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A DISCONTINUOUS SUBGRID EDDY VISCOSITY METHOD FOR THE TIME-DEPENDENT NAVIER–STOKES EQUATIONS*

SONGUL KAYA† AND BÉATRICE RIVIÈRE‡

Abstract. In this paper we provide an error analysis of a subgrid scale eddy viscosity method using discontinuous polynomial approximations for the numerical solution of the incompressible Navier–Stokes equations. Optimal continuous in time error estimates of the velocity are derived. The analysis is completed with some error estimates for two fully discrete schemes, which are first and second order in time, respectively.

Key words. error analysis, Navier–Stokes, discontinuous Galerkin, fully discrete scheme, high order method

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1. Introduction.

The goal of this paper is to formulate and analyze a subgrid eddy viscosity method for solving the incompressible time-dependent Navier–Stokes equations. If the separation point between large and small scales is held fixed, the model can be viewed as a large eddy simulation (LES) model. On the other hand, if the separation point is decreased as the mesh size tends to zero, the model can be viewed (and analyzed, as herein) as a numerical regularization of the Navier–Stokes equations.

For many flows in nature, capturing all the scales in a numerical simulation is an impossible task, since the scale separation may span several orders of magnitude. Global diffusion is the traditional phenomenology to model the dispersive effects of unresolved scales on resolved scales. The traditional approach for incorporating the effects of unresolved scales on the resolved ones for the Navier–Stokes equations utilizes eddy viscosity models. These models, first formulated by Boussinesq [5] and developed by Taylor and Prandlt [10], introduce a dissipation mechanism (Smagorinsky [29]). Standard eddy viscosity models act on all scales of motion, and their effects can be too diffusive on the coarse scales (Lewandowski [26] and Iliescu and Layton [19]). The idea of applying the eddy viscosity models on only the small scales results in the subgrid eddy viscosity method, introduced and analyzed by Guermond [14], Layton [24], and John and Kaya [20]. This subgrid eddy viscosity method can also be thought of as an extension to general domains and boundary conditions of the spectral vanishing viscosity idea of Maday and Tadmor [27]. Recently, Hughes, Mazzei, and Jansen [17] proposed a variational multiscale method (VMM) in which the diffusion acts only at the finest resolved scales. VMM is a promising approach in multiscale turbulence modelling. There are different choices on how to define coarse and small scales within the VMM framework. One approach is to define fluctuations via bubble functions and means via L^2 projection (Guermond [14] and Hughes [16]). Another possibility is to

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†Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616 (kaya@iit.edu).
‡Department of Mathematics, 301 Thackeray Hall, University of Pittsburgh, Pittsburgh, PA 15260 (riviere@math.pitt.edu).

define fluctuations via the finest resolved scales in a hierarchy of finite element spaces, and means via elliptic or Stokes projection (Layton [24], Kaya and Layton [22], and Hughes [18]).

For any numerical method, the error equation arising from the Navier–Stokes equations contains a convection-like term and a reaction (or stretching) term. Discontinuous Galerkin (DG) methods, first introduced in the work of Reed and Hill [28] and Lesaint and Raviart [25], are particularly efficient in controlling convective error terms. On the other hand, (generally nonlinear) eddy viscosity models are, in a sense, intended to give some control of the error's reaction-like terms. Indeed, the exponential sensitivity of trajectories of the Navier–Stokes equations (arising from reaction-like terms) is widely believed to be limited to the small scales. It is thus conjectured that by modelling their action on the large scales, the exponential sensitivity introduced by the reaction-like terms will be contained.

DG methods have recently become more popular in the science and engineering community. They use piecewise polynomial functions with no continuity constraint across element interfaces. As a result, variational formulations must include jump terms across interfaces [31]. The DG methods offers several advantages, including (i) flexibility in the design of the meshes and in the construction of trial and test spaces, (ii) local conservation of mass, (iii) h-p adaptivity, and (iv) higher order local approximations. DG methods have become widely used for solving computational fluid problems, especially diffusion and pure convection problems [3]. The reader should refer to Cockburn, Karniadakis, and Shu [6] for a historical review of DG methods. For the steady-state Navier–Stokes equations, a totally discontinuous finite element method is formulated in [12], while in [21], the velocity is approximated by discontinuous polynomials that are pointwise divergence-free, and the pressure by continuous polynomials.

Combining DG and eddy viscosity techniques is clearly advantageous. While convective effects are accurately modelled by DG, the dispersive effects of small scales on the large scales are correctly taken into account with the eddy viscosity model. Besides, due to the absence of continuity constraints, one can select various basis functions (such as hierarchical basis functions) for the coarse and refined scales. As an appropriate first step, we consider in this paper the combination of DG methods with a linear eddy viscosity model. We show that the errors are optimal with respect to the mesh size and depend on the Reynolds number in a reasonable fashion. The particular eddy viscosity model considered here was introduced in [24], and complete numerical analysis for Navier–Stokes equations was performed in [20] where it was combined with the classical finite element method.

The outline of the paper is as follows. The model problem and notation are presented in section 2. In section 3, a variational formulation and scheme are introduced. Section 4 contains the continuous in time algorithm, some stability results, and some error estimates. In section 5 , two fully discrete schemes are formulated and analyzed. Conclusions are given in the last section.

2. Notation and preliminaries.

We consider the time-dependent Navier–Stokes equations for incompressible flow as follows:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \text{ for } 0 < t \leq T, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega \text{ for } 0 < t \leq T, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{in } \Omega \text{ for } t = 0, \tag{2.3}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega \text{ for } 0 < t \leq T, \tag{2.4}$$

where $\boldsymbol{u}$ is the fluid velocity, p the pressure, $\boldsymbol{f}$ the external force, $\nu > 0$ the kinematic viscosity, and $\Omega \subset \mathbb{R}^2$ a bounded, simply connected domain with polygonal boundary $\partial\Omega$. We also impose the usual normalization condition on the pressure, namely, that $\int_\Omega p = 0$.

Let $\mathcal{K}_h = \{E_j, j = 1, \dots, N_h\}$ denote a nondegenerate triangulation of the domain Ω. Let h denote the maximum diameter of the elements E_j in $\mathcal{K}_h$. We denote the edges of $\mathcal{K}_h$ by $\{e_1, e_2, \dots, e_{P_h}, e_{P_h+1}, \dots, e_{M_h}\}$, where $e_k \subset \Omega$ for $1 \le k \le P_h$ and $e_k \subset \partial\Omega$ for $P_{h+1} \le k \le M_h$. With each edge we associate a normal unit vector $\mathbf{n}_k$. For $k > P_h$, the unit vector $\mathbf{n}_k$ is taken to be outward normal to $\partial\Omega$. Let e_k be an edge shared by elements E_i and E_j with $\mathbf{n}_k$ exterior to E_i. We define the jump $[\phi]$ and average $\{\phi\}$ of a function ϕ by

$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}.$$

If e belongs to the boundary $\partial\Omega$, the jump and average of ϕ coincide with its trace on e. We shall use standard notation for Sobolev spaces [1]. For any nonnegative integer s and $r \ge 1$, the classical Sobolev space on a domain $E \subset \mathbb{R}^2$ is

$$W^{s,r}(E) = \{v \in L^r(E) : \forall\, |m| \le s,\, \partial^m v \in L^r(E)\},$$

where $\partial^m v$ are the partial derivatives of v of order $|m|$. The usual norm in $W^{s,r}(E)$ is denoted by $\|\cdot\|_{s,r,E}$ and the seminorm by $|\cdot|_{s,r,E}$. The L^2 inner-product is denoted by $(\cdot,\cdot)_E$ and by $(\cdot,\cdot)$ if $E = \Omega$. For the Hilbert space $H^s(E) = W^{s,2}(E)$, the norm is denoted by $\|\cdot\|_{s,E}$. By $H^1_0(E)$ we shall understand the subspace of $H^1(E)$ functions that vanish on ∂E. Throughout the paper, boldface characters denote vector quantities. Define

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) : \quad \nabla\cdot\boldsymbol{v} = 0\}, \quad \boldsymbol{H} = \{\boldsymbol{v} \in L^2(\Omega)^2 : \quad \nabla\cdot\boldsymbol{v} = 0, \boldsymbol{v} = \boldsymbol{0}\}.$$

For any function $\boldsymbol{\phi}$ that depends on time t and space $\boldsymbol{x}$, denote

$$\boldsymbol{\phi}(t)(\boldsymbol{x}) = \boldsymbol{\phi}(t,\boldsymbol{x}) \quad \forall t \in [0,T], \forall \boldsymbol{x} \in \Omega.$$

If Y denotes a functional space in the space variable with the norm $\|\cdot\|_Y$ and if $\boldsymbol{\phi} = \boldsymbol{\phi}(t,\boldsymbol{x})$, then for $s > 0$

$$\|\boldsymbol{\phi}\|_{L^s(0,T;Y)} = \left[\int_0^T \|\boldsymbol{\phi}(t)\|_Y^s dt\right]^{1/s}, \quad \|\boldsymbol{\phi}\|_{L^\infty(0,T;Y)} = \max_{0\le t\le T} \|\boldsymbol{\phi}(t)\|_Y.$$

Recall that for a vector function $\boldsymbol{\phi}$, the tensor $\nabla\boldsymbol{\phi}$ is defined as $(\nabla\boldsymbol{\phi})_{i,j} = \frac{\partial\phi_i}{\partial x_j}$ and the tensor product of two tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ is defined as $\boldsymbol{T} : \boldsymbol{S} = \sum_{i,j} T_{ij} S_{ij}$. We define the following *broken* norm for positive s:

$$|\!|\!|\cdot|\!|\!|_s = \left[\sum_{j=1}^{N_h} \|\cdot\|^2_{s,E_j}\right]^{1/2}.$$

From [30], if $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$, there exists a solution $(\boldsymbol{u},p)$ of (2.1)–(2.4) such that $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;\boldsymbol{V})$. In addition, we will assume that $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{W}^{2,4/3}(\Omega))$ and $p \in L^\infty(0,T;W^{1,4/3}(\Omega))$ for the DG formulation to be well defined. For the analysis obtained in sections 4 and 5, we require extra regularity on the solution: $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^2(\Omega)), p \in L^2(0,T;H^1(\Omega))$. This assumption is valid if the data is more regular [30]: $\boldsymbol{f} \in L^\infty(0,T;\boldsymbol{H}), \boldsymbol{f}_t \in L^2(0,T;\boldsymbol{V}'), \boldsymbol{f}(0) \in \boldsymbol{H}$, $\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$. The following functional spaces are defined:

$$\boldsymbol{X} = \{\boldsymbol{v} \in (L^2(\Omega))^2 : \boldsymbol{v}|_{E_j} \in \boldsymbol{W}^{2,4/3}(E_j) \quad \forall E_j \in \mathcal{K}_h\},$$
$$Q = \{q \in L_0^2(\Omega) : q|_{E_j} \in W^{1,4/3}(E_j) \quad \forall E_j \in \mathcal{K}_h\},$$

where $L_0^2(\Omega)$ is given by

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}.$$

We associate to $(\boldsymbol{X}, Q)$ the following norms:

$$\|\boldsymbol{v}\|_X = (\|\!|\nabla \boldsymbol{v}|\!\|_0^2 + J(\boldsymbol{v},\boldsymbol{v}))^{\frac{1}{2}} \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \quad \|q\|_Q = \|q\|_{0,\Omega} \quad \forall q \in Q,$$

where the jump term J is defined as

$$J(\boldsymbol{u},\boldsymbol{v}) = \sum_{k=1}^{M_h} \frac{\sigma}{|e|} \int_{e_k} [\boldsymbol{u}] \cdot [\boldsymbol{v}]. \tag{2.5}$$

In this jump term, $|e|$ denotes the measure of the edge e and σ is a constant parameter that will be specified later.

