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A PRIORI ERROR ESTIMATES FOR FINITE ELEMENT METHODS BASED ON DISCONTINUOUS APPROXIMATION SPACES FOR ELLIPTIC PROBLEMS*

BÉATRICE RIVIÈRE†, MARY F. WHEELER†, AND VIVETTE GIRAULT‡

Abstract. We analyze three discontinuous Galerkin approximations for solving elliptic problems in two or three dimensions. In each one, the basic bilinear form is nonsymmetric: the first one has a penalty term on edges, the second has one constraint per edge, and the third is totally unconstrained. For each of them we prove *hp* error estimates in the H^1 norm, optimal with respect to h, the mesh size, and nearly optimal with respect to p, the degree of polynomial approximation. We establish these results for general elements in two and three dimensions. For the unconstrained method, we establish a new approximation result valid on simplicial elements. L^2 estimates are also derived for the three methods.

Key words. discontinuous Galerkin methods, elliptic equations, error estimates, penalties

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1. Introduction.

Over the past five years, discontinuous Galerkin methods have become very widely used for solving a large range of computational fluid problems. They are preferred over more standard continuous methods because of their flexibility in approximating globally rough solutions, their local mass conservation, their possible definition on unstructured meshes, their potential for error control and mesh adaptation, and their straightforward applications to anisotropic materials. Another interesting aspect of these methods is that they can employ polynomials of degree k ($\mathbb{P}_k$) on quadrilaterals, whose dimension is substantially lower than that of Q_k, the standard space of functions on quadrilaterals. Similarly, they lead to smaller systems of equations than the locally conservative mixed finite element methods and do not require the solution of saddle-point problems.

In this paper, we discuss three numerical algorithms for elliptic problems which employ discontinuous approximation spaces. The three methods are called the nonsymmetric interior penalty Galerkin (NIPG) method, the nonsymmetric constrained Galerkin (NCG) method, and the discontinuous Galerkin (DG) method. The three algorithms are closely related in that the underlying bilinear form for all three is the same and is nonsymmetric.

In the NIPG method, we modify the bilinear form of the interior penalty Galerkin method treated by Douglas and Dupont [9], Wheeler [18], Arnold [2], and Wheeler and Darlow [19]. Here, we antisymmetrize the bilinear form by changing the sign of one term and as a consequence we require only a positive penalty, whereas the proofs in [18] assume that the penalty is bounded below by a problem-dependent constant. We choose the penalty term in such a way that we can establish both H^1 and L^2 optimal convergence rates.

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†The Center for Subsurface Modeling, Texas Institute for Computational and Applied Mathematics, The University of Texas, Austin, TX 78712 (riviere@ticam.utexas.edu, mfw@ticam.utexas.edu).

‡Université Pierre et Marie Curie, Paris VI, Laboratoire Analyse Numérique, 4, place Jussieu, F-75230 Paris Cedex 05, France (girault@ann.jussieu.fr).

In the NCG method, we impose that the jumps on each edge (or face) of the subdivision have integral average zero. One constraint per edge (or face) is sufficient for again proving optimal rates of convergence in the H^1 and L^2 norms.

The DG method is totally unconstrained and for that reason its error analysis is more delicate. We recover full accuracy in the H^1 norm by introducing a special interpolant, local to each element, whose average flux is orthogonal to constants on each edge (or face). This interpolant is constructed first for constant tensors and then extended to $W^{1,\infty}$ tensors by an approximation result.

The results presented here extend significantly the work in [15]. To our knowledge these results are new.

The DG method with this bilinear form was first introduced by Baumann [6], and Oden, Babuška, and Baumann in [13]. In [6], Baumann showed that the method is elementwise conservative and he proved a stability result in one dimension for polynomials of at least degree three. However, his proof is based on an inf-sup condition, which seems problematic in higher dimensions. Our approach uses special approximation results and trace theorems but no isomorphisms.

This paper consists of five sections after this introduction. In section 2, we list the notation, state the problem, and describe the formulation of the three methods. In sections 3, 4, and 5, the proofs of the error estimates of the three methods described in section 2 are respectively given. A numerical example is presented in section 6. In the last section, we present some conclusions.

2. Statement of the problem. Finite element methods.

2.1. Notation and approximation properties. Let Ω be a polygonal domain in $\mathbb{R}^n$. Everything here applies also to $n = 1$, but to simplify we consider only $n = 2$ or 3. Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a nondegenerate quasi-uniform subdivision of Ω, where E_j is a triangle or a quadrilateral if $n = 2$, or a tetrahedron if $n = 3$. The nondegeneracy requirement (also called regularity) is that there exists $\rho > 0$ such that if h_j is the diameter of E_j, then E_j contains a ball of radius ρh_j in its interior. However, for quadrilaterals, the requirement is a little bit stronger: the quadrilateral is convex and each of its subtriangles contains a ball of radius ρh_j. Let $h = \max\{h_j, j = 1, \ldots, N_h\}$. The quasi-uniformity requirement is that there is $\tau > 0$ such that $\frac{h}{h_j} \leq \tau$ for all $j \in 1, \ldots, N_h$. This quasi-uniformity assumption is used for deriving error estimates in terms of the degree of polynomials (i.e., for the p-version). For deriving error estimates in terms of h (i.e., for the h-version), we need only a regular subdivision. We denote the edges (resp., faces for $n = 3$) of $\mathcal{E}_h$ by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_{h+1}}, \ldots, e_{M_h}\}$, where $e_k \subset \Omega, 1 \leq k \leq P_h$, and $e_k \subset \partial\Omega,\ P_h + 1 \leq k \leq M_h$. With each edge (or face) e_k, we associate a unit normal vector $\boldsymbol{\nu}_k$. For $k > P_h$, $\boldsymbol{\nu}_k$ is taken to be the unit outward vector normal to $\partial\Omega$. For $s \geq 0$ and $p \geq 1$, let

$$W^{s,p}(\mathcal{E}_h) = \{v \in L^p(\Omega) : v|_{E_j} \in W^{s,p}(E_j),\ j = 1, \ldots, N_h\},$$

and we denote it by $H^s(\mathcal{E}_h)$ when $p = 2$.

We now define the average and the jump for $\phi \in H^s(\mathcal{E}_h),\ s > \frac{1}{2}$. Let $1 \leq k \leq P_h$; for $e_k = \partial E_i \cap \partial E_j$ with $\boldsymbol{\nu}_k$ exterior to E_i, set

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \quad [\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}.$$

We consider a matrix-valued function $K = (k_{ij})_{1\le i,j\le n}$ and a nonnegative scalar function α. We assume

$$K \in W^{1,4}(\mathcal{E}_h), \quad \alpha \in L^\infty(\Omega).$$

We also assume that K is symmetric positive definite in $\bar{\Omega}$ uniformly with respect to x. This means that if $\gamma_{\min}(x) \le \gamma_{\max}(x)$ are the smallest and the largest eigenvalues at any point x of K, then there exist $\gamma_0 > 0$ and $\gamma_1 > 0$ such that

$$\forall\, x \in \bar{\Omega}, \quad \gamma_{\min}(x) \ge \gamma_0, \quad \gamma_{\max}(x) \le \gamma_1. \tag{2.1}$$

The L^2 inner product is denoted by $(.,.)$. The usual Sobolev norm of H^s on $E \subset \mathbb{R}^n$ is denoted by $\|\cdot\|_{s,E}$. We define the following "broken" norms for positive s:

$$|||\Phi|||_s = \left(\sum_{j=1}^{N_h} \|\Phi\|_{s,E_j}^2 \right)^{\frac{1}{2}}.$$

The reader can refer to Adams [1] and to Lions and Magenes [12] for the properties of Sobolev spaces.

Let r be a positive integer. The finite element subspace is taken to be

$$\mathcal{D}_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} \mathbb{P}_r(E_j),$$

where by $\mathbb{P}_r(E_j)$ we mean that the polynomials are defined on E_j and not on a reference set $\hat{E}$. This distinction is unnecessary when E_j is a triangle, a tetrahedron, or a parallelogram because the transformation from $\hat{E}$ to E_j is affine. But it is important when E_j is a quadrilateral: in this case, it is shown by [3] that $\mathbb{P}_r(\hat{E})$ does not have optimal approximation properties, whereas it is shown by [10] that $\mathbb{P}_r(E_j)$ has optimal approximation properties.

We use the following *hp* approximation properties, proven in [4, 5]. Let $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$. Then there exists a constant C depending on s, τ, ρ but independent of ϕ, r, and h and a sequence $z_r^h \in \mathbb{P}_r(E_j),\ r = 1, 2, \ldots,$ such that for any $0 \le q \le s$

$$\|\phi - z_r^h\|_{q,E_j} \le C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{s,E_j}, \qquad s \ge 0, \tag{2.2}$$

$$\|\phi - z_r^h\|_{0,\gamma_i} \le C \frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{1}{2}, \tag{2.3}$$

where $\mu = \min(r+1, s)$ and γ_i is an edge or a face on ∂E_j. Using the same technique as in [4], it can be shown that we have the additional approximation result

$$\|\phi - z_r^h\|_{1,\gamma_i} \le C \frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|\phi\|_{s,E_j}, \qquad s > \frac{3}{2}. \tag{2.4}$$

Finally, we shall often use the following bounds with respect to h that extend to three-dimensional formulae (2.4), (2.5) of [2]:

$$\forall\, \phi \in H^1(E_j), \quad \|\phi\|_{0,\gamma_i}^2 \le C \left(\frac{1}{h_j} |\phi|_{0,E_j}^2 + h_j |\phi|_{1,E_j}^2 \right), \tag{2.5}$$

$$\forall\, \phi \in H^2(E_j), \quad \left\| \frac{\partial \phi}{\partial \nu} \right\|_{0,\gamma_i}^2 \le C \left(\frac{1}{h_j} |\phi|_{1,E_j}^2 + h_j |\phi|_{2,E_j}^2 \right). \tag{2.6}$$

We shall also use the following inverse inequalities.

LEMMA 2.1. *Let E be an element in $\mathbb{R}^n$ $(n = 2, 3)$ of diameter h_E, let e_k be an edge or a face of E, and let $\boldsymbol{\nu}_k$ be a unit vector normal to e_k. Then, if χ is a polynomial of degree r on E, there exists a constant C independent of E and r such that*

$$\|\chi\|_{0,e_k} \le Crh_E^{-\frac{1}{2}}\|\chi\|_{0,E}, \tag{2.7}$$

$$\|\nabla\chi\cdot\boldsymbol{\nu}_k\|_{0,e_k} \le Crh_E^{-\frac{1}{2}}\|\nabla\chi\|_{0,E}. \tag{2.8}$$

Proof. Let us briefly recall the proof for the reader's convenience. In all the proofs, C will be a generic constant with different values on different places, that is, independent of h and r. We consider the case where $n = 2$; the proof extends easily to three dimensions. The factor $h_E^{-\frac{1}{2}}$ results from a straightforward scaling argument, and we have only to exhibit the factor r in a reference setting. First, let us consider the case where E is the reference square $[-1, 1]\times[-1, 1]$, e is the segment $[-1, 1]\times\{1\}$, and p is a polynomial of $\mathbb{Q}_r$, i.e., a polynomial of degree at most r in each variable. Then, expanding p in terms of Legendre polynomials

$$p(x_1, x_2) = \sum_{m,n}\alpha_{m,n}L_m(x_1)L_n(x_2),$$

we obtain on one hand

$$\|p\|_{0,E}^2 = \sum_{m,n}\frac{\alpha_{m,n}^2}{(m+\frac{1}{2})(n+\frac{1}{2})}.$$

On the other hand,

$$p(x_1, 1) = \sum_{m,n}\alpha_{m,n}L_m(x_1).$$

Therefore,

$$\|p\|_{0,e}^2 = \sum_m\left(\sum_n\alpha_{m,n}^2\right)\frac{1}{m+\frac{1}{2}}$$

and by the Cauchy–Schwarz inequality

$$\|p\|_{0,e}^2 \le \sum_m\frac{1}{m+\frac{1}{2}}\left(\sum_n\frac{\alpha_{m,n}^2}{n+\frac{1}{2}}\right)\left(\sum_n\left(n+\frac{1}{2}\right)\right) \le Cr^2\|p\|_{0,E}^2.$$

Hence,

$$\|p\|_{0,e} \le Cr\|p\|_{0,E}. \tag{2.9}$$

This inequality carries over to a polynomial of $\mathbb{P}_r$ in a quadrilateral because the image of $\mathbb{P}_r$ on the reference square belongs to $\mathbb{Q}_r$.

