8}\|\xi_2\|^2_{M_h^2}+Ch^{2k_2}|p|^2_{k_2+1,\Omega_2}+\frac{1}{8}\|\boldsymbol{\chi}_3\|^2_{X_h^3}+Ch^{2k_3+2}|\boldsymbol{u}|^2_{k_3+1,\Omega_3}.$$

Because of the L^2 projection, the first pressure term is reduced and bounded as follows:

$$b_1(\boldsymbol{\chi}_1, p_1 - \tilde{p}_1) = \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{p_1 - \tilde{p}_1\}[\boldsymbol{\chi}_1] \cdot \boldsymbol{n}_e \leq \frac{1}{8} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \|[\boldsymbol{\chi}_1]\|_{0,e}^2 + Ch^{2k_1} |p|_{k_1,\Omega_1}^2.$$

The remaining terms are the interface terms. Using the approximation result (25), the trace inequality (27) and Cauchy-Schwarz's inequality, we obtain

$$\int_{\Gamma_{12}} (p_2 - \tilde{p}_2)\boldsymbol{\chi}_1 \cdot \boldsymbol{n}_{12} \leq C\|\boldsymbol{\chi}_1\|_{0,\Omega_1} h^{k_2} |p|_{k_2+1,\Omega_2} \leq \frac{1}{8}\|\boldsymbol{\chi}_1\|_{X_h^1}^2 + Ch^{2k_2}|p|_{k_2+1,\Omega_2}^2.$$

Similarly,

$$\int_{\Gamma_{23}} (p_2 - \tilde{p}_2)\boldsymbol{\chi}_3 \cdot \boldsymbol{n}_{23} \leq \frac{1}{8}\|\boldsymbol{\chi}_3\|_{M_h^3}^2 + Ch^{2k_2}|p|_{k_2+1,\Omega_2}^2.$$

With the bound (11) and the approximation result (23), we have

$$\int_{\Gamma_{12}} \xi_2(\boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1) \cdot \boldsymbol{n}_{12} \leq \frac{1}{8}\|\xi_2\|_{M_h^2}^2 + Ch^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

The approximation result (23), the inverse inequality (27) and the bound (11) give

$$\int_{\Gamma_{23}} \xi_2(\boldsymbol{u}_3 - \tilde{\boldsymbol{u}}_3) \cdot \boldsymbol{n}_{23} \leq h^{-1/2}\|\xi_2\|_{0,\Omega_2} h^{k_3-1/2}|\boldsymbol{u}_3|_{k_3+1,\Omega_3} \leq \frac{1}{8}\|\xi_2\|_{M_h^2}^2 + Ch^{2k_3}|\boldsymbol{u}_3|_{k_3+1,\Omega_3}.$$

Define $\tilde{p}_3^e$ the L^2 projection of p_3 with respect to the L^2 inner product on the edge e. Since $\boldsymbol{\chi}$ belongs to $\boldsymbol{X}_h$ and by definition of the projection, we have

$$\int_{\Gamma_{13}} p_3(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_3) \cdot \boldsymbol{n}_{13} = \sum_{e \in \Gamma_{13}} \int_e (p_3 - \tilde{p}_3^e)(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_3) \cdot \boldsymbol{n}_{13} = \sum_{e \in \Gamma_{13}} \int_e (p_3 - \tilde{p}_3^e)\boldsymbol{\chi}_1 \cdot \boldsymbol{n}_{13}.$$

Assume that each edge e of Γ_{13} is shared by the elements $E_e^1 \in \mathcal{E}_h^1$ and $E_e^3 \in \mathcal{E}_h^3$. Define the constant $c_e = |e|^{-1} \int_e \boldsymbol{\chi}_3 \cdot \boldsymbol{n}_{13}$.

$$\begin{aligned}\sum_{e \in \Gamma_{13}} \int_e (p_3 - \tilde{p}_3^e)\boldsymbol{\chi}_1 \cdot \boldsymbol{n}_{13} &= \sum_{e \in \Gamma_{13}} \int_e (p_3 - \tilde{p}_3^e)(\boldsymbol{\chi}_1 \cdot \boldsymbol{n}_{13} - c_e) \leq Ch^{k_3} \sum_{e \in \Gamma_{13}} |p_3|_{k_3,E_e^3} \|\nabla\boldsymbol{\chi}_1\|_{0,E_e^1} \\ &\leq \frac{1}{8}\|\nabla\boldsymbol{\chi}\|_{0,\Omega_1}^2 + Ch^{2k_3}|p_3|_{k_3,\Omega_3}^2.\end{aligned}$$

The theorem is obtained by combining all bounds.