Recall the following property of norm $\|\cdot\|_X$ [12]: for each real number $p \in [2,\infty)$ there exists a constant $C(p)$ such that

$$\|\boldsymbol{v}\|_{L^p(\Omega)} \leq C(p)\|\boldsymbol{v}\|_X \quad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{2.6}$$

For any positive integer r, the finite-dimensional subspaces are

$$\boldsymbol{X}^h = \{\boldsymbol{v}^h \in \mathbf{X} : \boldsymbol{v}^h \in (\mathbb{P}_r(E_j))^2 \quad \forall E_j \in \mathcal{K}_h\},$$
$$Q^h = \{q^h \in Q : q^h \in \mathbb{P}_{r-1}(E_j) \quad \forall E_j \in \mathcal{K}_h\}.$$

We assume that for each integer $r \geq 1$, there exists an operator $R_h \in \mathcal{L}(\boldsymbol{H}^1(\Omega);\boldsymbol{X}^h)$ such that

$$\|R_h(\boldsymbol{v}) - \boldsymbol{v}\|_X \leq Ch^r|\boldsymbol{v}|_{r+1,\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega) \cap \boldsymbol{H}_0^1(\Omega), \tag{2.7}$$
$$\|\boldsymbol{v} - \boldsymbol{R}_h(\boldsymbol{v})\|_{0,E_j} \leq Ch_{E_j}^{r+1}|\boldsymbol{v}|_{r+1,\Delta_{E_j}} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega),\ 1 \leq j \leq N_h, \tag{2.8}$$

where Δ_{E_j} is a suitable macro element containing E_j. Note that for $r = 1, 2$, and 3, the existence of this interpolant follows from [8, 7, 9]. The bounds (2.7) and (2.8) are proved in [12] and in [13], respectively.

Also, for each integer $r \geq 1$, there is an operator $r_h \in \mathcal{L}(L_0^2(\Omega);Q_h)$ such that for any E_j in $\mathcal{K}_h$

$$\int_{E_j} z_h(r_h(q) - q) = 0 \quad \forall z_h \in \mathbb{P}_{r-1}(E_j), \forall q \in L_0^2(\Omega), \tag{2.9}$$
$$\|q - r_h(q)\|_{m,E_j} \leq Ch_{E_j}^{r-m}|q|_{r,E_j} \quad \forall q \in H^r(\Omega) \cap L_0^2(\Omega), m = 0, 1. \tag{2.10}$$

Finally, we recall some standard trace and inverse inequalities, which hold true on each element E in $\mathcal{K}_h$, with diameter h_E (see [11]):

$$\|\boldsymbol{v}\|_{0,e} \le C(h_E^{-1/2}\|\boldsymbol{v}\|_{0,E} + h_E^{1/2}\|\nabla\boldsymbol{v}\|_{0,E}) \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.11}$$

$$\|\nabla\boldsymbol{v}\|_{0,e} \le C(h_E^{-1/2}\|\nabla\boldsymbol{v}\|_{0,E} + h_E^{1/2}\|\nabla^2\boldsymbol{v}\|_{0,E}) \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.12}$$

$$\|\boldsymbol{v}\|_{L^4(e)} \le Ch_E^{-3/4}(\|\boldsymbol{v}\|_{0,E} + h_E\|\nabla\boldsymbol{v}\|_{0,E}) \quad \forall e \in \partial E, \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{2.13}$$

$$\|\boldsymbol{v}^h\|_{0,e} \le Ch_E^{-1/2}\|\boldsymbol{v}^h\|_{0,E} \quad \forall e \in \partial E, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{2.14}$$

$$\|\nabla\boldsymbol{v}^h\|_{0,e} \le Ch_E^{-1/2}\|\nabla\boldsymbol{v}^h\|_{0,E} \quad \forall e \in \partial E, \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{2.15}$$

$$\|\nabla\boldsymbol{v}^h\|_{0,E} \le Ch_E^{-1}\|\boldsymbol{v}^h\|_{0,E} \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{2.16}$$

$$\|\boldsymbol{v}^h\|_{L^4(E)} \le Ch_E^{-1/2}\|\boldsymbol{v}^h\|_{0,E} \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{2.17}$$

3. Variational formulation and scheme. Let us first define the bilinear forms $a : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ and $b : \boldsymbol{X} \times Q \to \mathbb{R}$:

$$a(\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^{N_h} \int_{E_j} \nabla\boldsymbol{v} : \nabla\boldsymbol{w} - \sum_{k=1}^{M_h} \int_{e_k} (\{\nabla\boldsymbol{v}\}\mathbf{n}_k \cdot [\boldsymbol{w}] - \epsilon_0\{\nabla\boldsymbol{w}\}\mathbf{n}_k \cdot [\boldsymbol{v}]), \tag{3.1}$$

$$b(\boldsymbol{v}, q) = -\sum_{j=1}^{N_h} \int_{E_j} q\nabla\cdot\boldsymbol{v} + \sum_{k=1}^{M_h} \int_{e_k} \{p\}[\boldsymbol{v}]\cdot\mathbf{n}_k, \tag{3.2}$$

where ϵ_0 takes the constant value 1 or -1. Throughout the paper, we will assume the following hypothesis: if $\epsilon_0 = 1$, the jump parameter σ is chosen to be equal to 1; if $\epsilon_0 = -1$, the jump parameter σ is bounded below by $\sigma_0 > 0$ and σ_0 is sufficiently large. Based on this assumption, we can easily prove the following lemma.

LEMMA 3.1. *There is a constant $\kappa > 0$ such that*

$$a(\boldsymbol{v}^h, \boldsymbol{v}^h) + J(\boldsymbol{v}^h, \boldsymbol{v}^h) \ge \kappa\|\boldsymbol{v}^h\|_X^2 \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{3.3}$$

In addition to these bilinear forms, we consider the following upwind discretization of the term $\boldsymbol{u}\cdot\nabla\boldsymbol{z}$:

$$\begin{aligned} c(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta}) = {} & \sum_{j=1}^{N_h} \left(\int_{E_j} (\boldsymbol{u}\cdot\nabla\boldsymbol{z})\cdot\boldsymbol{\theta} + \int_{\partial E_j^-} |\{\boldsymbol{u}\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{z}^{\text{int}} - \boldsymbol{z}^{\text{ext}})\cdot\boldsymbol{\theta}^{int} \right) \\ & + \frac{1}{2}\sum_{j=1}^{N_h} \int_{E_j} (\nabla\cdot\boldsymbol{u})\boldsymbol{z}\cdot\boldsymbol{\theta} - \frac{1}{2}\sum_{k=1}^{M_h} \int_{e_k} [\boldsymbol{u}]\cdot\boldsymbol{n}_k\{\boldsymbol{z}\cdot\boldsymbol{\theta}\} \end{aligned} \tag{3.4}$$

for all $\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta}$ in $\mathbf{X}$ and where on each element the inflow boundary is

$$\partial E_j^- = \{\boldsymbol{x} \in \partial E_j : \{\boldsymbol{u}\}\cdot\mathbf{n}_{E_j} < 0\},$$

and the superscript int (resp., ext) refers to the trace of the function on a side of E_j coming from the interior of E_j (resp., coming from the exterior of E_j on that side). Note that the form c is not linear with respect to its first argument but is linear with respect to its second and third arguments. To avoid any confusion, if necessary, in the analysis, we will explicitly write $c(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta}) = c_{\boldsymbol{w}}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\theta})$ when the inflow boundaries ∂E_j^- are defined with respect to the velocity $\{\boldsymbol{w}\}$. We finally recall the positivity of c proved in [12]:

$$c(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{z}) \ge 0 \quad \forall \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{X}. \tag{3.5}$$

With these forms, we consider a variational problem of (2.1)–(2.4): for all $t > 0$ find $\boldsymbol{u}(t) \in \boldsymbol{X}$ and $p(t) \in Q$ satisfying

$$(\boldsymbol{u}_t(t), \boldsymbol{v}) + \nu(a(\boldsymbol{u}(t), \boldsymbol{v}) + J(\boldsymbol{u}(t), \boldsymbol{v})) \\ + c(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}) + b(\boldsymbol{v}, p(t)) = (\boldsymbol{f}(t), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{3.6}$$

$$b(\boldsymbol{u}(t), q) = 0 \quad \forall q \in Q, \tag{3.7}$$

$$(\boldsymbol{u}(0), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{3.8}$$

We shall now show the equivalence of the strong and weak solutions.

LEMMA 3.2. *Every strong solution of* (2.1)–(2.4) *is also a solution of* (3.6)–(3.8) *and conversely.*

Proof. Fix $t > 0$. Let $(\boldsymbol{u}, p)$ be the solution of (2.1)–(2.4). Since $\boldsymbol{u}(t) \in \boldsymbol{H}_0^1(\Omega)$, by the trace theorem $[\boldsymbol{u}(t)] \cdot \boldsymbol{n}_k = 0$ on each edge. Also, $\nabla \cdot \boldsymbol{u}(t) = 0$; thus $\boldsymbol{u}$ satisfies (3.7). Multiplying the Navier–Stokes equation (2.1) by $\boldsymbol{v} \in \boldsymbol{X}$, integrating over each element, and summing over all elements yield

$$\sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{u}_t \cdot \boldsymbol{v} + \nu \nabla \boldsymbol{u} : \nabla \boldsymbol{v}) - \nu \sum_{k=1}^{M_h} \int_{e_k} [\nabla \boldsymbol{u} \mathbf{n}_k \cdot \boldsymbol{v}] + \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v} \\ - \sum_{j=1}^{N_h} \int_{E_j} p \nabla \cdot \boldsymbol{v} + \sum_{k=1}^{M_h} \int_{e_k} [p \boldsymbol{v} \cdot \mathbf{n}_k] = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}.$$

The boundary terms are rewritten as

$$\sum_{k=1}^{M_h} \int_{e_k} [\nabla \boldsymbol{u} \boldsymbol{n}_k . \boldsymbol{v}] = \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{u}\} \boldsymbol{n}_k \cdot [\boldsymbol{v}] + \sum_{k=1}^{M_h} \int_{e_k} [\nabla \boldsymbol{u}] \boldsymbol{n}_k \cdot \{\boldsymbol{v}\}.$$

The first part of the lemma is then obtained because the jumps of $\boldsymbol{u}, \nabla \boldsymbol{u} \boldsymbol{n}_k$, and p are zero almost everywhere.

Conversely, let $(\boldsymbol{u}, p)$ be a solution to (3.6)–(3.8). First, let E belong to $\mathcal{K}_h$ and choose $\boldsymbol{v} \in \mathcal{D}(E)^2$, extended by zero outside E. Then, $(\boldsymbol{u}, p)$ satisfy in the sense of distributions

$$\boldsymbol{u}_t - \nu \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in} \quad E. \tag{3.9}$$

Next consider $\boldsymbol{v} \in \mathcal{C}^1(\bar{E})$ such that $\boldsymbol{v} = \boldsymbol{0}$ on ∂E, extended by zero outside E, and $\nabla \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on ∂E except on one side e_k. We multiply (3.9) by $\boldsymbol{v}$ and integrate by parts. We then obtain

$$\int_{e_k} \{\nabla \boldsymbol{v}\} \boldsymbol{n}_k \cdot [\boldsymbol{u}] = 0,$$

which implies that $[\boldsymbol{u}] = \boldsymbol{0}$ almost everywhere on e_k. If e_k belongs to the boundary $\partial\Omega$, this implies that $\boldsymbol{u}|_{e_k} = \boldsymbol{0}$. Thus, $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$. Finally, choose $\boldsymbol{v} \in \mathcal{C}^1(\bar{E})$, with $\boldsymbol{v} = \boldsymbol{0}$ on ∂E except on one side e_k, extended by zero outside of E. Multiplying (3.9) by v and integrating by parts, we have

$$\int_{e_k} (-\nu \nabla \boldsymbol{u} \boldsymbol{n}_E + p \boldsymbol{n}_E) \cdot \boldsymbol{v} = \int_{e_k} \{-\nu \nabla \boldsymbol{u} \boldsymbol{n}_E + p \boldsymbol{n}_E\} \cdot \boldsymbol{v}.$$

Since $\boldsymbol{v}$ is arbitrary, this means that the quantity $-\nu\nabla \boldsymbol{u}\boldsymbol{n}_k + p\boldsymbol{n}_k$ is continuous across e_k. Therefore, (3.9) is satisfied over the entire domain Ω. The initial condition (2.3) is straightforward. ∎

We recall a discrete inf-sup condition and a property satisfied by R_h (see [12]).