Finally, in the case of triangles, we choose for reference triangle E the equilateral triangle with basis $e = [-1, 1]\times\{0\}$ and third vertex at $(0, \sqrt{3})$. Then we truncate this triangle by removing the smaller triangle above the line $x_2 = 1$. The remaining area is a trapezoid, say, $\tilde{T}$, that is mapped onto the reference square $[-1, 1]\times[-1, 1]$ by an invertible bilinear mapping. The image of p by this transformation, say, $\tilde{p}$, belongs to $\mathbb{Q}_r$, and therefore (2.9) is valid for $\tilde{p}$ in the square. Then passing back to the trapezoid $\tilde{T}$, we obtain

$$\|p\|_{0,e} \le Cr\|p\|_{0,\tilde{T}} \le Cr\|p\|_{0,E}. \qquad \square$$

2.2. Problem and nonsymmetric bilinear form. Let the boundary of the domain $\partial\Omega$ be the union of two disjoint sets Γ_D and Γ_N. We denote $\boldsymbol{\nu}$ the unit normal vector to each edge of $\partial\Omega$ exterior to Ω. For f given in $L^2(\Omega)$, p_0 given in $H^{\frac{1}{2}}(\Gamma_D)$, and g given in $L^2(\Gamma_N)$, we consider the following elliptic problem:

$$\begin{aligned} -\nabla\cdot(K\nabla p)+\alpha p &= f \quad \text{in} \quad \Omega, \\ p &= p_0 \quad \text{on} \quad \Gamma_D, \\ K\nabla p\cdot\boldsymbol{\nu} &= g \quad \text{on} \quad \Gamma_N. \end{aligned} \tag{2.10}$$

With the above assumptions on K and α, problem (2.10) has a unique solution in $H^1(\Omega)$ when $|\Gamma_D|>0$ or when $\alpha\neq 0$. On the other hand, when $\partial\Omega=\Gamma_N$ and $\alpha=0$, problem (2.10) has a solution in $H^1(\Omega)$ which is unique up to an additive constant, provided $\int_\Omega f=-\int_{\partial\Omega} g$ (see [11]) . For K in $W^{1,4}(\mathcal{E}_h)$ and $\psi,\phi\in H^2(\mathcal{E}_h)$, we consider the nonsymmetric bilinear form

$$\begin{aligned} a_{NS}(\psi,\phi)=&\sum_{j=1}^{N_h}\int_{E_j}(K\nabla\psi\nabla\phi+\alpha\psi\phi) \\ &-\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\boldsymbol{\nu}_k\}[\phi]+\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\boldsymbol{\nu}_k\}[\psi] \\ &-\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)\phi+\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\phi\cdot\boldsymbol{\nu}_k)\psi. \end{aligned}$$

We define the linear form

$$L(\phi)=(f,\phi)+\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\phi\cdot\boldsymbol{\nu}_k)p_0+\sum_{e_k\in\Gamma_N}\int_{e_k}\phi g.$$

2.3. Finite element schemes. First we introduce the following interior and boundary penalty term:

$$J_0^{\sigma,\beta}(\phi,\psi)=\sum_{k=1}^{P_h}\frac{r\sigma_k}{|e_k|^\beta}\int_{e_k}[\phi][\psi]+\sum_{e_k\in\Gamma_D}\frac{r\sigma_k}{|e_k|^\beta}\int_{e_k}\phi\psi,$$

where σ is a discrete positive function that takes the constant value σ_k on the edge or face e_k and is bounded below by $\sigma_0>0$, above by σ_m, $|e_k|$ denotes the measure of e_k, and $\beta\geq\frac{1}{2}$ is a real number. The Galerkin approximation P^{NIPG} in $\mathcal{D}_r(\mathcal{E}_h)$ solves the following discrete problem:

$$\begin{aligned} a_{NS}(P^{\mathrm{NIPG}},v)+J_0^{\sigma,\beta}(P^{\mathrm{NIPG}},v)&=L(v) \\ &+\sum_{e_k\in\Gamma_D}\frac{r\sigma_k}{|e_k|^\beta}\int_{e_k}p_0 v \quad \forall\, v\in\mathcal{D}_r(\mathcal{E}_h). \end{aligned} \tag{2.11}$$

The next lemma establishes the consistency of this scheme.

LEMMA 2.2. *Under the above assumptions on the data (including K), and if the solution p of problem* (2.10) *satisfies $p\in H^2(\mathcal{E}_h)$, then p satisfies*

$$a_{NS}(p,v)+J_0^{\sigma,\beta}(p,v)=L(v)+\sum_{e_k\in\Gamma_D}\frac{r\sigma_k}{|e_k|^\beta}\int_{e_k}p_0 v \quad \forall\, v\in H^2(\mathcal{E}_h). \tag{2.12}$$

Conversely, if $p \in H^1(\Omega) \cap H^2(\mathcal{E}_h)$ satisfies (2.12), *then p is the solution of problem* (2.10).

Proof. Since (2.10) is a linear problem, we can proceed as follows. First we observe that (2.12) has at most one solution p in $H^2(\mathcal{E}_h)$ (or $H^2(\mathcal{E}_h)/\mathbb{R}$ if $|\Gamma_D| = 0$ and $\alpha \equiv 0$). Indeed, if p solves (2.12) with zero data, then p is constant in each E_j and $[p] = 0$ on each interior edge e_k. Hence p is constant in Ω and if $|\Gamma_D| > 0$ or $\alpha \neq 0$, then this constant is zero. Otherwise, p is unique in $H^2(\mathcal{E}_h)/\mathbb{R}$. Thus, it suffices to prove that if the unique solution p of (2.10) belongs to $H^2(\mathcal{E}_h)$, then it also solves (2.12). For this, let v be an element in $H^2(\mathcal{E}_h)$. We multiply the first equation of (2.10) by v, integrate on E_j, and sum all over j.

$$\sum_{j=1}^{N_h} \int_{E_j} (K\nabla p \nabla v + \alpha p v) - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla p \cdot \boldsymbol{\nu}_k\}[v] - \int_{\partial\Omega} (K\nabla p \cdot \boldsymbol{\nu}) v = (f, v).$$

Using the Neumann boundary conditions, we get

$$\sum_{j=1}^{N_h} \int_{E_j} (K\nabla p \nabla v + \alpha p v) - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla p \cdot \boldsymbol{\nu}_k\}[v] - \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla p \cdot \boldsymbol{\nu}_k) v = (f, v) + \sum_{e_k \in \Gamma_N} \int_{e_k} g v.$$

We add $\sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla v \cdot \boldsymbol{\nu}_k) p$ and $\sum_{e_k \in \Gamma_D} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} pv$ to both sides, use the Dirichlet boundary condition, and note that $[p] = 0$. We clearly have (2.12). ∎

We note that on each element, the mass conservation for the NIPG method can be written as

$$\int_{E_j} \alpha P^{\mathrm{NIPG}} - \int_{\partial E_j \setminus \Gamma_N} \{K\nabla P^{\mathrm{NIPG}} \cdot \boldsymbol{\nu}_{\partial E_j}\} + \sum_{e_k \in \partial E_j \setminus \partial\Omega} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} [P^{\mathrm{NIPG}}][1] + \sum_{e_k \in \partial E_j \cap \Gamma_D} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} P^{\mathrm{NIPG}} = \int_{E_j} f + \int_{\partial E_j \cap \Gamma_N} g + \sum_{e_k \in \partial E_j \cap \Gamma_D} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} p_0.$$

The constrained discrete space is defined as follows:

$$\mathcal{D}_r^*(\mathcal{E}_h) = \left\{ v \in \prod_{j=1}^{N_h} P_r(E_j) : \int_{e_k} [v] = 0 \quad \forall\, k = 1, \ldots, P_h \right\}.$$

The discrete approximation P^{NCG} in $\mathcal{D}_r^*(\mathcal{E}_h)$ satisfies

$$a_{NS}(P^{\mathrm{NCG}}, v) = L(v) \quad \forall\, v \in \mathcal{D}_r^*(\mathcal{E}_h). \tag{2.13}$$

The consistency of this scheme is a consequence of Lemma 2.2.

The discontinuous Galerkin approximation P^{DG} in $\mathcal{D}_r(\mathcal{E}_h)$ satisfies the formulation

$$a_{NS}(P^{\mathrm{DG}}, v) = L(v) \quad \forall\, v \in \mathcal{D}_r(\mathcal{E}_h). \tag{2.14}$$

The fact that this scheme is consistent with problem (2.10) has been shown by Baumann [6].

LEMMA 2.3. *If $r \geq 1$, the discrete solution of each of the three methods exists and is unique, with one exception: if α is not bounded away from zero, then in the DG scheme, we assume that $r \geq 2$ and the elements are not quadrilaterals.*

Proof. Indeed, since it is a square system of linear equations in finite dimension, it suffices to show uniqueness of the solution.

In the NIPG case, choose $f = 0$, $p_0 = 0$, $g = 0$, and $v = P^{\mathrm{NIPG}}$ in (2.11). Then P^{NIPG} is piecewise constant on $\mathcal{E}_h$. The "interior" penalty terms imply that P^{NIPG} is one constant over Ω and the boundary penalty term allows us to conclude that P^{NIPG} is the zero constant if $|\Gamma_D| > 0$. If $|\Gamma_D| = 0$ and $\alpha \neq 0$, then we can also conclude that P^{NIPG} is zero. Otherwise, P^{NIPG} is unique up to an additive constant.

In the DG case, choose $f = 0$, $p_0 = 0$, $g = 0$, and $v = P^{\mathrm{DG}}$ in (2.14). Then, P^{DG} is piecewise constant on $\mathcal{E}_h$. If α is bounded below by a positive constant, then P^{DG} is the zero constant on Ω. If α is not bounded below by a positive constant, then in order to prove that P^{DG} is globally constant in Ω, we need to construct a test function v on a given element E_j such that the quantity $\int_{e_k} K\nabla v \cdot \boldsymbol{\nu}_k$ is given on one edge (or face) of E_j and vanishes on the other edges (or faces). If K is constant in each E_j, one can construct such a test function on a triangle, parallelogram, or tetrahedron, by following a similar argument as in Theorem 5.1. If K is not constant in each E_j, we assume that h is small enough and the existence of this test function follows from Corollary 5.2. Then, the proof ends as in the NIPG case.

Finally, in the NCG case, choose $f = 0$, $p_0 = 0$, $g = 0$, and $v = P^{\mathrm{NCG}}$ in (2.13). Then, P^{NCG} is piecewise constant on $\mathcal{E}_h$. The constraint on the discrete space implies that P^{NCG} is globally constant over Ω and we finish the proof as above.

Note that if $|\Gamma_D| = 0$ and $\alpha \equiv 0$, then for the three methods, the discrete solution is unique up to an additive constant. ∎

Remark. The statement of Lemma 2.3 is possibly true for the DG scheme on arbitrary quadrilaterals when α is not bounded away from zero; however, the construction of an adequate test function is more problematic. On the other hand, the statement is not true for the DG scheme with $r = 1$ and α not bounded away from zero.

3. A priori error estimates for NIPG method. In this section, we derive a priori energy error estimates for the problem with mixed boundary conditions. These estimates are optimal both in h and r if the continuous polynomials are contained in the discrete "broken" space (which is the case for triangles in two dimensions and tetrahedra in three dimensions). In the case of the Neumann problem ($\Gamma_N = \partial\Omega$) on triangles or tetrahedra, one can obtain fully optimal hr convergence results for any $\beta \geq (n-1)^{-1}$, $n = 2, 3$. In Theorem 3.2, L^2 error estimates are proved in the case of pure Neumann boundary conditions and they are optimal in h and suboptimal in r.

THEOREM 3.1. *Under the assumptions of Lemma 2.2 and if $p \in H^s(\mathcal{E}_h)$ and $\beta = (n-1)^{-1}$, we have the following: If $\alpha \equiv 0$, then*

$$\|\!| K^{\frac{1}{2}} \nabla (P^{\mathrm{NIPG}} - p) |\!\|_0 \leq C \left(\frac{1}{\sigma_0}, K \right) \frac{h^{\mu-1}}{r^{s-1-\delta}} |\!| p |\!|_s.$$

If $\alpha \geq \alpha_0 > 0$, then

$$|\!| P^{\mathrm{NIPG}} - p |\!|_1 \leq C \left(\frac{1}{\sigma_0}, K, \|\alpha\|_\infty \right) \frac{h^{\mu-1}}{r^{s-1-\delta}} |\!| p |\!|_s,$$

$$J_0^{\sigma,\beta}(P^{\mathrm{NIPG}} - p, P^{\mathrm{NIPG}} - p) \leq C \frac{h^{2\mu-2}}{r^{2s-2-2\delta}} |\!| p |\!|_s^2, \tag{3.1}$$

where $\mu = \min(r+1, s)$, $r \geq 1$, $s \geq 2$, *and* $\delta = 0$ *in the case of triangles or tetrahedra when* $|\Gamma_D| = 0$. *In the general case,* $\delta = 1/2$. *In the case of the pure Neumann problem on triangles or tetrahedra, these results are valid for any* $\beta \geq (n-1)^{-1}$.