Theorem 2. *Under the assumptions of Theorem 1 and the additional assumption that for any element $E \in \mathcal{E}_h^3$, the edge $\partial E \cap \Gamma_{23}$ belongs to only one element of $\mathcal{E}_h^2$, the approximation P satisfies the error estimate*

$$\|P - p\|_{0,\Omega} \leq Ch^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) + Ch^{k_2}|p|_{k_2+1,\Omega_2} + Ch^{k_3}(|\boldsymbol{u}|_{k_3+1,\Omega_3} + |p|_{k_3,\Omega_3}).$$

Proof. Subtracting (17) from (15), the error equation is

$$\begin{aligned}\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad B(\boldsymbol{v}, \xi) &= a_1(\boldsymbol{u}_1 - \boldsymbol{U}_1, \boldsymbol{v}_1) + a_3(\boldsymbol{u}_3 - \boldsymbol{U}_3, \boldsymbol{v}_3) \\ &+ \int_{\Gamma_{12}} (p_2 - P_2)\boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} - \int_{\Gamma_{23}} (p_2 - P_2)\boldsymbol{v}_3 \cdot \boldsymbol{n}_{23} + \int_{\Gamma_{12}} p_3(\boldsymbol{v}_1 - \boldsymbol{v}_3) \cdot \boldsymbol{n}_{13}. \qquad (29)\end{aligned}$$

Define $\bar{\xi} = (|\Omega_1| + |\Omega_3|)^{-1} \int_{\Omega_1 \cup \Omega_3} \xi$ and the function $\hat{\xi} \in L_0^2(\Omega_1 \cup \Omega_3)$ by $\hat{\xi} = \xi - \bar{\xi}$.

$$B(\boldsymbol{v}, \xi) = B(\boldsymbol{v}, \hat{\xi}) + B(\boldsymbol{v}, \bar{\xi}) = B(\boldsymbol{v}, \hat{\xi}) - \bar{\xi}\Big(\int_{\Gamma_{12}} \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12} - \int_{\Gamma_{23}} \boldsymbol{v}_3 \cdot \boldsymbol{n}_{23}\Big).$$

Let $\hat{\boldsymbol{v}} \in (H_0^1(\Omega_1 \cup \Omega_3))^2$ such that $-\nabla \cdot \hat{\boldsymbol{v}} = \hat{\xi}$ and $\|\hat{\boldsymbol{v}}\|_{X_h^1} + \|\hat{\boldsymbol{v}}\|_{X_h^3} \leq C\|\hat{\xi}\|_{0,\Omega_1 \cup \Omega_3}$. Choose in (29) $\boldsymbol{v} = \tilde{\boldsymbol{v}}^h$ the approximation of $\hat{\boldsymbol{v}}$, defined by (19)-(23):

$$\|\hat{\xi}\|_{0,\Omega_1 \cup \Omega_3}^2 = a_1(\boldsymbol{u}_1 - \boldsymbol{U}_1, \tilde{\boldsymbol{v}}_1^h) + a_3(\boldsymbol{u}_3 - \boldsymbol{U}_3, \tilde{\boldsymbol{v}}_3^h)$$
$$+ \int_{\Gamma_{12}} (p_2 - P_2)\tilde{\boldsymbol{v}}_1^h \cdot \boldsymbol{n}_{12} - \int_{\Gamma_{23}} (p_2 - P_2)\tilde{\boldsymbol{v}}_3^h \cdot \boldsymbol{n}_{23} + \int_{\Gamma_{12}} p_3(\tilde{\boldsymbol{v}}_1^h - \tilde{\boldsymbol{v}}_3^h) \cdot \boldsymbol{n}_{13}.$$

The additional assumption for the meshes at Γ_{23} is needed to bound $\int_{\Gamma_{23}} (p_2 - P_2)\tilde{\boldsymbol{v}}_3^h \cdot \boldsymbol{n}_{23}$. Using the fact that $\int_e \tilde{\boldsymbol{v}}_3^h \cdot \boldsymbol{n}_{23} = 0$ for $e = \partial E \cap \Gamma_{23}$ and $E \in \mathcal{E}_h^3$, and the inequality $\|\tilde{\boldsymbol{v}}^h\|_{X_h^1} + \|\tilde{\boldsymbol{v}}^h\|_{X_h^3} \leq C\|\hat{\xi}\|_{0,\Omega_1 \cup \Omega_3}$, we obtain:

$$\|\hat{\xi}\|_{0,\Omega_1 \cup \Omega_3} \leq C\|\boldsymbol{u}_1 - \boldsymbol{U}_1\|_{X_h^1} + C\|\boldsymbol{u}_3 - \boldsymbol{U}_3\|_{X_h^3} + C\|p_2 - P_2\|_{M_h^2} + Ch^{2k_3}|p_3|_{k_3,\Omega_3}^2.$$

We now bound $\|\bar{\xi}\|_{0,\Omega_2 \cup \Omega_3} = \bar{\xi}(|\Omega_1| + |\Omega_3|)^{1/2}$. Since $P \in L_0^2(\Omega)$, we have

$$\bar{\xi} = \frac{-1}{|\Omega_1| + |\Omega_3|} \int_{\Omega_2} (P_2 - \tilde{p}_2) \leq \frac{|\Omega_2|}{|\Omega_1| + |\Omega_3|} \|P_2 - \tilde{p}_2\|_{0,\Omega_2} \leq C\|P_2 - p_2\|_{X_h^2} + Ch^{k_2}|p_2|_{k_2+1,\Omega_2}.$$

Combining the results of Theorem 1 with the bounds above, give the optimal error bound.

As a concluding remark, one can introduce Lagrange multipliers on the interfaces, and obtain an equivalent method. This allows the decoupling of each subdomain, which may be advantageous for a parallel implementation.

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