LEMMA 3.3. *There exists a positive constant β_0, independent of h such that*

$$\inf_{q^h \in Q^h} \sup_{\boldsymbol{v}^h \in \mathbf{X}^h} \frac{b(\boldsymbol{v}^h, q^h)}{\|\boldsymbol{v}^h\|_X \, \|q^h\|_0} \geq \beta_0. \tag{3.10}$$

Furthermore, the operator R_h satisfies

$$b(R_h(\boldsymbol{v}) - \boldsymbol{v}, q^h) = 0 \quad \forall q^h \in Q^h, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{3.11}$$

In order to subtract the artificial diffusion introduced by the eddy viscosity on the coarse grid, we consider a coarsening of the mesh $\mathcal{K}_h$, namely $\mathcal{K}_H$, such that the fine mesh $\mathcal{K}_h$ is a refinement of $\mathcal{K}_H$ (so typically $h \ll H$). Denote by $\boldsymbol{L}$ the space of tensors $L^2(\Omega)^{2\times 2}$ and consider the finite-dimensional subspace of $\boldsymbol{L}$:

$$\boldsymbol{L}_H = \{\boldsymbol{S} \in \boldsymbol{L} : S_{ij}|_\Sigma \in \mathbb{P}_{r-1}(\Sigma) \, \forall \Sigma \in \mathcal{K}_H\}.$$

Let $P_H : \boldsymbol{L} \to \boldsymbol{L}_H$ denote the L^2 orthogonal projection on $\boldsymbol{L}_H$ and let I denote the identity mapping. Since P_H is a projection, we have the following properties:

$$\|I - P_H\| \leq 1, \tag{3.12}$$

$$\|(I - P_H)\nabla \boldsymbol{v}\|_{0,\Omega} \leq CH^r |\boldsymbol{v}|_{r+1,\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^{r+1}(\Omega). \tag{3.13}$$

Throughout the paper, the variable C will denote a generic positive constant that will take different values at different places but will be independent of h, H, ν, and ν_T. Define the following bilinear $g : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$:

$$g(\boldsymbol{v}, \boldsymbol{w}) = \sum_{j=1}^{N_h} \int_{E_j} (I - P_H)\nabla \boldsymbol{v} : (I - P_H)\nabla \boldsymbol{w} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}.$$

For all $t > 0$, we seek a discontinuous approximation $(\boldsymbol{u}^h(t), p^h(t)) \in \mathbf{X}^h \times Q^h$ such that

$$\begin{aligned}(\boldsymbol{u}_t^h(t), \boldsymbol{v}^h) &+ \nu(a(\boldsymbol{u}^h(t), \boldsymbol{v}^h) + J(\boldsymbol{u}^h(t), \boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}^h(t), \boldsymbol{v}^h) \\ &+ c(\boldsymbol{u}^h(t), \boldsymbol{u}^h(t), \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h(t)) = (\boldsymbol{f}(t), \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h,\end{aligned} \tag{3.14}$$

$$b(\boldsymbol{u}^h(t), q^h) = 0 \quad \forall q^h \in Q^h, \tag{3.15}$$

$$(\boldsymbol{u}^h(0), \boldsymbol{v}^h) = (\boldsymbol{u}_0, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h. \tag{3.16}$$

LEMMA 3.4. *There exists a unique solution to* (3.14)–(3.16).

Proof. Equations (3.14) and (3.15) reduce to the ordinary differential system

$$\frac{d\boldsymbol{u}^h}{dt} + \nu A\boldsymbol{u}^h + B\boldsymbol{u}^h + \nu_T G\boldsymbol{u}^h = \mathbf{F}.$$

By continuity, a solution exists. To prove uniqueness, we choose $\boldsymbol{v}^h = \boldsymbol{u}^h$ in (3.14) and $q^h = p^h$ in (3.15); we apply the coercivity equation (3.3) and the generalized Cauchy–Schwarz

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^h\|_{0,\Omega}^2 + \nu\kappa\|\boldsymbol{u}^h\|_X^2 \leq \|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\|\boldsymbol{u}^h\|_{L^4(\Omega)} \leq \frac{\nu\kappa}{2}\|\boldsymbol{u}^h\|_X^2 + \frac{C}{\nu\kappa}\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}^2.$$

Integrating over $[0,t]$ yields

$$\|\boldsymbol{u}^h(t)\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu\kappa\|\boldsymbol{u}^h\|^2_{L^2(0,T;X)} \leq \|\boldsymbol{u}^h(0)\|^2_0 + \frac{C}{\nu\kappa}\|\boldsymbol{f}\|^2_{L^2(0,T;L^{4/3}(\Omega))}.$$

Since $\boldsymbol{u}^h$ is bounded in $L^\infty(0,T;L^2(\Omega)^2)$, it is unique [4]. The existence and uniqueness of p^h are obtained from the inf-sup condition stated above. ∎

Remark 1. From a continuum mechanics point of view, it might be advantageous to consider the symmetrized velocity tensor. In this case, the bilinear form a is replaced by

$$a(\boldsymbol{v},\boldsymbol{w}) = \sum_{j=1}^{N_h}\int_{E_j}\nabla^s\boldsymbol{v}:\nabla^s\boldsymbol{w} - \sum_{k=1}^{M_h}\int_{e_k}(\{\nabla^s\boldsymbol{v}\}\mathbf{n}_k\cdot[\boldsymbol{w}] - \epsilon_0\{\nabla^s\boldsymbol{w}\}\mathbf{n}_k\cdot[\boldsymbol{v}]),$$

where $\nabla^s\boldsymbol{v} = 0.5(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)$ and the term relating the coarse and refined meshes is replaced by $\sum_{j=1}^{N_h}\int_{E_j}(I-P_H)\nabla^s\boldsymbol{u}:(I-P_H)\nabla^s\boldsymbol{v}^h$. It is easy to check that all the results proved in this paper also hold true for the symmetrized tensor formulation.

4. Semidiscrete a priori error estimate. In this section, a priori error estimates for the continuous in time problem are derived. The estimates are optimal in the fine mesh size h. The effects of the coarse scale appear as higher order terms.

THEOREM 4.1. *Let $(\boldsymbol{u},p)$ be the solution of* (2.1)–(2.4) *satisfying $\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^2(\Omega)), p\in L^2(0,T;H^1(\Omega))$. In addition, we assume that $\boldsymbol{u}_t\in L^2(0,T;\boldsymbol{H}^{r+1}(\Omega))$, $\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^{r+1}(\Omega))$, and $p\in L^2(0,T;\boldsymbol{H}^r(\Omega))$. Then, the continuous in time solution $\boldsymbol{u}_h$ satisfies*

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \kappa^{1/2}\nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^2(0,T;X)}\\
&\qquad + \nu_T^{1/2}\|(I-P_H)\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\|_{L^2(0,T;L^2(\Omega))}\\
&\leq Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}\\
&\qquad + |\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) + \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega},
\end{aligned}$$

where C is a positive constant independent of h,H,ν and ν_T.

Proof. We fix $t>0$ and for simplicity, we drop the argument in t. Defining $\boldsymbol{e}^h = \boldsymbol{u}-\boldsymbol{u}^h$ and subtracting (3.14), (3.15), (3.16) from (3.6), (3.7), (3.8), respectively, yields

$$\begin{aligned}
&(\boldsymbol{e}^h_t,\boldsymbol{v}^h) + \nu a(\boldsymbol{e}^h,\boldsymbol{v}^h) + \nu J(\boldsymbol{e}^h,\boldsymbol{v}^h) + \nu_T g(\boldsymbol{e}^h,\boldsymbol{v}^h) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^h)\\
&- c(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{v}^h) = -b(\boldsymbol{v}^h,p-p^h) + \nu_T g(\boldsymbol{u},\boldsymbol{v}^h) \quad \forall\boldsymbol{v}^h\in\boldsymbol{X}_h, \quad \forall t>0,
\end{aligned} \tag{4.1}$$

$$b(\boldsymbol{e}^h,q^h) = 0 \quad \forall q^h\in Q^h, \quad \forall t>0, \tag{4.2}$$

$$(\boldsymbol{e}^h(0),\boldsymbol{v}^h) = 0, \quad \forall\boldsymbol{v}^h\in\boldsymbol{X}^h. \tag{4.3}$$

Decompose the error $\boldsymbol{e}^h = \boldsymbol{\eta}-\boldsymbol{\phi}^h$, where $\boldsymbol{\phi}^h = \boldsymbol{u}^h - R_h(\boldsymbol{u})$ and $\boldsymbol{\eta}$ is the interpolation error $\boldsymbol{\eta} = \boldsymbol{u}-R_h(\boldsymbol{u})$. Set $\boldsymbol{v}^h = \boldsymbol{\phi}^h$ in (4.1) and $q^h = r_h(p)-p_h$ in (4.2):

$$\begin{aligned}
&(\boldsymbol{\phi}^h_t,\boldsymbol{\phi}^h) + \nu a(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + \nu J(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h) + \nu_T g(\boldsymbol{\phi}^h,\boldsymbol{\phi}^h)\\
&+ c_{\boldsymbol{u}^h}(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\phi}^h) - c_{\boldsymbol{u}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\phi}^h) = (\boldsymbol{\eta}_t,\boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta},\boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta},\boldsymbol{\phi}^h)\\
&+ \nu_T g(\boldsymbol{\eta},\boldsymbol{\phi}^h) + b(\boldsymbol{\phi}^h,p-r_h(p)) - \nu_T g(\boldsymbol{u},\boldsymbol{\phi}^h) \quad \forall t>0.
\end{aligned} \tag{4.4}$$

We now bound the terms on the right hand-side of (4.4). The first three terms are rewritten as

$$(\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + \nu a(\boldsymbol{\eta}, \boldsymbol{\phi}^h) + \nu J(\boldsymbol{\eta}, \boldsymbol{\phi}^h) = (\boldsymbol{\eta}_t, \boldsymbol{\phi}^h) + \nu \sum_{j=1}^{N_h} \int_{E_j} \nabla \boldsymbol{\eta} : \nabla \boldsymbol{\phi}^h$$
$$- \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\eta}\} \boldsymbol{n}_k \cdot [\boldsymbol{\phi}^h] + \nu \epsilon_0 \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\phi}^h\} \boldsymbol{n}_k \cdot [\boldsymbol{\eta}] + \nu J(\boldsymbol{\eta}, \boldsymbol{\phi}^h)$$
$$= S_1 + \cdots + S_5.$$

Using the Cauchy–Schwarz and Young's inequalities and the approximation result (2.7), the first two terms are bounded as follows:

$$S_1 \leq \|\boldsymbol{\eta}_t\|_{0,\Omega} \|\boldsymbol{\phi}^h\|_{0,\Omega} \leq \frac{1}{2} \|\boldsymbol{\phi}^h\|_{0,\Omega}^2 + Ch^{2r+2} |\boldsymbol{u}_t|_{r+1,\Omega}^2,$$
$$S_2 \leq \nu \sum_{j=1}^{N_h} \|\nabla \boldsymbol{\eta}\|_{0,E_j} \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \leq \frac{\kappa \nu}{8} \|\nabla \boldsymbol{\phi}^h\|_0^2 + C\nu h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

To bound the third term, we insert the standard Lagrange interpolant of degree r, denoted by $L_h(\boldsymbol{u})$:

$$- \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla \boldsymbol{\eta}\} \boldsymbol{n}_k \cdot [\boldsymbol{\phi}^h] = - \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla (\boldsymbol{u} - L_h(\boldsymbol{u}))\} \boldsymbol{n}_k \cdot [\boldsymbol{\phi}^h]$$
$$- \nu \sum_{k=1}^{M_h} \int_{e_k} \{\nabla (L_h(\boldsymbol{u}) - R_h(\boldsymbol{u}))\} \boldsymbol{n}_k \cdot [\boldsymbol{\phi}^h].$$

By using inequalities (2.12) and (2.15), the definition of the jump (2.5), and the approximation results (2.7), the third term can be bounded by

$$S_3 \leq \frac{\kappa \nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

Then, from the trace inequalities (2.11) and (2.15) and the approximation result (2.7), we have

$$S_4 \leq C\nu \left(\sum_{k=1}^{M_h} \frac{\sigma}{|e|} \|[\boldsymbol{\eta}]\|_{0,e_k}^2 \right)^{1/2} \left(\sum_{k=1}^{M_h} \frac{|e|}{\sigma} \|\{\nabla \boldsymbol{\phi}^h\}\|_{0,e_k}^2 \right)^{1/2}$$
$$\leq \frac{\kappa \nu}{8} \|\nabla \boldsymbol{\phi}^h\|_0^2 + C\nu h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