Proof. From Lemma 2.2, we have

$$a_{NS}(p, v) + J_0^{\sigma,\beta}(p, v) = L(v) + \sum_{e_k \in \Gamma_D} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} p_0 v \quad \forall\, v \in \mathcal{D}_r(\mathcal{E}_h).$$

Let $\tilde{p}$ be an interpolant of p having optimal *hp*-approximation errors (2.2)–(2.4) and denote $\chi = P^{\mathrm{NIPG}} - \tilde{p}$. We have, by consistency and definition,

$$\begin{aligned} a_{NS}(\chi, \chi) + J_0^{\sigma,\beta}(\chi, \chi) &= a_{NS}(p - \tilde{p}, \chi) + J_0^{\sigma,\beta}(p - \tilde{p}, \chi) \\ &= \sum_{j=1}^{N_h} \int_{E_j} (K\nabla(p - \tilde{p})\nabla\chi + \alpha(p - \tilde{p})\chi) \\ &\quad - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}[\chi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\chi \cdot \boldsymbol{\nu}_k\}[p - \tilde{p}] \\ &\quad - \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k)\chi + \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla\chi \cdot \boldsymbol{\nu}_k)(p - \tilde{p}) \\ &\quad + \sum_{k=1}^{P_h} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} [p - \tilde{p}][\chi] + \sum_{e_k \in \Gamma_D} \frac{r\sigma_k}{|e_k|^\beta} \int_{e_k} (p - \tilde{p})\chi. \end{aligned} \tag{3.2}$$

The first two terms in (3.2) can be bounded in the following way:

$$\begin{aligned} \left| \sum_{j=1}^{N_h} \int_{E_j} K\nabla(p - \tilde{p}) \cdot \nabla\chi \right| &\leq C |\!|\!| \nabla(p - \tilde{p}) |\!|\!|_0 |\!|\!| K^{\frac{1}{2}} \nabla\chi |\!|\!|_0, \\ &\leq \frac{1}{6} |\!|\!| K^{\frac{1}{2}} \nabla\chi |\!|\!|_0^2 + C |\!|\!| \nabla(p - \tilde{p}) |\!|\!|_0^2. \\ \left| \sum_{j=1}^{N_h} \int_{E_j} \alpha(p - \tilde{p})\chi \right| &\leq \|\alpha\|_\infty^{\frac{1}{2}} \|p - \tilde{p}\|_0 \|\alpha^{\frac{1}{2}}\chi\|_0, \\ &\leq \frac{1}{2} \|\alpha^{\frac{1}{2}}\chi\|_0^2 + \frac{1}{2} \|\alpha\|_\infty \|p - \tilde{p}\|_0^2. \end{aligned}$$

The third term is bounded by

$$\left| \int_{e_k} \{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}[\chi] \right| \leq \left(\frac{|e_k|^\beta}{r\sigma_k} \right)^{\frac{1}{2}} \|\{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}\|_{0,e_k} \left(\frac{r\sigma_k}{|e_k|^\beta} \right)^{\frac{1}{2}} \|[\chi]\|_{0,e_k},$$

$$\left| \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}[\chi] \right| \leq \frac{1}{8} J_0^{\sigma,\beta}(\chi, \chi) + C \sum_{k=1}^{P_h} \frac{|e_k|^\beta}{r\sigma_k} \|\{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}\|_{0,e_k}^2.$$

As e_k is an interior edge, we assume that $e_k = \partial E^1 \cap \partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$; we set $E_k^{12} = E^1 \cup E^2$ and $\bar{h}_k = \max(\mathrm{diam}(E^1), \mathrm{diam}(E^2))$. Using the approximation result (2.4), we induce

$$\sum_{k=1}^{P_h} \frac{|e_k|^\beta}{r\sigma_k} \|\{K\nabla(p - \tilde{p}) \cdot \boldsymbol{\nu}_k\}\|_{0,e_k}^2 \leq \frac{C}{\sigma_0} \sum_{k=1}^{P_h} |e_k|^\beta \frac{\bar{h}_k^{2\mu-3}}{r^{2s-2}} |\!|\!| p |\!|\!|_{s,E_k^{12}}^2.$$

Thus, the final bound for the third term in (3.2) is

$$\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\chi]\right| \le \frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi)+\frac{C}{\sigma_0}\sum_{k=1}^{P_h}|e_k|^{\beta}\frac{\bar{h}_k^{2\mu-3}}{r^{2s-2}}\|\!|p|\!\|_{s,E_k^{12}}^2. \tag{3.3}$$

In the case of triangles or tetrahedra, we can choose a continuous $\tilde{p}$ and for the pure Neumann problem, there are no other terms; thus we can conclude by choosing $\beta \ge (n-1)^{-1}$. In the general case, to bound the fourth term in (3.2), we deduce similarly by using the approximation result (2.3) and the inverse inequality (2.8)

$$\begin{aligned}\left|\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[p-\tilde{p}]\right| &\le \|\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}\|[p-\tilde{p}]\|_{0,e_k}\\ &\le C\frac{r}{\bar{h}_k^{\frac{1}{2}}}\|\!|K^{\frac{1}{2}}\nabla\chi|\!\|_{0,E_k^{12}}\frac{\bar{h}_k^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}}\|\!|p|\!\|_{s,E_k^{12}},\end{aligned}$$

$$\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[p-\tilde{p}]\right| \le \frac{1}{6}\|\!|K^{\frac{1}{2}}\nabla\chi|\!\|_0^2+C\frac{h^{2\mu-2}}{r^{2s-3}}\|\!|p|\!\|_s^2. \tag{3.4}$$

The interior penalty terms in (3.2) are bounded by using (2.3)

$$\begin{aligned}\left|\sum_{k=1}^{P_h}\frac{r\sigma_k}{|e_k|^{\beta}}\int_{e_k}[p-\tilde{p}][\chi]\right| &\le C\sum_{k=1}^{P_h}\frac{r\sigma_k}{|e_k|^{\beta}}\|[p-\tilde{p}]\|_{0,e_k}^2+\frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi)\\ &\le C\sum_{k=1}^{P_h}\frac{\bar{h}_k^{2\mu-1}}{r^{2s-2}}|e_k|^{-\beta}\|\!|p|\!\|_{s,E_k^{12}}^2+\frac{1}{8}J_0^{\sigma,\beta}(\chi,\chi).\end{aligned} \tag{3.5}$$

The boundary terms $|\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k)\chi|$, $|\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\chi\cdot\boldsymbol{\nu}_k)(p-\tilde{p})|$, and $|\sum_{e_k\in\Gamma_D}\frac{\sigma_k}{|e_k|^{\beta}}\int_{e_k}(p-\tilde{p})\chi|$ have similar bounds as the interior terms, using, respectively, (3.3), (3.4), and (3.5). By combining the bounds together, we obtain in the general case

$$\begin{aligned}\|\!|K^{\frac{1}{2}}\nabla\chi|\!\|_0^2+\|\alpha^{\frac{1}{2}}\chi\|_0^2+J_0^{\sigma,\beta}(\chi,\chi) &\le C\frac{h^{2\mu-2}}{r^{2s-2}}\|\!|p|\!\|_s^2+C\|\alpha\|_{\infty}\frac{h^{2\mu}}{r^{2s}}\|\!|p|\!\|_s^2,\\ &\quad +C\sum_{k=1}^{P_h}\left(|e_k|^{\beta}\frac{\bar{h}_k^{2\mu-3}}{r^{2s-3}}+|e_k|^{-\beta}\frac{\bar{h}_k^{2\mu-1}}{r^{2s-2}}\right)\|\!|p|\!\|_{s,E_k^{12}}^2\\ &\quad +C\sum_{e_k\in\Gamma_D}\left(|e_k|^{\beta}\frac{h_k^{2\mu-3}}{r^{2s-3}}+|e_k|^{-\beta}\frac{h_k^{2\mu-1}}{r^{2s-2}}\right)\|\!|p|\!\|_{s,E_k}^2.\end{aligned} \tag{3.6}$$

Thus, if $\beta = (n-1)^{-1}$, then we obtain optimal convergence rates with respect to h. ∎

Remark. When $|\Gamma_D| > 0$ and p_0 is a polynomial of degree r, for instance, $p_0 = 0$, in the case of triangles or tetrahedra, the continuous interpolant $\tilde{p}$ can be constructed so that $\tilde{p} = p_0$ on Γ_D. Then, the statement of Theorem 3.1 is valid for $\delta = 0$ and any $\beta \ge (n-1)^{-1}$.

In the following theorem, we assume that $\Gamma_N = \partial\Omega$ and that the subdivision of Ω consists of triangles or tetrahedra.

THEOREM 3.2. *Assume that Ω is convex, $\Gamma_N = \partial\Omega$, and K is sufficiently smooth so that for any $f \in L^2(\Omega)$, the solution ϕ of the dual problem*

$$\begin{aligned} -\nabla \cdot (K\nabla\phi) + \alpha\phi &= f \quad \text{in} \quad \Omega, \\ K\nabla\phi \cdot \boldsymbol{\nu} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

belongs to $H^2(\Omega)$, with continuous dependence on f. Then, in the case of triangles or tetrahedra, for any $\beta \geq (n-1)^{-1}$,

$$\|P^{\mathrm{NIPG}} - p\|_{0,\Omega} \leq C \frac{1}{r^{s-\frac{1}{2}}} h^{\min(\mu, \mu - \frac{3}{2} + \frac{\beta}{2}(n-1))} |\!|\!| p |\!|\!|_s,$$

where $\mu = \min(r+1, s)$ for $r \geq 1, s \geq 2$, and C is independent of h, r, p. In particular, optimal L^2 rates of convergence with respect to h are obtained if $\beta \geq 3$ for $n = 2$ and if $\beta \geq \frac{3}{2}$ for $n = 3$.

Proof. Consider the dual problem

$$\begin{aligned} -\nabla \cdot (K\nabla\phi) + \alpha\phi &= P^{\mathrm{NIPG}} - p \quad \text{in} \quad \Omega, \\ K\nabla\phi \cdot \boldsymbol{\nu} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

By assumption, $\phi \in H^2(\Omega)$ and there is a constant C that depends on Ω such that

$$\|\phi\|_{2,\Omega} \leq C \|P^{\mathrm{NIPG}} - p\|_{0,\Omega}. \tag{3.7}$$

Note that if $\alpha \equiv 0$, then since both P^{NIPG} and p are defined up to an additive constant, this constant can be chosen so that $P^{\mathrm{NIPG}} - p$ has zero mean value. This compatibility condition guarantees that the dual problem has a unique solution.

Denote $\chi = P^{\mathrm{NIPG}} - p$. Then

$$\|\chi\|_{0,\Omega}^2 = (-\nabla \cdot (K\nabla\phi) + \alpha\phi, \chi).$$

Integrating by parts on each element yields

$$\begin{aligned} \|\chi\|_{0,\Omega}^2 &= \sum_{j=1}^{N_h} \int_{E_j} (K\nabla\phi\nabla\chi + \alpha\phi\chi) - \sum_{j=1}^{N_h} \int_{\partial E_j} (K\nabla\phi \cdot \boldsymbol{\nu})\chi \\ &= \sum_{j=1}^{N_h} \int_{E_j} (K\nabla\phi\nabla\chi + \alpha\phi\chi) - \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \boldsymbol{\nu}_k\}[\chi] \end{aligned}$$

because the regularity of ϕ and K implies that the jumps $[K\nabla\phi \cdot \boldsymbol{\nu}_k]|_{e_k} = 0$. By subtracting the orthogonality equation for any $\phi^* \in \mathcal{D}_r(\mathcal{E}_h)$

$$a_{NS}(\chi, \phi^*) + J_0^{\sigma,\beta}(\chi, \phi^*) = 0,$$

and using the symmetry of K, the regularity of ϕ and K, and choosing for ϕ^* a continuous interpolant of ϕ satisfying (2.2)–(2.4), we obtain

$$\begin{aligned} \|\chi\|_{0,\Omega}^2 = &\sum_{j=1}^{N_h} \int_{E_j} (K\nabla(\phi - \phi^*)\nabla\chi + \alpha(\phi - \phi^*)\chi) \\ &+ \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(\phi - \phi^*) \cdot \boldsymbol{\nu}_k\}[\chi] - 2\sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\phi \cdot \boldsymbol{\nu}_k\}[\chi]. \end{aligned}$$

The first two terms are bounded in the following fashion by using (2.2) and (3.7):

$$\begin{aligned}\left|\sum_{j=1}^{N_h}\int_{E_j} K\nabla(\phi-\phi^*)\nabla\chi\right| &\le C\sum_{j=1}^{N_h}\|\phi-\phi^*\|_{1,E_j}\|K^{\frac{1}{2}}\nabla\chi\|_{0,E_j}\\ &\le C\frac{h}{r}\|\phi\|_{2,\Omega}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0\\ &\le C\frac{h}{r}\|\chi\|_{0,\Omega}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0;\\ \left|\sum_{j=1}^{N_h}\int_{E_j}\alpha(\phi-\phi^*)\chi\right| &\le C\|\alpha\|_\infty\frac{h^2}{r^2}\|\phi\|_{2,\Omega}\|\chi\|_{0,\Omega}\\ &\le C\|\alpha\|_\infty\frac{h^2}{r^2}\|\chi\|_{0,\Omega}|\!|\!|\chi|\!|\!|_1.\end{aligned}$$

By the Cauchy–Schwarz inequality, by (3.1), for any $\beta \ge (n-1)^{-1}$, and by properties (2.4) and (3.7), we have

$$\begin{aligned}\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}[\chi]\right| &\le \sum_{k=1}^{P_h}\left(\frac{|e_k|^\beta}{r\sigma_k}\right)^{\frac{1}{2}-\frac{1}{2}}\|\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}\|[\chi]\|_{0,e_k}\\ &\le J_0^{\sigma,\beta}(\chi,\chi)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{r\sigma_k}\|\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\\ &\le C\frac{h^{\mu-\frac{1}{2}}h^{\frac{\beta}{2}(n-1)}}{r^s}|\!|\!|p|\!|\!|_s\|\chi\|_{0,\Omega}.\end{aligned}$$

Similarly, by Cauchy–Schwarz, (3.1), and (2.6),

$$\begin{aligned}\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\phi\cdot\boldsymbol{\nu}_k\}[\chi]\right| &\le \sum_{k=1}^{P_h}\left(\frac{|e_k|^\beta}{r\sigma_k}\right)^{\frac{1}{2}}\|\{K\nabla\phi\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}\left(\frac{r\sigma_k}{|e_k|^\beta}\right)^{\frac{1}{2}}\|[\chi]\|_{0,e_k}\\ &\le J_0^{\sigma,\beta}(\chi,\chi)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\frac{|e_k|^\beta}{r\sigma_k}\|\{K\nabla\phi\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\\ &\le C\frac{h^{\mu-\frac{3}{2}}h^{\frac{\beta}{2}(n-1)}}{r^{s-\frac{1}{2}}}|\!|\!|p|\!|\!|_s\|\chi\|_{0,\Omega}.\end{aligned}$$

Combining the previous bounds and Theorem 3.1 applied in the pure Neumann case, we have for any $\beta \ge (n-1)^{-1}$ that

$$\|\chi\|_{0,\Omega} \le C\left(\frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}(n-1)}}{r^{s-\frac{1}{2}}}+\frac{h^\mu}{r^s}\right)|\!|\!|p|\!|\!|_s + C\|\alpha\|_\infty\frac{h^\mu}{r^s}|\!|\!|p|\!|\!|_s,$$

which concludes the proof. □

Remark. On parallelograms or quadrilaterals, we cannot use a continuous interpolant, and therefore the values of β in Theorems 3.1 and 3.2 are incompatible. Of course, we can have a continuous interpolant on the function space $\mathbb{Q}_k$, but the scheme is more costly.