The jump term is bounded by the approximation result (2.7) as follows:

$$S_5 \leq \frac{\kappa \nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu J(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \frac{\kappa \nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\nu h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

The eddy viscosity term in the right-hand side of (4.4) is bounded by (3.12) and (2.7):

$$\nu_T g(\boldsymbol{\eta}, \boldsymbol{\phi}^h) \leq \frac{\nu_T}{4} \|(I - P_H) \nabla \boldsymbol{\phi}^h\|_0^2 + C\nu_T h^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

Because of (2.9), the pressure term is reduced to

$$b(\boldsymbol{\phi}^h, p - r_h(p)) = \sum_{k=1}^{M_h} \int_{e_k} \{p - r_h(p)\}[\boldsymbol{\phi}^h] \cdot \boldsymbol{n}_k,$$

which is bounded by using the Cauchy–Schwarz inequality, trace inequality (2.11), and the approximation result (2.10):

$$b(\boldsymbol{\phi}^h, p - r_h(p)) \leq C \left(\|p - r_h(p)\|_0^2 + \sum_{j=1}^{N_h} h_{E_j}^2 |p - r_h(p)|_{1,E_j}^2 \right)^{1/2} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h)^{1/2}$$
$$\leq \frac{\kappa\nu}{12} J(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + C\frac{h^{2r}}{\nu} |p|_{r,\Omega}^2.$$

The last term on the right-hand side of (4.4), corresponding to the consistency error, is bounded using the Cauchy–Schwarz inequality and the bound (3.13):

$$\nu_T g(\boldsymbol{u}, \boldsymbol{\phi}^h) \leq \frac{\nu_T}{4} \|(I - P_H)\nabla \boldsymbol{\phi}^h\|_0^2 + C\nu_T H^{2r} |\boldsymbol{u}|_{r+1,\Omega}^2.$$

Thus far, the terms in the right-hand side of (4.4) are bounded by

$$\frac{1}{2}\|\boldsymbol{\phi}^h\|_0^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2 + C(\nu + \nu_T)h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2$$
$$+ C\nu_T H^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + \frac{\kappa\nu}{4}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\|(I - P_H)\nabla\boldsymbol{\phi}^h\|_0^2.$$

Consider now the nonlinear terms in (4.4). We first note that since $\boldsymbol{u}$ is continuous, the second term in (3.4) vanishes and can be replaced by a similar quantity with a different domain of integration:

$$c_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h) = c_{\boldsymbol{u}^h}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h).$$

Therefore, adding and subtracting the interpolant $R_h(\boldsymbol{u})$ yields

$$c_{\boldsymbol{u}^h}(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{\phi}^h) - c_{\boldsymbol{u}^h}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h) = c_{\boldsymbol{u}^h}(\boldsymbol{u}^h, \boldsymbol{\phi}^h, \boldsymbol{\phi}^h) + c_{\boldsymbol{u}^h}(\boldsymbol{\phi}^h, \boldsymbol{u}, \boldsymbol{\phi}^h)$$
$$- c_{\boldsymbol{u}^h}(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h) - c_{\boldsymbol{u}^h}(\boldsymbol{\eta}, R_h(\boldsymbol{u}), \boldsymbol{\phi}^h) - c_{\boldsymbol{u}^h}(\boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\phi}^h).$$

To simplify the writing, we drop the subscript $\boldsymbol{u}_h$ and write $c(\cdot,\cdot,\cdot)$ for $c_{\boldsymbol{u}_h}(\cdot,\cdot,\cdot)$. From inequality (3.5), the first term is positive. We then bound the other terms. We first note that we can rewrite the form c as

$$c(\boldsymbol{\phi}^h, \boldsymbol{u}, \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{\phi}^h - \frac{1}{2} b(\boldsymbol{\phi}^h, \boldsymbol{u} \cdot \boldsymbol{\phi}^h). \tag{4.5}$$

The first term, using the L^p bound (2.6), is bounded by

$$\sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{\phi}^h \leq \|\boldsymbol{\phi}^h\|_{L^4(\Omega)} \|\nabla \boldsymbol{u}\|_{L^4(\Omega)} \|\boldsymbol{\phi}^h\|_{L^2(\Omega)}$$
$$\leq \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Let $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ be the piecewise constant vectors such that

$$\boldsymbol{c}_1|_{E_j} = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{u}, \quad \boldsymbol{c}_2|_{E_j} = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{\phi}^h, \quad 1 \le j \le N_h.$$

We rewrite using (4.2) and (3.11):

$$b(\boldsymbol{\phi}^h, \boldsymbol{u}\cdot\boldsymbol{\phi}^h) = b(\boldsymbol{\phi}^h, \boldsymbol{u}\cdot\boldsymbol{\phi}^h - \boldsymbol{c}_1\cdot\boldsymbol{c}_2) = b(\boldsymbol{\phi}^h, (\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h) + b(\boldsymbol{\phi}^h, \boldsymbol{c}_1\cdot(\boldsymbol{\phi}^h - \boldsymbol{c}_2)).$$

Then, expanding the first term,

$$\begin{aligned} b(\boldsymbol{\phi}^h, (\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h) &= -\sum_{j=1}^{N_h}\int_E (\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h \nabla\cdot\boldsymbol{\phi}^h \\ &+ \sum_{k=1}^{M_h}\int_{e_k} \{(\boldsymbol{u}-\boldsymbol{c}_1)\cdot\boldsymbol{\phi}^h\}[\boldsymbol{\phi}^h]\cdot\boldsymbol{n}_k = S_6 + S_7. \end{aligned}$$

The first term is bounded, for $s > 2$, using the inverse inequality (2.16) and (2.6):

$$\begin{aligned} S_6 &\le C\sum_{j=1}^{N_h} \|\boldsymbol{u}-\boldsymbol{c}_1\|_{L^s(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)} \|\nabla\boldsymbol{\phi}^h\|_{L^2(E_j)} \\ &\le C\|\boldsymbol{\phi}^h\|_{0,\Omega} |\boldsymbol{u}|_{W^{1,s}(\Omega)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(\Omega)} \\ &\le C\|\boldsymbol{\phi}^h\|_{0,\Omega} |\boldsymbol{u}|_{W^{1,s}(\Omega)} \|\boldsymbol{\phi}^h\|_X \le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}\|\boldsymbol{\phi}^h\|_0^2. \end{aligned}$$

The bound for the second term is more technical. First, passing to the reference element $\hat{E}$ and using the trace inequality (2.14), we obtain

$$\begin{aligned} S_7 &\le C\sum_{k=1}^{M_h} |e_k||E|^{-1/2}\|\boldsymbol{\phi}^h\|_{0,E} \|(\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{\hat{e}} \\ &\le C\sum_{k=1}^{M_h} |e_k||E|^{-1/2}\|\boldsymbol{\phi}^h\|_{0,E} (\|(\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} + \|\hat{\nabla}((\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}}). \end{aligned}$$

The L^2 term is bounded, for $s > 2$, as

$$\begin{aligned} \|(\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} &\le \|\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1\|_{L^s(\hat{E})}\|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})} \\ &\le h|E|^{-1/s-(s-2)/(2s)}|\boldsymbol{u}|_{W^{1,s}(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \le C|\boldsymbol{u}|_{W^{1,s}(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}. \end{aligned}$$

Note that for the gradient term we write

$$\|\hat{\nabla}((\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\hat{\boldsymbol{\phi}}^h)\|_{0,\hat{E}} = \|(\hat{\nabla}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\phi}}^h + (\hat{\boldsymbol{u}}-\hat{\boldsymbol{c}}_1)\cdot\nabla\hat{\boldsymbol{\phi}}^h)\|.$$

Let us first bound

$$\begin{aligned} \|\hat{\nabla}\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} &\le \|\hat{\nabla}\hat{\boldsymbol{u}}\|_{L^s(\hat{E})}\|\hat{\boldsymbol{\phi}}^h\|_{L^{\frac{2s}{s-2}}(\hat{E})} \\ &\le Ch|E|^{-1/s}\|\nabla\boldsymbol{u}\|_{L^s(E)}|E|^{-(s-2)/2s}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)} \le C\|\nabla\boldsymbol{u}\|_{L^s(E)}\|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E)}. \end{aligned}$$

Now the other term is

$$\|(\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1) \cdot \hat{\nabla}\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq \|\hat{\boldsymbol{u}} - \hat{\boldsymbol{c}}_1\|_{L^\infty(\hat{E})} \|\hat{\nabla}\hat{\boldsymbol{\phi}}^h\|_{0,\hat{E}} \leq Ch\|\boldsymbol{u}\|_{L^\infty(E)} \|\nabla \boldsymbol{\phi}^h\|_{0,E}.$$

Combining all the bounds above and using (2.6), we have

$$S_7 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{\phi}^h\|_{0,E_j} \Big[|\boldsymbol{u}|_{W^{1,s}(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)}$$

$$+ \|\nabla \boldsymbol{u}\|_{L^s(E_j)} \|\boldsymbol{\phi}^h\|_{L^{\frac{2s}{s-2}}(E_j)} + h|\boldsymbol{u}|_{L^\infty(E_j)} \|\nabla \boldsymbol{\phi}^h\|_{L^2(E_j)} \Big] \leq \frac{\kappa\nu}{32} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{\phi}^h\|_0^2.$$

Now,

$$b(\boldsymbol{\phi}^h, \boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2)) = -\sum_{j=1}^{N_h} \int_E \boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2) \nabla \cdot \boldsymbol{\phi}^h$$

$$+ \sum_{k=1}^{M_h} \int_{e_k} \{\boldsymbol{c}_1 \cdot (\boldsymbol{\phi}^h - \boldsymbol{c}_2)\} [\boldsymbol{\phi}^h] \cdot \boldsymbol{n}_k = S_8 + S_9.$$

The first term is bounded by (2.16):

$$S_8 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\boldsymbol{\phi}^h - \boldsymbol{c}_2\|_{0,E_j} h^{-1} \|\boldsymbol{\phi}^h\|_{0,E_j}$$

$$\leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{0,E_j} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Similarly, the second term is bounded as

$$S_9 \leq C \sum_{j=1}^{N_h} \|\boldsymbol{c}_1\| \|\nabla \boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}_h\|_{0,E_j} \leq \frac{\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Thus,

$$c(\boldsymbol{\phi}^h, \boldsymbol{u}, \boldsymbol{\phi}^h) \leq \frac{5\kappa\nu}{64} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

Let us now bound $c(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h)$:

$$c(\boldsymbol{\phi}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h) = \sum_{j=1}^{N_h} \left(\int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h + \int_{\partial E_j^-} |\{\boldsymbol{\phi}^h\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{\eta}^{\text{int}} - \boldsymbol{\eta}^{\text{ext}}) \cdot \boldsymbol{\phi}^{h,int} \right)$$

$$- \frac{1}{2} b(\boldsymbol{\phi}^h, \boldsymbol{\eta} \cdot \boldsymbol{\phi}^h).$$

The first term is easily bounded:

$$\sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}^h \cdot \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi}^h \leq \sum_{j=1}^{N_h} \|\boldsymbol{\phi}^h\|_{0,E_j} \|\boldsymbol{\phi}^h\|_{L^4(E_j)} \|\nabla \boldsymbol{\eta}\|_{L^4(E_j)}$$

$$\leq \frac{\kappa\nu}{32} \|\boldsymbol{\phi}^h\|_X^2 + \frac{C}{\nu} \|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2 \|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

The second term is bounded using inequalities (2.13), (2.16), (2.6), and (2.8):

$$\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\phi}^h\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\eta}^{\rm int}-\boldsymbol{\eta}^{\rm ext})\cdot\boldsymbol{\phi}^{h,int} \le C\sum_{j=1}^{N_h}\|\boldsymbol{\phi}^h\|_{L^4(\partial E_j)}\|\boldsymbol{\eta}\|_{L^4(\partial E_j)}\|\boldsymbol{\phi}^h\|_{L^2(\partial E_j)}$$
$$\le C\sum_{j=1}^{N_h}h^{-3/2}h^{r+1}|\boldsymbol{u}|_{r+1,\Omega}\|\boldsymbol{\phi}^h\|_{0,\Omega}^2 \le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2 + C\|\boldsymbol{u}\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2\|\boldsymbol{\phi}^h\|_{0,\Omega}^2.$$

The last term in $c(\boldsymbol{\phi}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h)$ is bounded like the terms S_6, S_7, S_8, and S_9 of $c(\boldsymbol{\phi}^h,\boldsymbol{u},\boldsymbol{\phi}^h)$. The remaining nonlinear terms are bounded in a similar fashion:

$$c_{\boldsymbol{u}^h}(\boldsymbol{\eta},R_h(\boldsymbol{u}),\boldsymbol{\phi}^h)=\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{\eta}\cdot\nabla R_h(\boldsymbol{u}))\cdot\boldsymbol{\phi}^h$$
$$+\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{\eta}\}\cdot\boldsymbol{n}_{E_j}|(R_h(\boldsymbol{u})^{\rm int}-R_h(\boldsymbol{u})^{\rm ext})\cdot\boldsymbol{\phi}^{h,\rm int}+\frac{1}{2}\sum_{j=1}^{N_h}\int_{E_j}(\nabla\cdot\boldsymbol{\eta})R_h(\boldsymbol{u})\cdot\boldsymbol{\phi}^h$$
$$-\frac{1}{2}\sum_{k=1}^{M_h}\int_{e_k}[\boldsymbol{\eta}]\cdot\boldsymbol{n}_k\{R_h(\boldsymbol{u})\cdot\boldsymbol{\phi}^h\}=S_{10}+\cdots+S_{13}.$$

Using the bound (2.6) and the approximation result (2.7), we have

$$S_{10}\le\|\boldsymbol{\eta}\|_{L^2(\Omega)}\|\nabla R_h(\boldsymbol{u})\|_{L^4(\Omega)}\|\boldsymbol{\phi}^h\|_{L^4(\Omega)}\le\frac{\kappa\nu}{64}\|\boldsymbol{\phi}^h\|_X^2+C\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

The inequalities (2.11), (2.14), and (2.6) and the approximation result (2.7) yield

$$S_{11}\le C\sum_{j=1}^{N_h}h_{E_j}^{-1/2}(\|\boldsymbol{\eta}\|_{0,E_j}+h_{E_j}\|\nabla\boldsymbol{\eta}\|_{0,E_j})h_{E_j}^{-1/2}\|\boldsymbol{\phi}^h\|_{0,E_j}$$
$$\le C\|\boldsymbol{\phi}^h\|_{0,\Omega}^2+C\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Similarly, we have

$$S_{12}\le\sum_{j=1}^{N_h}\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}\|\boldsymbol{\phi}^h\|_{0,E_j}\|\nabla\cdot\boldsymbol{\eta}\|_{0,E_j}$$
$$\le C\|\boldsymbol{\phi}^h\|_{0,\Omega}^2+C\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Note that S_{13} is bounded exactly like S_{11}. The other nonlinear term is bounded using (2.7) and (2.14):

$$c_{\boldsymbol{u}^h}(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\phi}^h)=\sum_{j=1}^{N_h}\int_{E_j}(\boldsymbol{u}\cdot\nabla\boldsymbol{\eta})\cdot\boldsymbol{\phi}^h+\sum_{j=1}^{N_h}\int_{\partial E_j^-}|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_{E_j}|(\boldsymbol{\eta}^{\rm int}-\boldsymbol{\eta}^{\rm ext})\cdot\boldsymbol{\phi}^{h,\rm int}$$
$$\le C\sum_{j=1}^{N_h}\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}\|\nabla\boldsymbol{\eta}\|_{0,E_j}\|\boldsymbol{\phi}^h\|_{0,E_j}+C\sum_{j=1}^{N_h}\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}\|\boldsymbol{\eta}\|_{0,\partial E_j}\|\boldsymbol{\phi}^h\|_{0,\partial E_j}$$
$$\le C\|\boldsymbol{\phi}^h\|_{0,\Omega}^2+C\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Combining all bounds above and using (3.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_0^2 + \frac{\kappa\nu}{2}\|\boldsymbol{\phi}^h\|_X^2 + \frac{\nu_T}{2}\|(I-P_H)\nabla\boldsymbol{\phi}^h\|_0^2 \le C\left(\frac{1}{\nu}+1\right)\|\boldsymbol{\phi}^h\|_0^2$$
$$+ Ch^{2r}\left(\nu+\frac{1}{\nu}+\nu_T\right)|\boldsymbol{u}|_{r+1,\Omega}^2 + C\frac{h^{2r}}{\nu}|p|_{r,\Omega}^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2 + C\nu_T H^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2.$$

Integrating from 0 to t, noting that $\|\boldsymbol{\phi}^h(0)\|_0$ is of the order h^r, and using Gronwall's lemma, yield

$$\|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|_{L^2(0,t;X)}^2 + \nu_T\|(I-P_H)\nabla\boldsymbol{\phi}^h\|_{L^2(0,t;L^2(\Omega))}^2$$
$$\le Ce^{C(1+\nu^{-1})}h^{2r}[(\nu+\nu^{-1}+\nu_T)|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + \nu^{-1}|p|_{L^2(0,T;H^r(\Omega))}^2$$
$$+|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}^2 + \nu_T H^{2r}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}^2,$$

where the constant C is independent of ν, ν_T, h, H but depends on $\|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}$. The theorem is obtained using the approximation results (2.7) and (2.8) and the following inequality:

$$\|\boldsymbol{u}(t)-\boldsymbol{u}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{u}(t)-\boldsymbol{u}^h(t)\|_{L^2(0,T;X)}^2 + \nu_T\|(I-P_H)\nabla(\boldsymbol{u}(t)-\boldsymbol{u}^h(t))\|_{L^2(0,T;L^2(\Omega))}^2$$
$$\le \|\boldsymbol{\phi}^h(t)\|_0^2 + \kappa\nu\|\boldsymbol{\phi}^h\|_{L^2(0,T;X)}^2 + \nu_T\|(I-P_H)\nabla\boldsymbol{\phi}^h\|_{L^2(0,T;L^2(\Omega))}^2$$
$$+ \|\boldsymbol{\eta}(t)\|_0^2 + \kappa\nu\|\boldsymbol{\eta}\|_{L^2(0,T;X)}^2 + \nu_T\|(I-P_H)\nabla\boldsymbol{\eta}\|_{L^2(0,T;L^2(\Omega))}^2. \qquad \square$$

Remark 2. One of the most important properties of Theorem 4.1 is that the new method improves its robustness with respect to the Reynolds number. In most cases, error estimations of Navier–Stokes equations give a Gronwall constant that depends on the Reynolds number as $1/\nu^3$. In contrast, this approach leads to a better error estimate with a Gronwall constant depending on $1/\nu$. Optimal convergence rates are obtained for Theorem 4.1 if ν_T and H are appropriately chosen.

COROLLARY 4.2. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$. If the relation $\beta \ge 2r(\alpha-1)/\alpha$ is satisfied, then the estimate becomes*

$$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^2(0,T;X)} = \mathcal{O}(h^r).$$

For example, one may choose for a linear approximation the pair $(\nu_T, H) = (h, h^{1/2})$, for quadratic approximation $(\nu_T, H) = (h, h^{3/4})$ or $(\nu_T, H) = (h^2, h^{1/2})$, and for cubic approximation $(\nu_T, H) = (h, h^{5/6})$ or $(\nu_T, H) = (h^2, h^{2/3})$.

THEOREM 4.3. *Under the assumptions of Theorem* 4.1 *and if $a(\cdot,\cdot)$ is symmetric ($\epsilon_0 = -1$), the following estimate holds true:*

$$\|\boldsymbol{u}_t - \boldsymbol{u}_t^h\|_{L^2(0,T;L^2(\Omega))} + \nu^{1/2}\|\boldsymbol{u}-\boldsymbol{u}^h\|_{L^\infty(0,T;X)} \le Ce^{CT\nu^{-1}}[h^r|\boldsymbol{u}_0|_{r+1,\Omega}$$
$$+ h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + h^r|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu_T H^r h^{-1}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}],$$

where C is a positive constant independent of h, H, ν and ν_T. If $a(\cdot,\cdot)$ is nonsymmetric ($\epsilon_0 = 1$), the estimate is suboptimal, of order h^{r-1}.

Proof. We just give the outline of the proof. We introduce the modified Stokes problem: for any $t > 0$, find $(\boldsymbol{u}^S(t), p^S(t)) \in \boldsymbol{X}^h \times Q^h$ such that

$$\nu(a(\boldsymbol{u}^S(t),\boldsymbol{v}^h) + J(\boldsymbol{u}^S(t),\boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}^S(t),\boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^S(t))$$
$$= \nu(a(\boldsymbol{u}(t),\boldsymbol{v}^h) + J(\boldsymbol{u}(t),\boldsymbol{v}^h)) + \nu_T g(\boldsymbol{u}(t),\boldsymbol{v}^h) + b(\boldsymbol{v}^h, p(t)) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{4.6}$$
$$b(\boldsymbol{u}^S(t), q^h) = 0 \quad \forall q^h \in Q^h. \tag{4.7}$$

For any $t > 0$, there exists a unique solution to (4.6), (4.7). Furthermore, it is easy to show that the solution satisfies the error estimate

$$\kappa^{1/2}\nu^{1/2}\|\boldsymbol{u}(t)-\boldsymbol{u}^S(t)\|_X + \nu_T^{1/2}\|(I-P_H)\nabla(\boldsymbol{u}-\boldsymbol{u}^S)\|_{0,\Omega}$$
$$\leq h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{r+1,\Omega}+\nu^{-1/2}|p|_{r,\Omega}+|\boldsymbol{u}_t|_{r+1,\Omega})+\nu_T^{1/2}H^r|\boldsymbol{u}|_{r+1,\Omega} \quad \forall t>0.$$

Define $\boldsymbol{\eta} = \boldsymbol{u} - \boldsymbol{u}^S$ and $\boldsymbol{\xi} = \boldsymbol{u}^h - \boldsymbol{u}^S$, and choose the test function $\boldsymbol{v}^h = \boldsymbol{\xi}_t$. The resulting error equation is

$$\begin{aligned}&\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi},\boldsymbol{\xi}_t) + \frac{\nu}{2}\frac{d}{dt}J(\boldsymbol{\xi},\boldsymbol{\xi}) + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi},\boldsymbol{\xi})\\ &= (\boldsymbol{\eta}_t,\boldsymbol{\xi}_t) - \nu_T g(\boldsymbol{u},\boldsymbol{\xi}_t) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\xi}_t) - c(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\xi}_t).\end{aligned} \tag{4.8}$$

The first two terms in the right-hand side of (4.8) are bounded as in Theorem 4.1. A detailed argument is given in [23]. Let us rewrite the nonlinear terms

$$\begin{aligned}c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\xi}_t) - c(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{\xi}_t) &= c(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}_t) - c(\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\xi}_t) + c(\boldsymbol{\xi},\boldsymbol{u},\boldsymbol{\xi}_t)\\ &\quad - c(\boldsymbol{\eta},\boldsymbol{u}^h,\boldsymbol{\xi}_t) + c(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{\xi}_t) - c(\boldsymbol{u},\boldsymbol{\eta},\boldsymbol{\xi}_t).\end{aligned}$$

We assume that $\boldsymbol{\xi}$ belongs to $L^\infty((0,T)\times\Omega)$. L^p bounds, inverse inequality, and approximation results give the bounds for each nonlinear term as in Theorem 4.1. Collecting all the bounds with (4.8) gives

$$\begin{aligned}&\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \nu a(\boldsymbol{\xi},\boldsymbol{\xi}_t) + \frac{\nu}{2}\frac{d}{dt}J(\boldsymbol{\xi},\boldsymbol{\xi}) + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi},\boldsymbol{\xi})\\ &\leq \frac{1}{2}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + C\|\boldsymbol{\xi}\|_X^2 + Ch^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2 + C\nu_T^2 H^{2r}h^{-2}|\boldsymbol{u}|_{r+1,\Omega}^2.\end{aligned} \tag{4.9}$$