On the other hand, for the pure Dirichlet problem, even on triangles or tetrahedra, we must choose $\beta = (n-1)^{-1}$ in order to establish Theorem 3.1, and these are incompatible with the values of β in Theorem 3.2. However, if the Dirichlet datum p_0 on $\partial\Omega$ is a polynomial of degree r, in the case of triangles or tetrahedra, then the statement of Theorem 3.2 is valid.

Finally, it is well known that the duality argument of Theorem 3.2 is not well adapted to mixed boundary conditions (except in very particular cases) because the dual problem may have solutions that do not belong to $H^2(\Omega)$.

4. A priori error estimates for the NCG method. In this section, we derive an error estimate for the H^1 norm and the L^2 norm that are both h-optimal for the constrained Galerkin method applied to the pure Neumann problem.

THEOREM 4.1. *Under the assumptions of Lemma 2.2 and, if $p \in H^s(\mathcal{E}_h)$, we have*

$$\|\!|P^{\mathrm{NCG}} - p|\!\|_1 \le C(K, \|\alpha\|_\infty)\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}|\!|\!|p|\!|\!|_s,$$

where $\mu = \min(r+1, s)$, C is independent of h, r, p, and $r \ge 1, s \ge 2$.

Proof. We have the following orthogonality equation:

$$a_{NS}(P^{\mathrm{NCG}} - p, v) = 0 \quad \forall\, v \in \mathcal{D}_r^*(\mathcal{E}_h).$$

We can show [14] that there is an interpolant $\tilde{p} \in \mathcal{D}_r^*(\mathcal{E}_h)$ that satisfies the approximation properties (2.2)–(2.4). Let $\chi = P^{\mathrm{NCG}} - \tilde{p}$; then

$$\begin{aligned} a_{NS}(\chi,\chi) &= a_{NS}(p-\tilde{p},\chi) \\ &= \sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p})\nabla\chi + \alpha\chi(p-\tilde{p})) \\ &\quad - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[p-\tilde{p}]. \end{aligned} \tag{4.1}$$

The first two terms are easily bounded by Cauchy–Schwarz and by using (2.2):

$$\begin{aligned} \left|\sum_{j=1}^{N_h}\int_{E_j} K\nabla(p-\tilde{p})\nabla\chi\right| &\le |\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0\,|\!|\!|K^{\frac{1}{2}}\nabla(p-\tilde{p})|\!|\!|_0 \\ &\le C|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0\left(\sum_{j=1}^{N_h}\frac{h_j^{2\mu-2}}{r^{2s-2}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

In a similar manner, we have

$$\begin{aligned} \left|\sum_{j=1}^{N_h}\int_{E_j}\alpha\chi(p-\tilde{p})\right| &\le \|\alpha^{\frac{1}{2}}\chi\|_0\|\alpha^{\frac{1}{2}}(p-\tilde{p})\|_0 \\ &\le C\|\alpha\|_\infty^{\frac{1}{2}}\|\alpha^{\frac{1}{2}}\chi\|_0\left(\sum_{j=1}^{N_h}\frac{h_j^{2\mu}}{r^{2s}}\|p\|_{s,E_j}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

Now we estimate the third term in (4.1), which we denote by A. Let c_k be any constant:

$$A \equiv \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\chi] = \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\chi - c_k].$$

We have by Cauchy–Schwarz and (2.4)

$$|A| \le C\frac{h^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}|||p|||_s \left(\sum_{k=1}^{P_h} \|[\chi - c_k]\|_{0,e_k}^2\right)^{\frac{1}{2}}. \tag{4.4}$$

The other factor is bounded in the following way; here again, we assume that e_k is the interface between two elements E^1 and E^2 of $\mathcal{E}_h$:

$$\|[\chi]\|_{0,e_k} = \|[\chi - c_k]\|_{0,e_k} \le \|(\chi - c_k)_{E^1}\|_{0,e_k} + \|(\chi - c_k)_{E^2}\|_{0,e_k}.$$

Since P^{NCG} belongs to $\mathcal{D}_r^*(\mathcal{E}_h)$, we have

$$\int_{e_k} (\chi)_{E^1} d\sigma = \int_{e_k} (\chi)_{E^2} d\sigma.$$

Therefore, it suffices to estimate $\|(\chi - c_k)_{E^1}\|_{0,e_k}$ with the choice

$$c_k = \frac{1}{|e_k|}\int_{e_k} (\chi)_{E^1} d\sigma.$$

Passing to the reference element, we note that $c_k = \frac{1}{|\hat{e}|}\int_{\hat{e}} \hat{\chi} d\hat{\sigma}$ and that the mapping $\hat{f} \mapsto \hat{f} - \frac{1}{|\hat{e}|}\int_{\hat{e}} \hat{f} d\hat{\sigma}$ is continuous on $H^1(\hat{e})$ and vanishes on constant functions. Thus,

$$\|\hat{\chi} - c_k\|_{0,\hat{e}} \le \hat{C}\|\hat{\nabla}_{\hat{e}}\hat{\chi}\|_{0,\hat{e}},$$

where $\hat{\nabla}_{\hat{e}}$ denotes the tangential gradient on $\hat{e}$. However, since $\hat{\nabla}_{\hat{e}}\hat{\chi}$ belongs to a finite-dimensional space, on which all norms are equivalent, and since the subdivision of Ω is regular, we get the following by applying the inverse estimate (2.7) on the reference element:

$$\|\hat{\chi} - c_k\|_{0,\hat{e}} \le \hat{C}r\|\hat{\nabla}_{\hat{e}}\hat{\chi}\|_{0,\hat{E}}.$$

Therefore,

$$\|(\chi - c_k)_{E^1}\|_{0,e_k} \le \hat{C}|e_k|^{\frac{1}{2}} h_1 |E^1|^{-\frac{1}{2}} r\|\nabla\chi\|_{0,E^1} \le \hat{C}h_1^{\frac{1}{2}} r\|\nabla\chi\|_{0,E^1}.$$

Thus, summing on k, we have

$$\sum_{k=1}^{P_h} \|[\chi]\|_{0,e_k}^2 = \sum_{k=1}^{P_h} \|[\chi - c_k]\|_{0,e_k}^2 \le C\sum_{j=1}^{N_h} h_j r^2 \|\nabla\chi\|_{0,E_j}^2. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain a bound for A:

$$|A| \le C\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}|||p|||_s |||K^{\frac{1}{2}}\nabla\chi|||_0. \tag{4.6}$$

The last term in (4.1) is bounded as in Theorem 3.1:

$$\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[p-\tilde{p}]\right| \le C|\!|\!| K^{\frac{1}{2}}\nabla\chi|\!|\!|_0 \frac{h^{\mu-1}}{r^{s-\frac{3}{2}}}|\!|\!| p|\!|\!|_s. \tag{4.7}$$

Combining (4.2), (4.3), (4.6), and (4.7), we obtain

$$a_{NS}(\chi,\chi) \le C\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}|\!|\!| p|\!|\!|_s |\!|\!| K^{\frac{1}{2}}\nabla\chi|\!|\!|_0 + C\|\alpha\|_\infty^{\frac{1}{2}}\frac{h^\mu}{r^s}\|\alpha^{\frac{1}{2}}\chi\|_0|\!|\!| p|\!|\!|_s,$$

which concludes the proof. □

Remark. If we add the condition on P^{NCG},

$$\int_{e_k} P^{\mathrm{NCG}} = \int_{e_k} p_0 \quad \forall\, e_k \in \Gamma_D, \tag{4.8}$$

then we can construct $\tilde{p}$ such that

$$\int_{e_k}(P^{\mathrm{NCG}} - \tilde{p}) = 0 \quad \forall\, e_k \in \Gamma_D,$$

and therefore the error terms arising from the Dirichlet boundary condition have the same estimates as the interior jump terms. Hence, if we impose (4.8), the statement of Theorem 4.1 is also valid for the problem with mixed boundary conditions.

The next theorem shows that the NCG scheme gives optimal results with respect to h in the L^2 norm. For simplicity, we do not specify the dependence with respect to r.

THEOREM 4.2. *Under the assumptions of Theorem* 3.2, *we have*

$$\|P^{\mathrm{NCG}} - p\|_{0,\Omega} \le Ch^\mu |\!|\!| p|\!|\!|_s$$

for $r \ge 1, s \ge 2$, *and* C *independent of* h *and* p.

Proof. As in the proof of Theorem 3.2, we consider the following dual problem:

$$\begin{cases} -\nabla\cdot(K\nabla\psi) + \alpha\psi = P^{\mathrm{NCG}} - p & \text{in} \quad \Omega, \\ K\nabla\psi\cdot\boldsymbol{\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Denote $\chi = P^{\mathrm{NCG}} - p$. Thus, we have

$$\begin{aligned} \|\chi\|_{0,\Omega}^2 &= (-\nabla\cdot(K\nabla\psi) + \alpha\psi, \chi) \\ &= \sum_{j=1}^{N_h}\int_{E_j}(K\nabla\psi\nabla\chi + \alpha\psi\chi) - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi\cdot\boldsymbol{\nu}_k\}[\chi]. \end{aligned} \tag{4.9}$$

Let ψ^* be in $\mathcal{D}_r^*(\mathcal{E}_h)\cap C^0(\bar{\Omega})$. The orthogonality condition implies that

$$0 = \sum_{j=1}^{N_h}\int_{E_j}(K\nabla\chi\nabla\psi^* + \alpha\chi\psi^*) + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\psi^*\cdot\boldsymbol{\nu}_k\}[\chi]. \tag{4.10}$$

Now, we subtract (4.10) from (4.9) and use the regularity of $K\nabla\psi\cdot\boldsymbol{\nu}_k$ on e_k:

$$\begin{aligned} \|\chi\|_{0,\Omega}^2 &= \sum_{j=1}^{N_h}\int_{E_j}(K\nabla(\psi-\psi^*)\nabla\chi + \alpha(\psi-\psi^*)\chi) \\ &\quad -2\sum_{k=1}^{P_h}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\psi-\psi^*)\cdot\boldsymbol{\nu}_k\}[\chi]. \end{aligned} \tag{4.11}$$

As in the proof of Theorem 4.1, the first two terms are easily bounded by using Cauchy–Schwarz, Hölder inequalities, and the approximation property (2.2):

$$\left|\sum_{j=1}^{N_h}\int_{E_j} K\nabla(\psi-\psi^*)\nabla\chi\right| \le Ch\|\chi\|_{0,\Omega}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0,$$

$$\left|\sum_{j=1}^{N_h}\int_{E_j} \alpha(\psi-\psi^*)\chi\right| \le C\|\alpha\|_\infty h^2\|\chi\|_{0,\Omega}|\!|\!|\chi|\!|\!|_1,$$

and we apply Theorem 4.1 to bound the factor $|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0$ in the first term and $|\!|\!|\chi|\!|\!|_{1,\Omega}$ in the second term. To bound the third term in (4.11), let $\tilde{p}$ be an element of $\mathcal{D}_r^*(\mathcal{E}_h)\cap C^0(\bar{\Omega})$ and $\vec{c}$ be any constant vector; we can write

$$\begin{aligned}2\sum_{k=1}^{P_h}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)[\chi] &= 2\sum_{k=1}^{P_h}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)[P^{\mathrm{NCG}}-\tilde{p}]\\ &= 2\sum_{k=1}^{P_h}\int_{e_k}((K\nabla\psi-\vec{c})\cdot\boldsymbol{\nu}_k)[P^{\mathrm{NCG}}-\tilde{p}].\end{aligned}$$

As it was proved in Theorem 4.1, (4.5), and from the approximation result (2.2), we have

$$\begin{aligned}\sum_{k=1}^{P_h}\|[P^{\mathrm{NCG}}-\tilde{p}]\|_{0,e_k}^2 &\le C\sum_{j=1}^{N_h}h_j\|\nabla(P^{\mathrm{NCG}}-\tilde{p})\|_{0,E_j}^2\\ &\le C\sum_{j=1}^{N_h}h_j\|\nabla(P^{\mathrm{NCG}}-p)\|_{0,E_j}^2 + C\sum_{j=1}^{N_h}h_j\|\nabla(p-\tilde{p})\|_{0,E_j}^2\\ &\le Ch^{2\mu-1}|\!|\!|p|\!|\!|_s^2.\end{aligned}$$

On the other hand, by choosing

$$\vec{c}=\frac{1}{|E^1|}\int_{E^1}K\nabla\psi,$$

we have by applying (2.6)

$$\|(K\nabla\psi-\vec{c})\cdot\boldsymbol{\nu}_k\|_{0,e_k}\le\|(K\nabla\psi-\vec{c})|_{E^1}\|_{0,e_k}\le Ch_1^{\frac{1}{2}}|K\nabla\psi|_{1,E^1}.$$

By assumption, $K\in W^{1,4}(\mathcal{E}_h)$ and $\psi\in H^2(\mathcal{E}_h)$. Therefore, $K\nabla\psi$ belongs to $H^1(\mathcal{E}_h)$ with

$$|\!|\!|K\nabla\psi|\!|\!|_1\le C\|\chi\|_{0,\Omega};$$

then

$$\left|2\sum_{k=1}^{P_h}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)[\chi]\right|\le Ch^\mu\|\chi\|_{0,\Omega}|\!|\!|p|\!|\!|_s.$$

Let A denote the last term in (4.11):

$$|A|\le C\left(\sum_{k=1}^{P_h}\|\{\nabla(\psi-\psi^*)\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{P_h}\|[\chi]\|_{0,e_k}^2\right)^{\frac{1}{2}}.$$

As previously, we have by (2.4) and regularity of ψ

$$\sum_{k=1}^{P_h} \|\{\nabla(\psi-\psi^*)\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}^2 \le Ch|\!|\!|\psi|\!|\!|_2^2 \le Ch\|\chi\|_{0,\Omega}^2,$$

and according to (4.5),

$$\sum_{k=1}^{P_h} \|[\chi]\|_{0,e_k}^2 \le Ch|\!|\!|\nabla\chi|\!|\!|_0^2 \le Ch^{2\mu-1}|\!|\!|p|\!|\!|_s^2.$$

Thus, we obtain

$$|A| \le h^\mu \|\chi\|_{0,\Omega} |\!|\!|p|\!|\!|_s.$$

The theorem is obtained by combining all the above results. □

Remark. In the case of the pure Dirichlet problem, i.e., $p = p_0$ on $\partial\Omega$, we can obtain the same order of convergence in the L^2 norm if we impose (4.8) on P^{NCG}. Indeed, the corresponding dual problem is

$$\begin{cases} -\nabla\cdot(K\nabla\psi) + \alpha\psi = P^{\mathrm{NCG}} - p & \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then, we interpolate ψ by ψ^* in $\mathcal{D}_r^*(\mathcal{E}_h)\cap H_0^1(\Omega)$ and (4.11) is replaced by

$$\begin{aligned} \|\chi\|_{0,\Omega}^2 = &\sum_{j=1}^{N_h}\int_{E_j} (K\nabla(\psi-\psi^*)\nabla\chi + \alpha(\psi-\psi^*)\chi) \\ &-2\sum_{k=1}^{P_h}\int_{e_k} (K\nabla\psi\cdot\boldsymbol{\nu}_k)[\chi] + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\psi-\psi^*)\cdot\boldsymbol{\nu}_k\}[\chi] \\ &-2\sum_{e_k\in\Gamma_D}\int_{e_k}(K\nabla\psi\cdot\boldsymbol{\nu}_k)\chi + \sum_{e_k\in\Gamma_D}\int_{e_k}\{K\nabla(\psi-\psi^*)\cdot\boldsymbol{\nu}_k\}\chi. \end{aligned}$$

Since $\tilde{p}$ can be constructed so that $\int_{e_k}\chi = 0$ for all e_k in Γ_D, the above boundary terms have the same estimates as the interior jump terms.

5. A priori error estimate for the DG method. In this section, we derive an a priori optimal error estimate in two or three dimensions for the problem with Neumann, or Dirichlet, or mixed boundary conditions, on triangles, parallelograms, or tetrahedra. We make several additional assumptions:

- $K \in [W^{1,\infty}(E_j)]^{n\times n} \quad \forall\, j = 1,\dots,N_h$, uniformly in E_j, i.e., $\exists\,\gamma_2 > 0$,

$$\max_{E_j}\left(\sum_{i,m=1}^{n}\|\nabla k_{im}\|_{\infty,E_j}^2\right)^{\frac{1}{2}} \le \gamma_2.$$

- We denote $\bar{K} = (\bar{k}_{ij})$, where

$$\bar{k}_{ij} = \frac{1}{|E|}\int_E k_{ij} \quad \forall\, E\in\mathcal{E}_h. \tag{5.1}$$

  It is easy to prove that $\bar{K}$ is also symmetric positive definite and that the largest eigenvalue of $\bar{K}$ is $\le\gamma_1$ and its smallest eigenvalue is $\ge\gamma_0$, where γ_1 and γ_0 are the constants of (2.1).

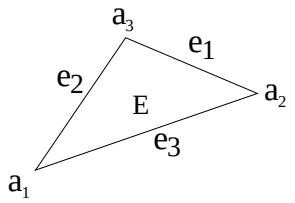

FIG. 5.1. *Triangular element.*

We first prove a local approximation result that holds for $n = 2$ or 3.

THEOREM 5.1. *Let E be an element of the subdivision $\mathcal{E}_h$, that is, a triangle or a parallelogram in two dimensions or a tetrahedron in three dimensions. Let h_E be the diameter of E, let $p \in H^s(E)$, for $s \geq 2$, let $r \geq 2$, and let $\bar{K}$ be the constant tensor defined by* (5.1). *There exists an interpolant of p, $\tilde{p}^I \in \mathbb{P}_r(E)$ satisfying*

$$\int_{e_k} \bar{K}\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k = 0 \quad \forall\, e_k \in \partial E, \tag{5.2}$$

$$\|\nabla^i(\tilde{p}^I - p)\|_{0,E} \leq C \frac{h_E^{\mu-i}}{r^{s-\frac{3}{2}-\delta}} \|p\|_{s,E} \quad \textit{for} \quad i = 0, 1, 2, \tag{5.3}$$

$$\int_{e_k} |\bar{K}\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k| \leq C \frac{h_E^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|p\|_{s,E}, \tag{5.4}$$

where $\delta = 0$ for $i = 0, 1$, $\delta = \frac{1}{2}$ for $i = 2$, $\mu = \min(r + 1, s)$, and C is independent of h_E and r.

Proof. The case of triangles. Let E be a triangle with vertices a_1, a_2, a_3, opposite sides e_1, e_2, e_3, and unit exterior normal vectors $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ as in Figure 5.1. Let $\lambda_1, \lambda_2, \lambda_3$ be the barycentric coordinates of a_1, a_2, and a_3 in E. First, we will show that given f in $H^s(E)$ with $s \geq 2$, there is a polynomial q_1 in $\mathbb{P}_2(E)$ such that $\int_{e_1} \bar{K}\nabla(q_1 - f) \cdot \boldsymbol{\nu}_1 = 0$, $\int_{e_2} \bar{K}\nabla q_1 \cdot \boldsymbol{\nu}_2 = 0$, and $\int_{e_3} \bar{K}\nabla q_1 \cdot \boldsymbol{\nu}_3 = 0$. For this, consider the polynomial

$$q_1 = 4q_1(a_{12})\lambda_1(1 - \lambda_1),$$

where a_{12} is the midpoint of $e_3 = [a_1, a_2]$. It is easy to check that each component of $\nabla q_1 = 4q_1(a_{12})\nabla\lambda_1(1 - 2\lambda_1)$ has zero mean-value on e_2 and e_3, and $(\nabla\lambda_1)\lambda_1$ vanishes on e_1. Therefore, $q_1(a_{12})$ is determined by the conditions

$$4q_1(a_{12}) \int_{e_1} \bar{K}\nabla\lambda_1 \cdot \boldsymbol{\nu}_1 = \int_{e_1} \bar{K}\nabla f \cdot \boldsymbol{\nu}_1.$$

However,

$$\nabla\lambda_1 = -\frac{\boldsymbol{\nu}_1}{2} \frac{|e_1|}{|E|}. \tag{5.5}$$

Therefore,

$$q_1(a_{21}) = -\frac{1}{2} \frac{|E|}{|e_1|^2} \frac{1}{(\bar{K}\boldsymbol{\nu}_1, \boldsymbol{\nu}_1)} \int_{e_1} \bar{K}\nabla f \cdot \boldsymbol{\nu}_1.$$

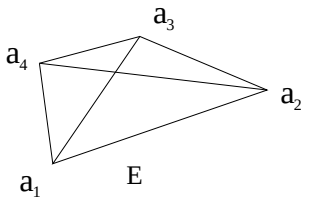

FIG. 5.2. *Tetrahedron element.*

Hence,

$$|q_1(a_{12})| \le \frac{C}{\gamma_0}\left|\int_{e_1} \bar{K}\nabla f\cdot\boldsymbol{\nu}_1\right| \le C\frac{\gamma_1}{\gamma_0}h_E^{\frac{1}{2}}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}.$$

Therefore, for $i=0,1,2$,

$$\|\nabla^i q_1\|_{0,E} \le C|E|^{\frac{1}{2}}h_E^{-i}|q_1(a_{12})| \le C|E|^{\frac{1}{2}}h_E^{\frac{1}{2}-i}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}. \tag{5.6}$$

Similarly, we construct polynomials q_2 and q_3 in $\mathbb{P}_2(E)$ such that

$$\int_{e_\delta}\bar{K}\nabla q_\delta\cdot\boldsymbol{\nu}_\delta = \int_{e_\delta}\bar{K}\nabla f\cdot\boldsymbol{\nu}_\delta \quad \text{for} \quad \delta=2,3,$$
$$\int_{e_1}\bar{K}\nabla q_2\cdot\boldsymbol{\nu}_1 = \int_{e_3}\bar{K}\nabla q_2\cdot\boldsymbol{\nu}_3 = 0,$$
$$\int_{e_1}\bar{K}\nabla q_3\cdot\boldsymbol{\nu}_1 = \int_{e_2}\bar{K}\nabla q_3\cdot\boldsymbol{\nu}_2 = 0,$$

and such that the inequalities of (5.6) hold for q_2 and q_3, with respect to e_2 and e_3. Let $q=q_1+q_2+q_3$, constructed with $f=p-p^I$, where p^I is an approximation of p satisfying (2.2)–(2.4), and set $\tilde{p}^I=q+p^I$. Then $\tilde{p}^I$ satisfies (5.2), and applying (2.4) to (5.6) we derive, for $i=0,1,2$,

$$\begin{aligned}|\tilde{p}^I-p|_{i,E} &\le |q|_{i,E}+|p^I-p|_{i,E}\\ &\le C\frac{h_E^{\mu-i}}{r^{s-\frac{3}{2}}}\|p\|_{s,E}+|p^I-p|_{i,E},\end{aligned}$$

which has the same order of approximation as $|p^I-p|_{i,E}$. Similarly,

$$\int_{e_k}|\bar{K}\nabla(\tilde{p}^I-p)\cdot\boldsymbol{\nu}_k| \le \int_{e_k}|\bar{K}\nabla q\cdot\boldsymbol{\nu}_k| + \int_{e_k}|\bar{K}\nabla(p^I-p)\cdot\boldsymbol{\nu}_k|,$$

and applying (2.4) to the second term, (2.6) to the first term, and the above estimate for $|q|_{i,E}$, we obtain (5.4).