In the case where the bilinear form a is symmetric ($\epsilon_0 = -1$), the inequality becomes

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{\xi}_t\|_{0,\Omega}^2 + \frac{\nu}{2}\frac{d}{dt}\|\boldsymbol{\xi}\|_X^2 + \frac{\nu_T}{2}\frac{d}{dt}g(\boldsymbol{\xi},\boldsymbol{\xi})\\ &\leq C\|\boldsymbol{\xi}\|_X^2 + Ch^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + Ch^{2r}|\boldsymbol{u}_t|_{r+1,\Omega}^2 + C\nu_T^2 H^{2r}h^{-2}|\boldsymbol{u}|_{r+1,\Omega}^2.\end{aligned} \tag{4.10}$$

Integrating from 0 to t and using Gronwall's lemma yield

$$\|\boldsymbol{\xi}_t\|_{L^2(0,T;L^2(\Omega))}^2 + \nu\|\boldsymbol{\xi}\|_{L^\infty(0,T;X)}^2 + \nu_T\max_{0\leq t\leq T} g(\boldsymbol{\xi},\boldsymbol{\xi}) \leq Ce^{CT\nu^{-1}}[h^{2r}|\boldsymbol{u}_0|_{r+1,\Omega}^2$$
$$+ Ch^{2r}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2 + Ch^{2r}|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}^2 + C\nu_T^2H^{2r}h^{-2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}^2].$$

In the case where the bilinear form a is nonsymmetric, we rewrite (4.9) as

$$a(\boldsymbol{\xi},\boldsymbol{\xi}_t) = \frac{1}{2}\frac{d}{dt}|||\nabla\boldsymbol{\xi}|||_0^2 - \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}\}\boldsymbol{n}_k\cdot[\boldsymbol{\xi}_t] + \sum_{k=1}^{M_h}\int_{e_k}\{\nabla\boldsymbol{\xi}_t\}\boldsymbol{n}_k\cdot[\boldsymbol{\xi}].$$

The bound is then suboptimal: $\mathcal{O}(h^{r-1})$. ☐

We now derive an error estimate for the pressure.

THEOREM 4.4. *We keep the assumptions of Theorem* 4.1 *and we consider the case where* $a(\cdot,\cdot)$ *is symmetric* ($\epsilon_0 = -1$) *and* $\nu \leq 1$. *Then the solution* p^h *satisfies* *the following error estimate:*

$$\|p^h - r_h(p)\|_{L^2(0,T;L^2(\Omega))} \le Ce^{CT\nu^{-1}}[\nu h^r|\boldsymbol{u}_0|_{r+1,\Omega}$$
$$+ \nu h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu h^r|\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu\nu_T H^r h^{-1}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}]$$
$$+ C\nu^{1/2}h^r|\boldsymbol{u}_0|_{r+1,\Omega} + C\nu h^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + C\nu h^r|p|_{L^2(0,T;H^r(\Omega))}$$
$$+ C\nu_T H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$
$$+ Ce^{CT(\nu^{-1}+1)}[h^r((\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}$$
$$+ |\boldsymbol{u}_t|_{L^2(0,T;H^{r+1}(\Omega))}) + \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega},$$

where C is independent of h, H, ν, and ν_T. Again, if $a(\cdot,\cdot)$ is nonsymmetric ($\epsilon_0 = 1$), the estimate is suboptimal.

Proof. The error equation can be written for all $\boldsymbol{v}^h$ in $\boldsymbol{X}^h$:

$$-b(\boldsymbol{v}^h, p^h - r_h(p)) = (\boldsymbol{u}_t^h - \boldsymbol{u}_t, \boldsymbol{v}^h) + \nu a(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) + \nu J(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h)$$
$$+ \nu_T g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) + c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) + \nu_T g(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p - r_h(p)).$$

From the inf-sup condition (3.10), there is $\boldsymbol{v}^h \in \boldsymbol{X}^h$ such that

$$b(\boldsymbol{v}^h, p^h - r_h(p)) = -\|p^h - r_h(p)\|_0^2, \quad \|\boldsymbol{v}^h\|_X \le \frac{1}{\beta_0}\|p^h - r_h(p)\|_{0,\Omega}.$$

Thus, we have

$$\|p^h - r_h(p)\|_{0,\Omega}^2 = (\boldsymbol{u}_t^h - \boldsymbol{u}_t, \boldsymbol{v}^h) + \nu \sum_{j=1}^{N_h}\int_{E_j} \nabla(\boldsymbol{u}^h - \boldsymbol{u}) : \nabla \boldsymbol{v}^h$$
$$-\nu \sum_{k=1}^{M_h}\int_{e_k}\{\nabla(\boldsymbol{u}^h - \boldsymbol{u})\}\boldsymbol{n}_k \cdot [\boldsymbol{v}^h] + \nu\epsilon_0 \sum_{k=1}^{M_h}\int_{e_k}\{\nabla \boldsymbol{v}^h\}\boldsymbol{n}_k \cdot [\boldsymbol{u}^h - \boldsymbol{u}] + \nu J(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h)$$
$$+ \nu_T g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{v}^h) + c(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}^h) + \nu_T g(\boldsymbol{u}, \boldsymbol{v}^h) - b(\boldsymbol{v}^h, p - r_h(p)).$$

All the terms above can be handled as in Theorem 4.1. The resulting inequality is

$$\|p^h - r_h(p)\|_{0,\Omega}^2 \le C\nu^2\|\boldsymbol{u}_t^h - \boldsymbol{u}_t\|_{0,\Omega}^2 + C\nu^2\|\boldsymbol{u}^h - \boldsymbol{u}\|_X^2 + C\nu^2 h^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2$$
$$+ C\nu^2 h^{2r}|p|_{r,\Omega}^2 + C\nu_T^2 H^{2r}|\boldsymbol{u}|_{r+1,\Omega}^2 + C\nu_T^2 g(\boldsymbol{u}^h - \boldsymbol{u}, \boldsymbol{u}^h - \boldsymbol{u}) + C\|\boldsymbol{u}^h - \boldsymbol{u}\|_{0,\Omega}^2.$$

We now integrate from 0 to T and use Theorem 4.1 and Theorem 4.3 to conclude. ∎

5. Fully discrete scheme. In this section, we formulate two fully discrete finite element schemes for the discontinuous eddy viscosity method. Let Δt denote the time step, let $M = T/\Delta t$, and let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the interval $(0, T)$. We denote the function ϕ evaluated at the time t_m by ϕ_m and the average of ϕ at two successive time levels by $\phi_{m+\frac{1}{2}} = \frac{1}{2}(\phi_m + \phi_{m+1})$.

Scheme 1: Given $\boldsymbol{u}_0^h$, find $(\boldsymbol{u}_m^h)_{m\ge1}$ in $\boldsymbol{X}^h$ and $(p_m^h)_{m\ge1}$ in Q^h such that

$$\frac{1}{\Delta t}(\boldsymbol{u}_{m+1}^h - \boldsymbol{u}_m^h, \boldsymbol{v}^h) + \nu(a(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h) + J(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h)) + c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h)$$
$$+ \nu_T g(\boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p_{m+1}^h) = (\boldsymbol{f}_{m+1}, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{5.1}$$
$$b(\boldsymbol{u}_{m+1}^h, q^h) = 0 \quad \forall q^h \in Q^h. \tag{5.2}$$

Scheme 2: Given $\tilde{\boldsymbol{u}}_0^h, \tilde{\boldsymbol{u}}_1^h, \tilde{p}_1^h$, find $(\tilde{\boldsymbol{u}}_m^h)_{m\geq 2}$ in $\boldsymbol{X}^h$ and $(\tilde{p}_m^h)_{m\geq 2}$ in Q^h such that

$$\frac{1}{\Delta t}(\tilde{\boldsymbol{u}}_{m+1}^h - \tilde{\boldsymbol{u}}_m^h, \boldsymbol{v}^h) + \nu(a(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h) + J(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h)) + c(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h)$$

$$+ \nu_T g(\tilde{\boldsymbol{u}}_{m+\frac{1}{2}}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, \tilde{p}_{m+\frac{1}{2}}^h) = (\boldsymbol{f}_{m+\frac{1}{2}}, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{5.3}$$

$$b(\tilde{\boldsymbol{u}}_{m+1}^h, q^h) = 0 \quad \forall q^h \in Q^h. \tag{5.4}$$

For both schemes, the initial velocity is defined to be the L^2 projection of $\boldsymbol{u}_0$. Scheme 1 is based on a backward Euler discretization. Scheme 2 is based on a Crank–Nicolson discretization, and requires the velocity and pressure at the first step. The approximations $\tilde{\boldsymbol{u}}_1^h$ and $\tilde{p}_1^h$ can be obtained by a first order scheme (see [2]). We will show that Scheme 1 is first order in time and Scheme 2 is second order in time. First, we prove the stability of the schemes.

LEMMA 5.1. *The solution* $(\boldsymbol{u}_m^h)_m$ *of* (5.1), (5.2) *remains bounded in the following sense:*

$$\|\boldsymbol{u}_m^h\|_{0,\Omega}^2 \leq K, \quad m = 0, \ldots, M,$$

$$\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1}^h\|_X^2 \leq \frac{K}{2\nu}, \quad \Delta t \sum_{m=0}^{M-1} \|(I - P_H)\nabla \boldsymbol{u}_{m+1}^h\|_0^2 \leq \frac{K}{2\nu_T},$$

where $K = \|\boldsymbol{u}_0\|_{0,\Omega}^2 + \|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}^2$.

The solution $(\tilde{\boldsymbol{u}}_m^h)_m$ *of* (5.3), (5.4) *remains bounded in the following sense:*

$$\|\tilde{\boldsymbol{u}}_m^h\|_{0,\Omega}^2 \leq \tilde{K}, \quad m = 0, \ldots, M,$$

$$\Delta t \sum_{m=0}^{M-1} \|\tilde{\boldsymbol{u}}_{m+1}^h\|_X^2 \leq \frac{\tilde{K}}{2\nu}, \quad \Delta t \sum_{m=0}^{M-1} \|(I - P_H)\nabla \tilde{\boldsymbol{u}}_{m+1}^h\|_{0,\Omega}^2 \leq \frac{\tilde{K}}{2\nu_T},$$

where $\tilde{K} = \|\boldsymbol{u}_0\|_{0,\Omega}^2 + 2\|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}^2$.

Proof. Choose $\boldsymbol{v}^h = \boldsymbol{u}_{m+1}^h$ in (5.1) and $q^h = p_{m+1}^h$ in (5.2). We multiply by $2\Delta t$ and sum over m. Then, from the positivity of c and (3.3), we have

$$\|\boldsymbol{u}_m^h\|_{0,\Omega}^2 - \|\boldsymbol{u}_0^h\|_{0,\Omega}^2 + 2\kappa\nu\Delta t \sum_{j=0}^{m-1} \|\boldsymbol{u}_{j+1}^h\|_X^2 + 2\nu_T \Delta t \sum_{j=0}^{m-1} \|(I - P_H)\nabla \boldsymbol{u}_{j+1}^h\|_0^2$$
$$\leq \Delta t \sum_{j=0}^{m-1} \|\boldsymbol{f}_{j+1}\|_{0,\Omega}^2 + \Delta t \sum_{j=0}^{m-1} \|\boldsymbol{u}_{j+1}^h\|_{0,\Omega}^2.$$

The result is obtained by using a discrete version of Gronwall's lemma [15] and the fact that $\|\boldsymbol{u}_0^h\|_{0,\Omega} \leq \|\boldsymbol{u}_0\|_{0,\Omega}$.