The case of tetrahedra. The situation is much the same for tetrahedra. Let E be a tetrahedron with vertices a_1,a_2,a_3,a_4, opposite faces e_1,e_2,e_3,e_4, and unit exterior normal vectors $\boldsymbol{\nu}_1,\boldsymbol{\nu}_2,\boldsymbol{\nu}_3,\boldsymbol{\nu}_4$ as in Figure 5.2. Let $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ be the barycentric coordinates of a_1,a_2,a_3, and a_4 in E. Again, we will show that given f in $H^s(E)$ with $s\ge 2$, there is a polynomial q_1 in $\mathbb{P}_2(E)$ such that $\int_{e_1}K\nabla(q_1-f)\cdot\boldsymbol{\nu}_1=0$, $\int_{e_\delta}K\nabla q_1\cdot\boldsymbol{\nu}_\delta=0$ for $\delta=2,3,4$. For this, consider the polynomial

$$q_1 = q_1(a_1)\lambda_1(3\lambda_1-2).$$

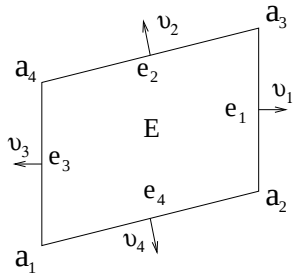

FIG. 5.3. *Parallelogram element.*

It is easy to check that each component of ∇q_1,

$$\nabla q_1 = q_1(a_1)\nabla\lambda_1(6\lambda_1 - 2),$$

has zero mean-value on e_2, e_3, e_4 and $(\nabla\lambda_1)\lambda_1$ vanishes on e_1. Therefore, $q_1(a_1)$ is determined by the condition

$$-2q_1(a_1)\int_{e_1}\bar{K}\nabla\lambda_1\cdot\boldsymbol{\nu}_1 = \int_{e_1}\bar{K}\nabla f\cdot\boldsymbol{\nu}_1,$$

and since (5.5) is valid in three dimensions, this holds if

$$q_1(a_1) = \frac{|E|}{|e_1|^2}\frac{1}{(\bar{K}\boldsymbol{\nu}_1,\boldsymbol{\nu}_1)}\int_{e_1}\bar{K}\nabla f\cdot\boldsymbol{\nu}_1.$$

Hence

$$|q_1(a_1)| \le C\frac{\gamma_1}{\gamma_0}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1},$$

and for $i = 0, 1, 2$, we obtain the analogue of (5.6):

$$\|\nabla^i q_1\|_{0,E} \le C|E|^{\frac{1}{2}}h_E^{-i}|q_1(a_1)| \le C|E|^{\frac{1}{2}}h_E^{-i}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_1}.$$

Then, the proof finishes as in the triangular case.

The case of parallelograms. Let E be a parallelogram with vertices a_1, a_2, a_3, a_4, opposite sides e_1, e_2, e_3, e_4, and unit exterior normal vectors $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3, \boldsymbol{\nu}_4$ as in Figure 5.3. We want to construct a polynomial $q \in \mathbb{P}_2(E)$ such that $\int_{e_3}\bar{K}\nabla q\cdot\boldsymbol{\nu}_3 = \int_{e_3}\bar{K}\nabla f\cdot\boldsymbol{\nu}_3$ and $\int_{e_\delta}\bar{K}\nabla q\cdot\boldsymbol{\nu}_\delta = 0$ for $\delta = 1, 2, 4$. For this, we replace the barycentric coordinates with the functions $\phi_i \in \mathbb{P}_1(E)$, $1 \le i \le 4$, defined by $\phi_i|_{e_i} = 0$ for $1 \le i \le 4$ and $\phi_1|_{e_3} = 1, \phi_2|_{e_4} = 1, \phi_3|_{e_1} = 1, \phi_4|_{e_2} = 1$. Then, we set

$$q = \sum_{i=1}^{4} 2q_i\phi_i\left(\phi_i - \frac{1}{2}\right), \quad q_i \in \mathbb{R},$$

and we want to adjust the numbers q_i so that q satisfies the above conditions. Considering that E is a parallelogram, it is easy to check that

$$\nabla\phi_i = -\frac{|e_i|}{|E|}\boldsymbol{\nu}_i.$$

Set $c_3 = |e_3|(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_3), c_4 = |e_4|(\bar{K}\boldsymbol{\nu}_4, \boldsymbol{\nu}_4), c_{34} = |e_3|(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4), c_{43} = |e_4|(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)$. Then the numbers q_i must satisfy the following system of equations:

$$\begin{aligned}
\frac{|e_3|}{|E|}(3c_3q_1 + c_{43}q_2 + c_3q_3 - c_{43}q_4) &= \int_{e_3} \bar{K}\nabla f \cdot \boldsymbol{\nu}_3,\\
c_3q_1 - c_{43}q_2 + 3c_3q_3 + c_{43}q_4 &= 0,\\
-c_{34}q_1 + c_4q_2 + c_{34}q_3 + 3c_4q_4 &= 0,\\
c_3q_1 + 3c_4q_2 - c_{34}q_3 + c_4q_4 &= 0.
\end{aligned}$$

In order to solve this system of equations and derive an upper bound for q_i, set

$$\alpha_{34} = (\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_3)(\bar{K}\boldsymbol{\nu}_4, \boldsymbol{\nu}_4) - (\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)^2,$$

and observe that

$$\alpha_{34} = \|\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_3\|^2\|\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_4\|^2 \sin^2\theta,$$

where θ is the angle between $\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_3$ and $\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_4$. Owing to the regularity of $\mathcal{E}_h$, this angle is bounded below for all $E \in \mathcal{E}_h$ by a fixed angle $\theta_0 > 0$:

$$\forall\, E \in \mathcal{E}_h, \quad \alpha_{34} \geq \|\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_3\|^2\|\bar{K}^{\frac{1}{2}}\boldsymbol{\nu}_4\|^2 \sin^2\theta_0 \geq \gamma_0^2 \sin^2\theta_0.$$

Then, the solution of the above system is

$$\begin{aligned}
q_4 &= \frac{b}{2}\frac{|E|}{|e_3||e_4|}\frac{(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)}{\alpha_{34}}, \quad q_2 = -q_4,\\
q_3 &= -\frac{b}{2}\frac{|E|}{|e_3|^2}\frac{1}{(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_3)}\left(\frac{(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)^2}{\alpha_{34}} + \frac{1}{4}\right),\\
q_1 &= \frac{b}{2}\frac{|E|}{|e_3|^2}\frac{1}{(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_3)}\left(\frac{(\bar{K}\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)^2}{\alpha_{34}} + \frac{3}{4}\right),
\end{aligned}$$

where $b = \int_{e_3} \bar{K}\nabla f \cdot \boldsymbol{\nu}_3$. We can easily check that

$$\begin{aligned}
|q_2| = |q_4| &\leq \frac{C}{2}\frac{\gamma_1^2}{\gamma_0^2}\frac{1}{\sin^2\theta_0}h_E^{\frac{1}{2}}\left\|\frac{\partial f}{\partial n}\right\|_{0,e_3},\\
|q_3| &\leq \frac{C}{2}\frac{\gamma_1}{\gamma_0}h_E^{\frac{1}{2}}\left(\frac{\gamma_1^2}{\gamma_0^2\sin^2\theta_0} + \frac{1}{4}\right)\left\|\frac{\partial f}{\partial n}\right\|_{0,e_3},\\
|q_1| &\leq \frac{C}{2}\frac{\gamma_1}{\gamma_0}h_E^{\frac{1}{2}}\left(\frac{\gamma_1^2}{\gamma_0^2\sin^2\theta_0} + \frac{3}{4}\right)\left\|\frac{\partial f}{\partial n}\right\|_{0,e_3}.
\end{aligned}$$

From these bounds, we immediately deduce that $\nabla^i q$ satisfies (5.6) and the proof finishes as in the triangular case. □

The next corollary extends the result of Theorem 5.1 to nonconstant tensors.

COROLLARY 5.2. *We retain the notation of Theorem* 5.1. *Let K be a tensor-valued function in $[W^{1,\infty}(E)]^{n\times n}$. There exists $h_0 > 0$ independent of E such that for all $h_E \leq h_0$, there exists an interpolant of p, $\tilde{p}^I \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying*

$$\begin{aligned}
\int_{e_k} K\nabla(\tilde{p}^I - p)\cdot\boldsymbol{\nu}_k &= 0 \quad \forall\, e_k \in \partial E,\\
\|\nabla^i(\tilde{p}^I - p)\|_{0,E} &\leq C\frac{h_E^{\mu-i}}{r^{s-\frac{3}{2}-\delta}}\|p\|_{s,E}, \quad i = 0,1,2,\\
\int_{e_k} |K\nabla(\tilde{p}^I - p)\cdot\boldsymbol{\nu}_k| &\leq C\frac{h_E^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}\|p\|_{s,E},
\end{aligned}$$

where $\delta = 0$ if $i = 0, 1$, $\delta = \frac{1}{2}$ if $i = 2$, $\mu = \min(r+1, s)$, and C is independent of h and r.

Proof. Let $\bar{K}$ be defined by (5.1). Let

$$\|K - \bar{K}\|_{\infty,E} = \sup_{x \in E} \|K(x) - \bar{K}\|,$$

where $\|\cdot\|$ denotes the matrix norm subordinated to the Euclidean norm. It is easy to check that

$$\sup_{x \in E} \|K(x) - \bar{K}\| \leq \sup_{x \in E} \|K(x) - \bar{K}\|_S,$$

where $\|K(x)\|_S^2 = \sum_{i,j=1}^n k_{ij}^2(x)$. Since $\bar{k}_{ij}$ is also the average of $\hat{k}_{ij} = k_{ij} \circ F$ on the reference square $\hat{E}$, the mapping $\hat{k}_{ij} \mapsto \hat{k}_{ij} - \bar{k}_{ij}$ vanishes if $\hat{k}_{ij}$ is a constant function. Thus,

$$\|k_{ij} - \bar{k}_{ij}\|_{\infty,E} \leq C \|\hat{\nabla} \hat{k}_{ij}\|_{\infty,\hat{E}} \leq C h_E \|\nabla k_{ij}\|_{\infty,E} \quad \forall\, E \in \mathcal{E}_h.$$

Therefore,

$$\|K - \bar{K}\|_{\infty,E} \leq C_\infty h_E \|\nabla K\|_{\infty,E}. \tag{5.7}$$

In the course of the proof of Theorem 5.1, for any $f \in H^s(E)$, we have constructed a polynomial $q \in \mathbb{P}_2(E)$ such that

$$\forall\, e \in \partial E, \quad \int_e \bar{K} \nabla q \cdot \boldsymbol{\nu} = \int_e \bar{K} \nabla f \cdot \boldsymbol{\nu}.$$

We propose to achieve the same result, with K instead of $\bar{K}$, by means of the following iterative procedure. Starting with $q^0 = 0$, construct $q^m \in \mathbb{P}_2(E)$, as in Theorem 5.1, the solution of

$$\forall\, e \in \partial E, \quad \int_e \bar{K} \nabla q^m \cdot \boldsymbol{\nu} = \int_e K \nabla f \cdot \boldsymbol{\nu} - \int_e (K - \bar{K}) \nabla q^{m-1} \cdot \boldsymbol{\nu}. \tag{5.8}$$

Theorem 5.1 guarantees the existence of q^m; moreover, combining (5.6), (5.7), and (5.8) we obtain in the two-dimensional (2D) case

$$\|\nabla q^m\|_{0,E} \leq C_2 |E|^{\frac{1}{2}} h_E^{-\frac{1}{2}} (\|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} + C_\infty h_E \|\nabla K\|_{\infty,E} \|\nabla q^{m-1} \cdot \boldsymbol{\nu}\|_{0,\partial E}) \tag{5.9}$$

and in the three-dimensional (3D) case

$$\|\nabla q^m\|_{0,E} \leq C_3 |E|^{\frac{1}{2}} h_E^{-1} (\|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} + C_\infty h_E \|\nabla K\|_{\infty,E} \|\nabla q^{m-1} \cdot \boldsymbol{\nu}\|_{0,\partial E})$$

with constants independent of m, h, and E. Let us prove by induction on m that for small enough h_E,

$$\|\nabla q^m\|_{0,E} \leq \begin{cases} 2C_2 |E|^{\frac{1}{2}} h_E^{-\frac{1}{2}} \|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} & \text{in two dimensions,} \\ 2C_3 |E|^{\frac{1}{2}} h_E^{-1} \|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} & \text{in three dimensions.} \end{cases} \tag{5.10}$$

This is trivially true for $m = 0$. Assume that (5.10) holds for $m - 1$. Applying (2.8), we have in the 2D case

$$\|\nabla q^m\|_{0,E} \leq C_2 |E|^{\frac{1}{2}} h_E^{-\frac{1}{2}} (\|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} + C C_\infty h_E^{\frac{1}{2}} \|\nabla K\|_{\infty,E} \|\nabla q^{m-1}\|_{0,E})$$

with a similar inequality in the 3D case. Therefore, in both cases, there exists a constant C_0 that depends on the above constants but is independent of h, m, and E, such that if

$$h_0 = \frac{1}{2C_0\|\nabla K\|_{\infty,E}} \tag{5.11}$$

and $h_E \leq h_0$, then the perturbation is bounded by $\|\nabla q^{m-1}\|_{0,E}/2$. But either $\|\nabla q^{m-1}\|_{0,E} \leq \|\nabla q^m\|_{0,E}$, in which case (5.10) is established for $\|\nabla q^m\|_{0,E}$, or

$$\|\nabla q^m\|_{0,E} \leq \|\nabla q^{m-1}\|_{0,E}$$

and the induction hypothesis implies (5.10). Therefore, in both cases, (5.10) is proven by induction. Furthermore, substituting (5.10) and (2.8) into the analogue of (5.9) for $\nabla^i q^m$, we easily derive that, for $i = 0, 1, 2$,

$$\|\nabla^i q^m\|_{0,E} \leq \begin{cases} C|E|^{\frac{1}{2}} h_E^{\frac{1}{2}-i} \|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} & \text{in two dimensions,} \\ C|E|^{\frac{1}{2}} h_E^{-i} \|\nabla f \cdot \boldsymbol{\nu}\|_{0,\partial E} & \text{in three dimensions.} \end{cases}$$

It follows from these bounds, uniform with respect to m, that a subsequence of q^m (still denoted by q^m) converges to a polynomial of $\mathbb{P}_2(E)$, and passing to the limit with respect to m in (5.8), q satisfies

$$\forall\, e \in \partial E, \quad \int_e K\nabla q \cdot \boldsymbol{\nu} = \int_e K\nabla f \cdot \boldsymbol{\nu}.$$

The estimates for q follow from the previous bounds and the argument of Theorem 5.1. ∎

COROLLARY 5.3. *Let $h \leq h_0$, where h_0 is defined by (5.11), let $\mathcal{E}_h$ consist of triangles, parallelograms, or tetrahedra (in three dimensions), and let $p \in H^s(\mathcal{E}_h)$, for $s \geq 2$, and let $r \geq 2$. There exists an interpolant of p, $\tilde{p}^I \in \mathcal{D}_r(\mathcal{E}_h)$ satisfying*

$$\int_{e_k} \{K\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k\} = 0 \quad \forall\, k = 1, \ldots, P_h, \tag{5.12}$$

$$\int_{e_k} K\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k = 0 \quad \forall\, e_k \in \Gamma_D, \tag{5.13}$$

$$|||\nabla^i(\tilde{p}^I - p)|||_0 \leq C\frac{h^{\mu-i}}{r^{s-\frac{3}{2}-\delta}}|||p|||_s, \quad i = 0, 1, 2, \tag{5.14}$$

$$\sum_{k=1}^{P_h}\int_{e_k} |\{K\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k\}| + \sum_{k=P_h+1}^{M_h}\int_{e_k} |K\nabla(\tilde{p}^I - p) \cdot \boldsymbol{\nu}_k| \leq C(K)\frac{h^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}, \tag{5.15}$$

where $\delta = 0$ if $i = 0, 1$, $\delta = \frac{1}{2}$ if $i = 2$, and $\mu = \min(r+1, s)$, and C is independent of h and r.

Proof. The proof is a straightforward application of Corollary 5.2. ∎

THEOREM 5.4. *Assume $\Omega \subset \mathbb{R}^n$ for $n = 2, 3$. If $\alpha \equiv 0$, then there is a constant C independent of h, r, p such that for $s \geq 2$ and $r \geq 2$*

$$|||K^{\frac{1}{2}}\nabla(p - P^{\mathrm{DG}})|||_0 \leq C(K)\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}|||p|||_s.$$

If $\alpha \geq \alpha_0 > 0$, then the following inequality holds:

$$|||p - P^{\mathrm{DG}}|||_1 \leq C(K, \|\alpha\|_\infty) \frac{h^{\mu-1}}{r^{s-\frac{5}{2}}} |||p|||_s,$$

where $\mu = \min(r+1, s)$.

Proof. The difference between P^{DG} and p satisfies

$$a_{NS}(P^{\mathrm{DG}} - p, v) = 0 \quad \forall\, v \in \mathcal{D}_r(\mathcal{E}_h).$$

We take $v = P^{\mathrm{DG}} - \tilde{p}^I$, where $\tilde{p}^I$ is the interpolant of p constructed in Corollary 5.3 and denote $\chi = P^{\mathrm{DG}} - \tilde{p}^I$. We have

$$\begin{aligned} a_{NS}(\chi, \chi) = &\sum_{j=1}^{N_h} \int_{E_j} (K\nabla(p - \tilde{p}^I) \cdot \nabla\chi + \alpha(p - \tilde{p}^I)\chi) \\ &- \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p - \tilde{p}^I) \cdot \boldsymbol{\nu}_k\}[\chi] + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\chi \cdot \boldsymbol{\nu}_k\}[p - \tilde{p}^I] \\ &- \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla(p - \tilde{p}^I) \cdot \boldsymbol{\nu}_k)\chi + \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla\chi \cdot \boldsymbol{\nu}_k)(p - \tilde{p}^I). \end{aligned}$$

In view of (5.12) and (5.13), we can write

$$\begin{aligned} a_{NS}(\chi, \chi) = &\sum_{j=1}^{N_h} \int_{E_j} (K\nabla(p - \tilde{p}^I) \cdot \nabla\chi + \alpha(p - \tilde{p}^I)\chi) \\ &- \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p - \tilde{p}^I) \cdot \boldsymbol{\nu}_k\}([\chi] - c_k) + \sum_{k=1}^{P_h} \int_{e_k} \{K\nabla\chi \cdot \boldsymbol{\nu}_k\}[p - \tilde{p}^I] \\ &- \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla(p - \tilde{p}^I) \cdot \boldsymbol{\nu}_k)(\chi - c_k) + \sum_{e_k \in \Gamma_D} \int_{e_k} (K\nabla\chi \cdot \boldsymbol{\nu}_k)(p - \tilde{p}^I), \end{aligned} \tag{5.16}$$

where c_k is chosen as follows: as previously, we assume that $e_k = \partial E^1 \cap \partial E^2$, where E^1 and E^2 are elements of $\mathcal{E}_h$ adjacents to e_k; then we take $c_k = c_1 - c_2$, where $c_i = \frac{1}{|E^i|} \int_{E^i} \chi, i = 1, 2$. Then

$$\|[\chi] - c_k\|_{0,e_k} \leq \|\chi|_{E^1} - c_1\|_{0,e_k} + \|\chi|_{E^2} - c_2\|_{0,e_k}$$

and applying (2.5) and (2.2),

$$\|\chi|_{E^1} - c_1\|_{0,e_k} \leq C(h_1^{-\frac{1}{2}} \|\chi - c_1\|_{0,E^1} + h_1^{\frac{1}{2}} |\chi|_{1,E^1}) \leq C h_1^{\frac{1}{2}} |\chi|_{1,E^1}.$$

Hence,

$$\|[\chi] - c_k\|_{0,e_k} \leq C(h_1^{\frac{1}{2}} |\chi|_{1,E^1} + h_2^{\frac{1}{2}} |\chi|_{1,E^2}).$$

Therefore, (5.15) gives

$$\sum_{k=1}^{P_h} \int_{e_k} \{K\nabla(p - \tilde{p}^I) \cdot \boldsymbol{\nu}_k\}([\chi] - c_k) \leq C(K) \frac{h^{\mu-1}}{r^{s-\frac{3}{2}}} |||K^{\frac{1}{2}} \nabla\chi|||_0 |||p|||_s.$$

The first two terms and the fourth term are estimated as in Theorem 4.1 (the same result as in (3.4) except for the dependence with respect to r that becomes r^{2s-5}). Finally, the boundary terms are estimated as the interior terms, and the theorem is obtained by combining all the results together. ∎

So far, we do not know how to derive a sharper error estimate for the DG method in the L^2 norm. As we have seen in Theorem 5.4, if α is bounded away from zero, the error in the H^1 norm gives automatically a bound for the error in the L^2 norm, but we lose a power of h with respect to the interpolation error.

The next proposition extends this result in two dimensions for the pure Neumann problem to the case where α is not bounded away from zero. The same argument can be applied to the pure Dirichlet problem. In the case of mixed boundary conditions, the same result also holds, provided the corresponding dual problem satisfies the regularity assumption of Proposition 5.5.

PROPOSITION 5.5. *Let the dimension $n = 2$. In addition to the assumptions made at the beginning of this section, suppose K is such that for any f in $L^{\frac{4}{3}}(\Omega)$, the solution $\phi \in H^1(\Omega)$ of the dual problem defined as*

$$-\nabla \cdot K\nabla\phi + \alpha\phi = f \quad \text{in} \quad \Omega, \tag{5.17}$$

$$K\nabla\phi \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \partial\Omega \tag{5.18}$$

belongs to $W^{2,\frac{4}{3}}(\mathcal{E}_h)$, with continuous dependence on f. Then,

$$\|P^{\mathrm{DG}} - p\|_{0,\Omega} \leq C\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}(C_1 + C_2 h^{\frac{1}{2}} r)$$

for $r \geq 2, s \geq 2$; C_1 and C_2 are independent of h, r, p.

Proof. Let $\tilde{p}$ be an interpolant of p in $\mathcal{D}_r(\mathcal{E}_h) \cap C^0(\bar{\Omega})$ that satisfies the approximation properties (2.2)–(2.4) and denote $\chi = P^{\mathrm{DG}} - \tilde{p}$. Consider the dual problem (5.17)–(5.18) with $f = \chi$. By assumption, ϕ belongs to $H^1(\Omega) \cap W^{2,\frac{4}{3}}(\mathcal{E}_h)$ and there exists a constant C that depends on Ω such that

$$|||\phi|||_{2,\frac{4}{3}} \leq C\|\chi\|_{\frac{4}{3},\Omega},$$

where $|||\cdot|||_{k,m}$ denotes the Sobolev broken norm on $W^{k,m}(\mathcal{E}_h)$. We observe that the arguments in the proof of Lemma 2.2 are valid for ϕ in $H^1(\Omega) \cap W^{2,\frac{4}{3}}(\mathcal{E}_h)$:

$$a_{NS}(\phi, v) = \int_\Omega \chi v \quad \forall\, v \in H^2(\mathcal{E}_h).$$

Therefore, we obtain

$$\|\chi\|_{0,\Omega}^2 = a_{NS}(\phi, \chi).$$

We note that by symmetry of K, the following property holds for the nonsymmetric bilinear form a_{NS}:

$$a_{NS}(\phi, \chi) + a_{NS}(\chi, \phi) = 2\sum_{j=1}^{N_h} \int_{E_j} K\nabla\chi\nabla\phi + 2\sum_{j=1}^{N_h} \int_{E_j} \alpha\chi\phi.$$

Thus, we obtain

$$\|\chi\|_{0,\Omega}^2 = 2\sum_{j=1}^{N_h} \int_{E_j} (K\nabla\chi\nabla\phi + \alpha\chi\phi) - a_{NS}(\chi, \phi). \tag{5.19}$$

We now choose ϕ^* in $\mathcal{D}_2(\mathcal{E}_h)$ an interpolant of ϕ constructed in Corollary 5.3, and we add the orthogonality equation $a_{NS}(\chi,\phi^*) - a_{NS}(p-\tilde{p},\phi^*) = 0$:

$$\begin{aligned}
\|\chi\|_{0,\Omega}^2 &= 2\sum_{j=1}^{N_h}\int_{E_j}(K\nabla\phi\nabla\chi+\alpha\chi\phi) - a_{NS}(\chi,\phi-\phi^*) - a_{NS}(p-\tilde{p},\phi^*)\\
&= 2\sum_{j=1}^{N_h}\int_{E_j}(K\nabla\phi\nabla\chi+\alpha\chi\phi) - \sum_{j=1}^{N_h}\int_{E_j}K\nabla\chi\nabla(\phi-\phi^*) - \sum_{j=1}^{N_h}\int_{E_j}\alpha\chi(\phi-\phi^*)\\
&\quad - \sum_{j=1}^{N_h}\int_{E_j}K\nabla(p-\tilde{p})\nabla\phi^* - \sum_{j=1}^{N_h}\int_{E_j}\alpha(p-\tilde{p})\phi^*\\
&\quad + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[\phi-\phi^*] - \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}[\chi]\\
&\quad + \sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\phi^*]. \qquad (5.20)
\end{aligned}$$

Using the fact that ϕ belongs to $W^{2,\frac{4}{3}}(\mathcal{E}_h) \hookrightarrow H^{\frac{3}{2}}(\mathcal{E}_h)$, we bound the volume terms as follows:

$$\begin{aligned}
\left|2\sum_{j=1}^{N_h}\int_{E_j}(K\nabla\phi\nabla\chi+\alpha\phi\chi)\right| &\le C(|||K^{\frac{1}{2}}\nabla\chi|||_0 + |||\alpha^{\frac{1}{2}}\chi|||_0)\|\chi\|_0,\\
\left|\sum_{j=1}^{N_h}\int_{E_j}(K\nabla\chi\nabla(\phi-\phi^*)+\alpha\chi(\phi-\phi^*))\right| &\le Ch^{\frac{1}{2}}(|||K^{\frac{1}{2}}\nabla\chi|||_0 + h|||\alpha^{\frac{1}{2}}\chi|||_0)|||\phi|||_{\frac{3}{2}}\\
&\le Ch^{\frac{1}{2}}(|||K^{\frac{1}{2}}\nabla\chi|||_0 + h|||\alpha^{\frac{1}{2}}\chi|||_0)|||\phi|||_{2,\frac{4}{3}}\\
&\le Ch^{\frac{1}{2}}(|||K^{\frac{1}{2}}\nabla\chi|||_0 + h|||\alpha^{\frac{1}{2}}\chi|||_0)\|\chi\|_{0,\Omega},\\
\left|\sum_{j=1}^{N_h}\int_{E_j}(K\nabla(p-\tilde{p})\nabla\phi^*+\alpha(p-\tilde{p})\phi^*)\right| &\le C\frac{h^{\mu-1}}{r^{s-1}}|||p|||_s\left(|||\nabla\phi^*|||_0 + \frac{h}{r}|||\phi^*|||_0\right)\\
&\le C\frac{h^{\mu-1}}{r^{s-1}}\left(1+\frac{h}{r}\right)|||p|||_s\|\chi\|_{0,\Omega}.
\end{aligned}$$