For Scheme 2, the proof is similar. Choose $\boldsymbol{v}^h = \tilde{\boldsymbol{u}}_{m+\frac{1}{2}}$ in (5.3) and $q^h = \tilde{p}_{m+\frac{1}{2}}^h$ in (5.4). The rest of the proof follows as above. See [23] for more details. ∎

THEOREM 5.2. *Under the assumptions of Theorem* 4.1 *and if* $\boldsymbol{u}_t$ *and* $\boldsymbol{u}_{tt}$ *belong to* $L^\infty(0,T;L^2(\Omega))$, *there is a constant* C *independent of* h, H, ν, *and* ν_T *such that*

$$\max_{m=0,\dots,M} \|\boldsymbol{u}_m - \boldsymbol{u}_m^h\|_{0,\Omega} + \left(\nu\kappa\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1} - \boldsymbol{u}_{m+1}^h\|_X^2\right)^{1/2}$$
$$+ \left(\nu_T \Delta t \sum_{m=0}^{M} |\!|\!|(I-P_H)(\nabla \boldsymbol{u}_{m+1} - \boldsymbol{u}_{m+1}^h)|\!|\!|_0^2\right)^{1/2} \le Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}$$
$$+ Ce^{CT\nu^{-1}}[h^r(\nu+\nu^{-1}+\nu_T)^{1/2}|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T^{1/2}H^r|\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}$$
$$+ \nu^{-1/2}\Delta t(\|\boldsymbol{u}_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}) + h^r\nu^{-1/2}|p|_{L^2(0,T;H^r(\Omega))}].$$

Proof. As in the continuous case, we set $\boldsymbol{e}_m = \boldsymbol{u}_m - \boldsymbol{u}_m^h$. We subtract from (5.1) and (5.2) equations (3.14) and (3.15) evaluated at time $t = t_{m+1}$.

$$(\boldsymbol{u}_t(t_{m+1}), \boldsymbol{v}^h) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1}^h - \boldsymbol{u}_m^h, \boldsymbol{v}^h) + \nu[a(\boldsymbol{e}_{m+1}, \boldsymbol{v}^h) + J(\boldsymbol{e}_{m+1}, \boldsymbol{v}^h)]$$
$$+ \nu_T g(\boldsymbol{e}_{m+1}, \boldsymbol{v}^h) + c(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{v}^h) - c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{v}^h)$$
$$+ b(\boldsymbol{v}^h, p_{m+1} - p_{m+1}^h) = \nu_T g(\boldsymbol{u}_{m+1}, \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{X}^h, \tag{5.5}$$
$$b(\boldsymbol{e}_{m+1}, q^h) = 0 \quad \forall q^h \in Q^h. \tag{5.6}$$

Define $\boldsymbol{\phi}_m = \boldsymbol{u}_m^h - (R_h(\boldsymbol{u}))_m$, $\boldsymbol{\eta}_m = \boldsymbol{u}_m - (R_h(\boldsymbol{u}))_m$. Choose $\boldsymbol{v}^h = \boldsymbol{\phi}_{m+1}$ in (5.5) and $q^h = p_{m+1}^h$ in (5.6). Adding and subtracting the interpolant and using (3.3) yield the following error equation:

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0^2$$
$$+ c(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}) + b(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})$$
$$\le \left\|\frac{\partial \boldsymbol{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m)\right\|_{0,\Omega} \|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \frac{1}{\Delta t}\left\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\right\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$
$$+ \nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{\eta}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0$$
$$+ \nu_T|\!|\!|(I-P_H)\nabla\boldsymbol{u}_{m+1}|\!|\!|_0|\!|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!|\!|_0.$$

We rewrite the nonlinear terms

$$c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_{m+1}}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}).$$

We now drop the subscript $\boldsymbol{u}_m^h$:

$$c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_m^h, \boldsymbol{u}_{m+1}^h, \boldsymbol{\phi}_{m+1}) - c_{\boldsymbol{u}_m^h}(\boldsymbol{u}_{m+1}, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$= c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})$$
$$- c(\boldsymbol{\eta}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) - c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1}).$$

Thus, we rewrite the error equation as

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+1}\|_X^2 + \nu_T \|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!\|_0^2$$
$$+ c(\boldsymbol{u}_m^h, \boldsymbol{\phi}_{m+1}, \boldsymbol{\phi}_{m+1}) \le |c(\boldsymbol{\phi}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + |c(\boldsymbol{\phi}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})|$$
$$+ |c(\boldsymbol{\eta}_m, \boldsymbol{u}_{m+1}^I, \boldsymbol{\phi}_{m+1})| + |c(\boldsymbol{u}_m, \boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})| + |c(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m, \boldsymbol{u}_{m+1}, \boldsymbol{\phi}_{m+1})|$$
$$+ |b(\boldsymbol{\phi}_{m+1}, p_{m+1}^h - p_{m+1})| + \left\| \frac{\partial \boldsymbol{u}}{\partial t}(t_{m+1}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m) \right\|_{0,\Omega} \|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}$$
$$+ \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} + \nu|a(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{\eta}_{m+1}, \boldsymbol{\phi}_{m+1})|$$
$$+ \nu_T \|\!|(I-P_H)\nabla\boldsymbol{\eta}_{m+1}|\!\|_0 \|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!\|_0$$
$$+ \nu_T \|\!|(I-P_H)\nabla\boldsymbol{u}_{m+1}|\!\|_0 \|\!|(I-P_H)\nabla\boldsymbol{\phi}_{m+1}|\!\|_0 \le |T_0| + \cdots + |T_{10}|.$$

We want to bound the terms $T_0, T_2, \ldots, T_{10}$. T_0 can be handled as in Theorem 4.1. Then, T_0 is bounded as

$$T_0 \le \frac{\kappa\nu}{6}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}(\|\boldsymbol{u}\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2 + \|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2)\|\boldsymbol{\phi}_m\|_{0,\Omega}^2.$$

Also, the term T_1 is bounded exactly like the term (4.5) in the proof of Theorem 4.1. Here, the constant vectors are

$$\boldsymbol{c}_1 = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{u}_{m+1}, \quad \boldsymbol{c}_2 = \frac{1}{|E_j|}\int_{E_j} \boldsymbol{\phi}_{m+1}.$$

Then, T_1 can be rewritten as

$$T_1 = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\phi}_m \cdot \nabla \boldsymbol{u}_{m+1}) \cdot \boldsymbol{\phi}_{m+1} - \frac{1}{2} b(\boldsymbol{\phi}_m, (\boldsymbol{u}_{m+1} - \boldsymbol{c}_1) \cdot \boldsymbol{\phi}_{m+1})$$
$$- \frac{1}{2} b(\boldsymbol{\phi}_m, \boldsymbol{c}_1 \cdot (\boldsymbol{\phi}_{m+1} - \boldsymbol{c}_2)) \le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\|\boldsymbol{\phi}_m\|_{0,\Omega}^2.$$

Expanding T_2, we obtain

$$T_2 = \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{\eta}_m \cdot \nabla \boldsymbol{u}_{m+1}^I) \cdot \boldsymbol{\phi}_{m+1} + \sum_{j=1}^{N_h} \int_{\partial E_j^-} |\{\boldsymbol{\eta}_m\} \cdot \boldsymbol{n}_{E_j}| (\boldsymbol{u}_{m+1}^{I,\mathrm{int}} - \boldsymbol{u}_{m+1}^{I,\mathrm{ext}}) \cdot \boldsymbol{\phi}_{m+1}^{int}$$
$$+ \frac{1}{2}\sum_{j=1}^{N_h} \int_{E_j} (\nabla \cdot \boldsymbol{\eta}_m) \boldsymbol{u}_{m+1}^I \cdot \boldsymbol{\phi}_{m+1} - \frac{1}{2}\sum_{k=1}^{P_h} \int_{e_k} [\boldsymbol{\eta}_m] \cdot \boldsymbol{n}_k \{\boldsymbol{u}_{m+1}^I \cdot \boldsymbol{\phi}_{m+1}\}$$
$$= T_{21} + \cdots + T_{24}.$$

The bound for T_{21} is obtained using (2.6) and (2.8):

$$T_{21} \le \|\boldsymbol{\eta}_m\|_{0,\Omega} \|\nabla \boldsymbol{u}_{m+1}^I\|_{L^4(\Omega)} \|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)}$$
$$\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|_{L^\infty(0,T;W^{2,4/3}(\Omega))}^2 |\boldsymbol{u}_m|_{r+1,\Omega}^2.$$

Similarly for the term T_{22}, the inequalities (2.7) and (2.14) give

$$T_{22} \le C\sum_{j=1}^{N_h} \|\boldsymbol{\eta}_m\|_{L^2(\partial E_j)} \|\boldsymbol{u}_{m+1}^I\|_{L^\infty(\Omega)} \|\boldsymbol{\phi}_{m+1}\|_{L^2(\partial E_j)}$$
$$\le \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|_{L^\infty([0,T]\times\Omega)}^2 |\boldsymbol{u}_m|_{r+1,\Omega}^2.$$

The estimate of T_{23} is obtained by using a bound on interpolant, the Cauchy–Schwarz inequality, the approximation result (2.7), Young's inequality, and L^p bound (2.6):

$$T_{23} \leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)}|\boldsymbol{u}_m|^2_{r+1,\Omega}.$$

The term T_{24} is bounded exactly as for T_{22}. Because of the regularity of $\boldsymbol{u}$ and the approximation result (2.7), we can bound T_3:

$$\begin{aligned} T_3 &\leq C\|\boldsymbol{u}_m\|_{L^\infty(\Omega)}h^r|\boldsymbol{u}_{m+1}|_{r+1,\Omega}\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega} \\ &\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}\|\boldsymbol{u}\|^2_{L^\infty([0,T]\times\Omega)}|\boldsymbol{u}_m|^2_{r+1,\Omega}. \end{aligned}$$

The term T_4 is bounded using the estimate (2.6):

$$\begin{aligned} T_4 &\leq \Delta t\|\boldsymbol{u}_t\|_{L^\infty(t_m,t_{m+1};L^2(\Omega))}\|\nabla\boldsymbol{u}_{m+1}\|_{L^4(\Omega)}\|\boldsymbol{\phi}_{m+1}\|_{L^4(\Omega)} \\ &\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2\|\boldsymbol{u}_t\|^2_{L^\infty(t_m,t_{m+1};L^2(\Omega))}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}. \end{aligned}$$

By property of the interpolant (3.11) and properties of $r_h(p)$, (2.9), and (2.10), we now bound T_5:

$$\begin{aligned} T_5 &= b(\boldsymbol{\phi}_{m+1}, p^h_{m+1} - (r_h(p))_{m+1}) - b(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1}) \\ &= -b(\boldsymbol{\phi}_{m+1}, p_{m+1} - (r_h(p))_{m+1}) = \sum_{k=1}^{M_h}\int_{e_k}\{p_{m+1} - (r_h(p))_{m+1}\}[\boldsymbol{\phi}_{m+1}]\cdot\boldsymbol{n}_k \\ &\leq \sum_{k=1}^{M_h}\|[\boldsymbol{\phi}_{m+1}]\|_{0,e_k}|e_k|^{1/2-1/2}\|p_{m+1}\|_{0,e_k} \leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}|p_{m+1}|^2_{r,\Omega}. \end{aligned}$$

From a Taylor expansion, we have

$$T_6 \leq C\Delta t\|\boldsymbol{\phi}_{m+1}\|_X\|\boldsymbol{u}_{tt}(t^*)\|_{0,\Omega} \leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\Delta t^2\|\boldsymbol{u}_{Tm}\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

To bound T_7, we assume that $h \leq \Delta t$ and we use (2.8) and (2.6):

$$\begin{aligned} T_7 &\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}\frac{h^{2r+2}}{\Delta t^2}(|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega}) \\ &\leq \frac{\kappa\nu}{24}\|\boldsymbol{\phi}_{m+1}\|_X^2 + C\nu^{-1}h^{2r}(|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega}). \end{aligned}$$

The terms T_8, T_9, and T_{10} are exactly bounded as in Theorem 4.1. (See [23] for details.) Combining all the bounds of the terms $T_0, \ldots, T_{10}$, multiplying by $2\Delta t$, and summing over m, we obtain

$$\begin{aligned} &\|\boldsymbol{\phi}_{m+1}\|^2_{0,\Omega} - \|\boldsymbol{\phi}_0\|^2_{0,\Omega} + \nu\kappa\Delta t\sum_{i=0}^{m}\|\boldsymbol{\phi}_{i+1}\|_X^2 + \nu_T\Delta t\sum_{i=0}^{m}\|(I-P_H)\nabla\boldsymbol{\phi}_{i+1}\|_0^2 \\ &\leq Ce^{CT\nu^{-1}}[h^{2r}(\nu + \nu^{-1} + \nu_T)|\boldsymbol{u}|^2_{L^2(0,T;H^{r+1}(\Omega))} + \nu_T H^{2r}|\boldsymbol{u}|^2_{L^2(0,T;H^{r+1}(\Omega))} \\ &+ \nu^{-1}\Delta t^2(\|\boldsymbol{u}_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}) + h^{2r}\nu^{-1}|p|^2_{L^2(0,T;H^r(\Omega))}]. \end{aligned}$$