The first edge term in (5.20) is bounded by the inverse estimate (2.8), the trace theorem (2.5) combined with Corollary 5.3, with the same notation as in Theorem 3.1:

$$\begin{aligned}
\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla\chi\cdot\boldsymbol{\nu}_k\}[\phi-\phi^*]\right| &\le C\sum_{k=1}^{P_h}\bar{h}_k^{-\frac{1}{2}}r\|K^{\frac{1}{2}}\nabla\chi\|_{0,E_k^{12}}\bar{h}_k|||\phi|||_{\frac{3}{2},E_k^{12}}\\
&\le Crh^{\frac{1}{2}}|||K^{\frac{1}{2}}\nabla\chi|||_0\|\chi\|_{0,\Omega}. \qquad (5.21)
\end{aligned}$$

Similarly as in Theorem 5.4, we define a constant on each edge (or face) e_k by

$$c_k = \frac{1}{|E^1|}\int_{E^1}\chi - \frac{1}{|E^2|}\int_{E^2}\chi.$$

To bound the next term in (5.20), we use the imbedding $W^{\frac{1}{4},\frac{4}{3}}(\hat{e}) \hookrightarrow L^2(\hat{e})$ and the trace theorem $\|\hat{f}\|_{W^{\frac{1}{4},\frac{4}{3}}(\hat{e})} \leq \hat{C}\|\hat{f}\|_{W^{1,\frac{4}{3}}(\hat{E})}$. Thus, by Corollary 5.3, we derive

$$\begin{aligned}\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}[\chi]\right| &\leq \sum_{k=1}^{P_h}\|\{K\nabla(\phi-\phi^*)\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}\|[\chi]-c_k\|_{0,e_k}\\ &\leq C\sum_{k=1}^{P_h}|\!|\!|\phi|\!|\!|_{2,\frac{4}{3},E_k^{12}}\bar{h}_k^{\frac{1}{2}}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_{0,E_k^{12}}\\ &\leq Ch^{\frac{1}{2}}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0\left(\sum_{k=1}^{P_h}|\!|\!|\phi|\!|\!|^2_{2,\frac{4}{3},E_k^{12}}\right)^{\frac{1}{2}}\\ &\leq Ch^{\frac{1}{2}}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0|\!|\!|\phi|\!|\!|_{2,\frac{4}{3}}\\ &\leq Ch^{\frac{1}{2}}|\!|\!|K^{\frac{1}{2}}\nabla\chi|\!|\!|_0\|\chi\|_{0,\Omega}.\end{aligned} \tag{5.22}$$

Using the approximation properties, we can bound the last term in (5.20)

$$\begin{aligned}\left|\sum_{k=1}^{P_h}\int_{e_k}\{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}[\phi^*]\right| &\leq \sum_{k=1}^{P_h}\|\{K\nabla(p-\tilde{p})\cdot\boldsymbol{\nu}_k\}\|_{0,e_k}\|[\phi^*-\phi]\|_{0,e_k}\\ &\leq C\sum_{k=1}^{P_h}\frac{\bar{h}_k^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}|\!|\!|p|\!|\!|_{s,E_k^{12}}\bar{h}_k|\!|\!|\phi|\!|\!|_{\frac{3}{2},E_k^{12}}\\ &\leq C\frac{h^{\mu-\frac{1}{2}}}{r^{s-\frac{3}{2}}}|\!|\!|p|\!|\!|_s\|\chi\|_{0,\Omega}.\end{aligned} \tag{5.23}$$

The theorem is obtained by combining all the bounds above. ∎

Remark. In three dimensions, the duality argument used in proving Proposition 5.5 is more delicate. The regularity of the dual problem (5.17)–(5.18) should be as follows: for any $f \in L^2(\Omega)$, the solution ϕ of (5.17)–(5.18) belongs to $W^{2,\frac{3}{2}}(\mathcal{E}_h)$ with continuous dependence on f. But, as it is shown in [8], for K sufficiently smooth, ϕ belongs to $W^{2,p}(\Omega)$ for any $p < p_0$ and $p_0 \geq \frac{4}{3}$, and this is substantially smaller than $\frac{3}{2}$. To simplify the discussion, let us assume that ϕ belongs to $W^{2,\frac{4}{3}}(\mathcal{E}_h)$. Then on one hand $W^{2,\frac{4}{3}}(E) \hookrightarrow H^{\frac{5}{4}}(E)$ and on the other hand, the trace of $\nabla\phi$ on a face e belongs to $W^{\frac{1}{4},\frac{4}{3}}(e) \hookrightarrow L^{\frac{3}{5}}(e)$. Thus, by using Hölder's inequality, we can modify the proof of Proposition 5.5 and derive in three dimensions that, under the very mild regularity assumption $\phi \in W^{2,\frac{4}{3}}(\mathcal{E}_h)$, we have

$$\|P^{\mathrm{DG}} - p\|_{0,\Omega} \leq C\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}(C_1 + C_2h^{\frac{1}{2}}r).$$

6. Numerical example. The reader will find in [6, 13] several examples of numerical computations with the DG method that confirm our theoretical results in the case of smooth solutions. In contrast, the experiment in this section investigates the robustness of the DG method when solving problem (2.10) with discontinuous coefficients and a nonsmooth solution. The domain $\Omega = (-1,1)^2$ is divided into four subdomains Ω_i as shown in Figure 6.1. The tensor K is a constant tensor K_i over each subdomain and we assume that $K_1 = K_3$ and $K_2 = K_4$. We consider

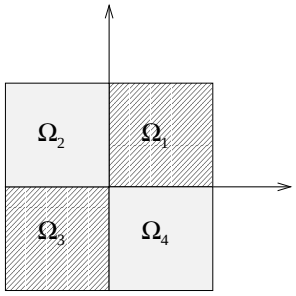

FIG. 6.1. *Initial domain for the problem with discontinuous coefficients.*

the case where $K_1 = 5Id$ and $K_2 = Id$. We seek an analytical solution of the form $r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$, where (r, θ) are the polar coordinates of a given point in Ω and a_i, b_i are constants that depend on the subdomain Ω_i.

We can easily show that with the choice $\alpha = 0.53544094560$, the analytical solution p_{ex} satisfies the usual interface conditions: p_{ex} and $K\nabla p_{ex} \cdot \boldsymbol{\nu}$ are continuous across the interfaces and $-\nabla \cdot K\nabla p = 0$ on Ω. In addition, the solution has a singularity at the origin, so that it belongs to $H^1(\Omega)$ but not to $H^2(\Omega)$. We first analyze the convergence of the DG solution on a sequence of uniformly refined meshes. The coarse mesh consists of the four subdomains Ω_i. The relative error in the L^2 norm, defined as $\frac{\|p - P^{\mathrm{DG}}\|_0}{\|p\|_0}$, is plotted against the number of degrees of freedom in Figure 6.2. First, we choose finite elements of degree two, and then of degree three. We see, for these elements and in this example, that the convergence rate does not depend on the degree of the approximation, but that it depends on the regularity of the solution. In that case, the convergence rate in the L^2 norm is $O(h^{2\alpha}) = O(h^{1.07})$, where h is the

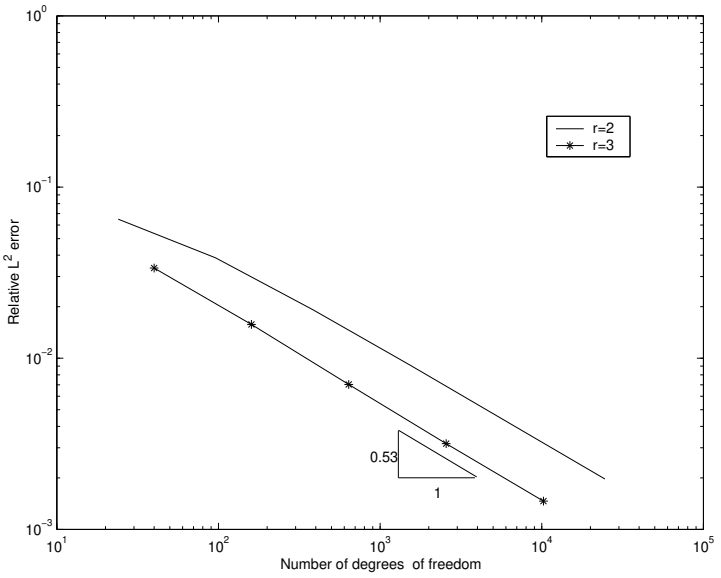

FIG. 6.2. *Relative error in the L^2 norm versus the number of degrees of freedom for uniform meshes in the case $r = 2, 3$.*

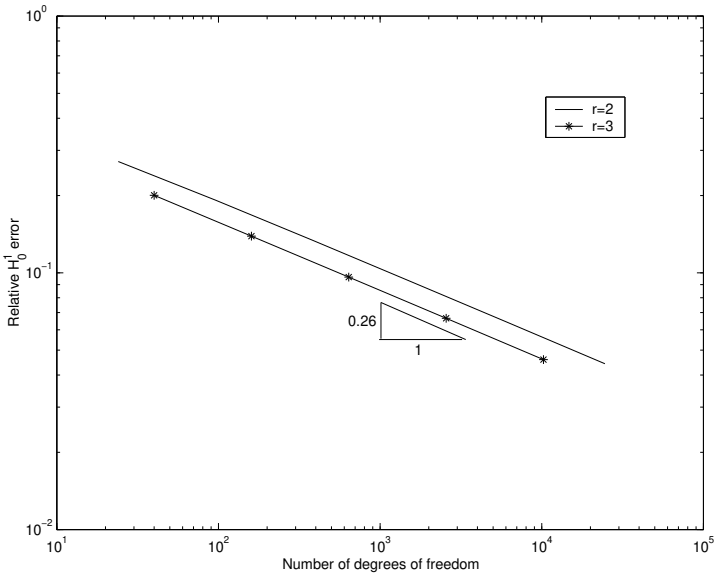

FIG. 6.3. *Relative error in the H^1 norm versus the number of degrees of freedom for uniform meshes in the case $r = 2, 3$.*

mesh size, or $O(N^\alpha)$, where N is the number of degrees of freedom. We also study the numerical error in the energy norm and we see in Figure 6.3 that the convergence rate of the error in the energy norm is $O(h^\alpha) = O(N^{\frac{\alpha}{2}})$, which is a little bit better than expected.

We now study the influence of adaptivity on the numerical DG solution. We refer to [16] for more details on the mesh adaptation strategy that we use here. Here, the error indicators are the norms of solutions of a discrete problem, for which the approximation order is $r + q$. Therefore, we denote by q the degree of enrichment. We begin with a uniform mesh of size $h = 1$ and we adaptively refine the mesh. We fix $r = 2$ and $q = 5$ and we see that adaptive mesh refinements lead to optimal convergence rates (Figure 6.4).

7. Conclusion and perspectives. The energy estimates we have derived in this paper are local in the H^1 norm, i.e., they involve only the local regularity of the solution element per element, and not its global regularity. Of course, for recovering full accuracy in the L^2 norm, we need the global H^2 regularity of the dual problem. But this is always necessary for a duality argument.

These discontinuous Galerkin methods have a very wide range of applications. The most interesting one from the computational point of view is the DG method because it is completely unconstrained. As a future extension, we propose to investigate the numerical analysis of the DG method for problems of miscible displacement in porous media, which involves the analysis of time-dependent convection-diffusion equations. Numerical simulations with the DG method give very promising results [17], especially compared to other methods. Another extension concerns subdivisions with nonmatching grids. The advantage of the DG method is that it does not seem to require the introduction of mortar elements or penalties.

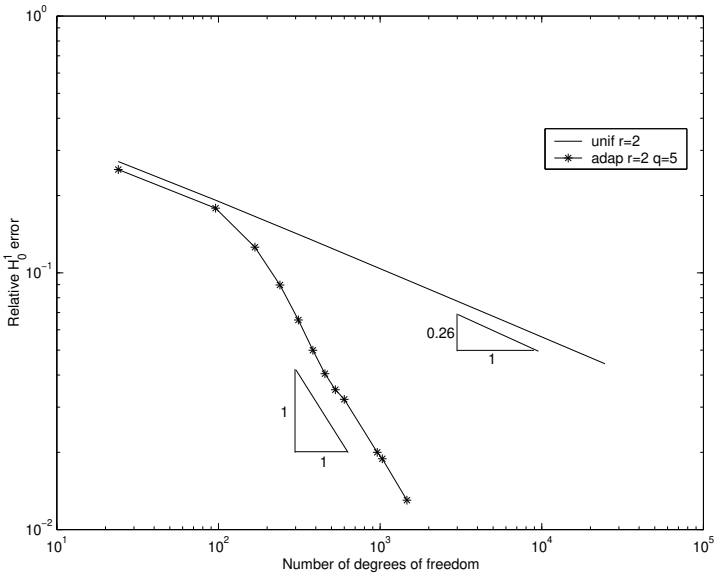

FIG. 6.4. *Relative error in the H^1 norm versus the number of degrees of freedom for uniformly and adaptively refined meshes in the case $r = 2$, $q = 5$.*

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