The final result is obtained by noting that $\|\boldsymbol{\phi}_0\|_{0,\Omega}$ is of order h^r and by using approximation results and a triangle inequality. ∎

THEOREM 5.3. *Assume that* $\boldsymbol{u}_{tt} \in L^\infty(0,T;(H^1(\Omega))^2)$, $p_{tt} \in L^\infty(0,T;H^1(\Omega))$, $\boldsymbol{u}_{ttt} \in L^\infty(0,T;(H^2(\Omega))^2)$, *and* $\boldsymbol{f}_{tt} \in L^\infty(0,T;(L^2(\Omega))^2)$. *Under the assumptions of Theorem* 4.1, *there is a constant* C *independent of* h, H, ν, *and* ν_T *such that*

$$\max_{m=0,\dots,M} \|\boldsymbol{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + \left(\nu\kappa\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_X^2\right)^{1/2}$$
$$+ \left(\nu_T \Delta t \sum_{m=0}^{M-1} \|(I-P_H)\nabla \boldsymbol{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_0^2\right)^{1/2} \le Ce^{CT\nu^{-1}}[h^r \nu^{-1/2}\|p\|_{L^2(0,T;H^r(\Omega))}$$
$$+ h^r(\nu + \nu^{-1} + \nu_T)^{1/2}\|\boldsymbol{u}\|_{L^2(0,T;H^{r+1}(\Omega))} + \Delta t^2 \nu^{1/2}\|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;H^2(\Omega))}$$
$$+ \Delta t^2 \nu^{-1/2}(\|\boldsymbol{u}_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|p_{tt}\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\boldsymbol{f}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}) + \nu_T^{1/2} H^r |\boldsymbol{u}|_{L^2(0,T;H^{r+1}(\Omega))}] + Ch^r|\boldsymbol{u}_0|_{r+1,\Omega}.$$

Proof. The proof is derived in a similar fashion as for the backward Euler scheme. Using the same notation, the error equation is obtained by subtracting (3.6) evaluated at the time $t = t_{m+1/2}$ from (5.3) and adding and subtracting the interpolant $(R_h(\boldsymbol{u}))_{m+1/2}$. After some manipulation, we obtain

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|_{0,\Omega}^2 - \|\boldsymbol{\phi}_m\|_{0,\Omega}^2) + \nu\kappa\|\boldsymbol{\phi}_{m+\frac12}\|_X^2 + \nu_T\|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac12}\|_0^2$$
$$+ c(\tilde{\boldsymbol{u}}^h_{m+\frac12}, \boldsymbol{\phi}^h_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12}) \le |c(\boldsymbol{\phi}_{m+\frac12}, \boldsymbol{\eta}_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12})| + |c(\boldsymbol{\phi}_{m+\frac12}, \boldsymbol{u}_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12})|$$
$$+ |c(\boldsymbol{\eta}_{m+\frac12}, \boldsymbol{u}^I_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12})| + |c(\boldsymbol{u}_{m+\frac12}, \boldsymbol{\eta}_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12})|$$
$$+ |c(\boldsymbol{u}_{m+\frac12} - \boldsymbol{u}(t_{m+\frac12}), \boldsymbol{u}_{m+\frac12}, \boldsymbol{\phi}_{m+\frac12})| + |c(\boldsymbol{u}(t_{m+\frac12}), \boldsymbol{u}_{m+\frac12} - \boldsymbol{u}(t_{m+\frac12}), \boldsymbol{\phi}_{m+\frac12})|$$
$$+ |b(\boldsymbol{\phi}_{m+\frac12}, \tilde{p}^h_{m+\frac12} - p(t_{m+\frac12}))| + \left\|\boldsymbol{u}_t(t_{m+\frac12}) - \frac{1}{\Delta t}(\boldsymbol{u}_{m+1} - \boldsymbol{u}_m)\right\|_{0,\Omega} \|\boldsymbol{\phi}_{m+\frac12}\|_{0,\Omega}$$
$$+ \frac{1}{\Delta t}\|\boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac12}\|_{0,\Omega} + \|\boldsymbol{f}_{m+\frac12} - \boldsymbol{f}(t_{m+\frac12})\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac12}\|_{0,\Omega}$$
$$+ \nu|a(\boldsymbol{u}(t_{m+\frac12}) - \boldsymbol{u}^I_{m+\frac12}, \boldsymbol{\phi}_{m+1}) + J(\boldsymbol{u}(t_{m+\frac12}) - \boldsymbol{u}^I_{m+\frac12}, \boldsymbol{\phi}_{m+1})|$$
$$+ \nu_T\|(I-P_H)\nabla\boldsymbol{\eta}_{m+\frac12}\|_0\|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac12}\|_0$$
$$+ \nu_T\|(I-P_H)\nabla\boldsymbol{u}_{m+\frac12}\|_0\|(I-P_H)\nabla\boldsymbol{\phi}_{m+\frac12}\|_0 \le A_0 + \cdots + A_{13}.$$

The terms $A_0, A_1, A_2, A_3, A_8, A_{11}$, and A_{12} are bounded exactly like the terms $T_0, T_1, T_2, T_3, T_7, T_9$, and T_{10}, respectively. From a Taylor expansion, we bound the terms A_4 and A_5:

$$A_4 + A_5 = \sum_{j=1}^{N_h} \int_{E_j} ((\boldsymbol{u}_{m+\frac12} - \boldsymbol{u}(t_{m+\frac12})) \cdot \nabla \boldsymbol{u}_{m+\frac12}) \cdot \boldsymbol{\phi}_{m+\frac12}$$
$$+ \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{u}(t_{m+\frac12}) \cdot \nabla(\boldsymbol{u}_{m+\frac12} - \boldsymbol{u}(t_{m+\frac12})) \cdot \boldsymbol{\phi}_{m+\frac12}$$
$$= \frac{\Delta t^2}{8} \sum_{j=1}^{N_h} \int_{E_j} (\boldsymbol{u}_{tt}(t^*) \cdot \nabla \boldsymbol{u}_{m+\frac12}) \cdot \boldsymbol{\phi}_{m+\frac12} + \frac{\Delta t^2}{8} \sum_{j=1}^{N_h} \int_{E_j} \boldsymbol{u}(t_{m+\frac12}) \cdot \nabla(\boldsymbol{u}_{tt}(t^*)) \cdot \boldsymbol{\phi}_{m+\frac12}$$
$$\le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac12}\|_X^2 + C\nu^{-1}\Delta t^4 \|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{2,4/3}(\Omega))}.$$

With (3.7), (3.11), and (5.4), the pressure term can be rewritten as

$$A_6 = b(\boldsymbol{\phi}_{m+\frac{1}{2}}, \tilde{p}^h_{m+\frac{1}{2}} - p_{m+\frac{1}{2}}) + b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))$$
$$= -\, b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}) + b(\boldsymbol{\phi}_{m+\frac{1}{2}}, p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))$$
$$= \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - (r_h(p))_{m+\frac{1}{2}}\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \boldsymbol{n}_k - \sum_{j=1}^{N_h} \int_{E_j} (p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}}))\nabla \cdot \boldsymbol{\phi}_{m+\frac{1}{2}}$$
$$+ \sum_{k=1}^{M_h} \int_{e_k} \{p_{m+\frac{1}{2}} - p(t_{m+\frac{1}{2}})\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \cdot \mathbf{n}_k$$
$$\le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}h^{2r}(|p_{m+1}|_{r,\Omega}^2 + |p_m|_{r,\Omega}^2) + C\nu^{-1}\Delta t^4\|p_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}.$$

We now bound A_7, using a Taylor expansion:

$$A_7 \le C\Delta t^2\|\boldsymbol{u}_{ttt}(t^*)\|_{0,\Omega}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_{0,\Omega} \le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + C\nu^{-1}\Delta t^4\|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

Also using a Taylor expansion, we bound A_9:

$$A_9 \le C\nu^{-1}\Delta t^4\|\boldsymbol{f}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2.$$

Finally the last term A_{10} is handled as follows:

$$A_{10} = \nu[a(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\boldsymbol{\eta}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})]$$
$$+ \nu[a(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}}) + J(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}, \boldsymbol{\phi}_{m+\frac{1}{2}})] = A_{101} + A_{102}.$$

The term A_{101} is bounded like T_8. The term A_{102} reduces to

$$A_{102} = \nu\sum_{j=1}^{N_h}\int_{E_j}\nabla(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}}) : \nabla\boldsymbol{\phi}_{m+\frac{1}{2}}$$
$$- \nu\sum_{k=1}^{M_h}\int_{e_k}\{\nabla(\boldsymbol{u}(t_{m+\frac{1}{2}}) - \boldsymbol{u}_{m+\frac{1}{2}})\boldsymbol{n}_k\}[\boldsymbol{\phi}_{m+\frac{1}{2}}] \le \frac{\kappa\nu}{64}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2$$
$$+ C\nu\Delta t^4\|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;H^2(\Omega))}.$$

Combining all the bounds above yields

$$\frac{1}{2\Delta t}(\|\boldsymbol{\phi}_{m+1}\|^2_{0,\Omega} - \|\boldsymbol{\phi}_m\|^2_{0,\Omega}) + \frac{\nu\kappa}{2}\|\boldsymbol{\phi}_{m+\frac{1}{2}}\|_X^2 + \frac{\nu_T}{2}\|(I - P_H)\nabla\boldsymbol{\phi}_{m+\frac{1}{2}}\|_0^2$$
$$\le C\nu^{-1}(\|\boldsymbol{\phi}_m\|^2_{0,\Omega} + \|\boldsymbol{\phi}_{m+1}\|^2_{0,\Omega}) + Ch^{2r}(\nu + \nu^{-1} + \nu_T)(|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega})$$
$$+ Ch^{2r}\nu^{-1}(|p_{m+1}|^2_{r,\Omega} + |p_m|^2_{r,\Omega}) + C\Delta t^4\nu\|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;H^2(\Omega))}$$
$$+ C\Delta t^4\nu^{-1}(\|\boldsymbol{u}_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|p_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\boldsymbol{f}_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}) + C\nu_T H^{2r}(|\boldsymbol{u}_{m+1}|^2_{r+1,\Omega} + |\boldsymbol{u}_m|^2_{r+1,\Omega}).$$

The end of the proof is similar to that of Theorem 5.2. ☐

COROLLARY 5.4. *Assume that $\nu_T = h^\beta$ and $H = h^{1/\alpha}$, where $\beta \geq 2r(\alpha - 1)/\alpha$ (see Corollary 4.2); then the estimates in Theorems 5.2 and 5.3 are optimal:*

$$\max_{m=0,\dots,M} \|\boldsymbol{u}_m - \boldsymbol{u}_m^h\|_{0,\Omega} + \left(\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1} - \boldsymbol{u}_{m+1}^h\|_X^2\right)^{1/2} = \mathcal{O}(h^r + \Delta t),$$

$$\max_{m=0,\dots,M} \|\boldsymbol{u}_m - \tilde{\boldsymbol{u}}_m\|_{0,\Omega} + \left(\Delta t \sum_{m=0}^{M-1} \|\boldsymbol{u}_{m+1} - \tilde{\boldsymbol{u}}_{m+1}\|_X^2\right)^{1/2} = \mathcal{O}(h^r + \Delta t^2).$$

Remark 3. The analysis presented in this paper is applicable to the three-dimensional Navier–Stokes equations assuming that the L^p bound (2.6) and the inf-sup condition (3.10) hold true.

6. Conclusion. In this paper, we have analyzed the stability and convergence of totally discontinuous schemes for solving the time-dependent Navier–Stokes equations. Both semidiscrete approximation and fully discrete approximation are constructed for velocity. In addition, semidiscrete approximation of pressure is obtained. We showed that these estimations are optimal. Numerical experiments are currently under investigation.

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