Non Conforming Methods for Transport with Nonlinear Reaction

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Abstract. The transport equation is solved by a discontinuous Galerkin method, that is locally conservative and that allows for non-conforming meshes. The convective fluxes are upwinded. *hp* error estimates are derived in $L^\infty(L^2)$ and $L^2(H^1)$ for the continuous in time scheme. A class of fully discrete schemes is presented and analyzed.

1. Introduction

There is a need for efficient and accurate algorithms for simulating the transport of species through a porous medium on general geometries. Applications include study of radioactive nuclear decay, which is essential in the performance assessment of nuclear storage facilities, remediation of industrial pollutants contaminating the ground, and oil recovery processes.

Currently, there exists several well known schemes such as higher order Godunov [**I, E, K**], MUSCL [**Y, J**], Essentially Non-Oscillatory (ENO) [**P, S**], control volume [**F, N, Q**], and characteristic [**M, O, X, L, B**]. These schemes possess one or more deficiencies. These schemes are either not extendible to unstructured and/or non-conforming grids, are at best second order convergent in regions with smooth solutions, involve dual grids which are very complicated in three dimensional simulations, or are not locally conservative.

In this paper, we formulate and analyze a family of methods known discontinuous Galerkin (DG) methods for solving the transport problem with nonlinear reactions. These methods have the following appealing features: 1) they are element-wise conservative; 2) they support local approximations of high order; 3) they are robust and local oscillations can be eliminated by the introduction of slope limiters; 4) they are implementable on unstructured and even non matching meshes; and, 5) with the appropriate meshing, they are capable of delivering exponential rates of convergence.

There are a variety of methods using discontinuous discrete spaces such as the Bassi and Rebay method [**D**] and the Local Discontinuous Galerkin (LDG) [**H, A**] method, the Oden, Babuška and Baumann method [**R**], the interior penalty Galerkin methods [**Z**] [Wheeler and Douglas], and the NIPG methods [**V, W**]. In Arnold, Brezzi, Cockburn and Marini [**C**] a general framework of these methods is

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1991 *Mathematics Subject Classification.* Primary 65M15; Secondary 65M60.

Key words and phrases. discontinuous Galerkin methods, *hp* estimates, upwind.

presented. Application of these methods to a wide variety of problems can be found in [**AA**]. In this paper we restrict our attention to the Oden, Babuška, Baumann formulation. With minor modifications, the theorems proved in this paper apply to NIPG and to the interior penalty Galerkin method.

The paper is organized as follows. In the following section, we describe the mathematical model and define notation. The DG scheme is introduced in section 3. Section 4 contains the analysis of the semidiscrete solution. In section 5, the error estimates for a class of fully discrete schemes are derived. The last section contains some concluding remarks.

2. Model Problem and Scheme

The domain Ω is polygonal and bounded in $\mathbb{R}^d, d=2,3$. Let $\boldsymbol{u}$ be a velocity field that satisfies $\nabla\cdot\boldsymbol{u}=0$ and that varies in space. We decompose the boundary of the domain into an inflow part Γ_{in} and an outflow part Γ_{out}, $\partial\Omega=\Gamma_{\text{in}}\cup\Gamma_{\text{out}}$, where $\Gamma_{\text{in}}=\{x\in\partial\Omega:\boldsymbol{u}\cdot\boldsymbol{n}<0\}$, and $\Gamma_{\text{out}}=\{x\in\partial\Omega:\boldsymbol{u}\cdot\boldsymbol{n}\geq 0\}$. The transport of a contaminant through a porous medium is modelized by the following partial differential equations.

$$\begin{aligned}
&(2.1) & \phi c_t+\nabla\cdot(\boldsymbol{u}c-D(\boldsymbol{u})\nabla c) &= f(c), \text{ in } \Omega\times(0,T],\\
&(2.2) & (\boldsymbol{u}c-D(\boldsymbol{u})\nabla c)\cdot\boldsymbol{n} &= \boldsymbol{u}\, c_{\text{in}}\cdot\boldsymbol{n}, \quad \text{on } \Gamma_{\text{in}}\times(0,T],\\
&(2.3) & -D(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} &= 0, \quad \text{on } \Gamma_{\text{out}}\times(0,T],\\
&(2.4) & c(0,\cdot) &= c_0, \quad \text{in } \Omega.
\end{aligned}$$

Here, c is the concentration of the contaminant, $f(c)$ a general nonlinear reaction source function, $D(\boldsymbol{u})$ a diffusion-dispersion tensor. We assume that f is Lipschitz in c and that $D(\boldsymbol{u})$ is symmetric, positive definite in $\overline{\Omega}$ uniformly with respect to x. The porosity ϕ is the fraction of the volume of the medium occupied by pores, and it is assumed to be bounded below and above by positive constants. The concentrations c_{in} and c_0 are respectively the concentration at the inflow boundary and the concentration at the initial time.

We now establish some notation for the spatial discretization. Let $\mathcal{E}_h=\{E\}_E$ be a non degenerate subdivision of Ω, made of triangles in 2D and tetrahedra in 3D. We allow for a non conforming partition of the domain. Let h be the maximum diameter of the elements. Let Γ be the skeleton of the mesh of Ω, that is the union of the open sets that coincide with interior edges (or faces) of elements. We also associate with each set γ_k in Γ, a unit normal vector $\boldsymbol{n}_k$. For γ_k in $\partial\Omega$, the vector $\boldsymbol{n}_k$ is outward to $\partial\Omega$. We define for $s\geq 0$ and $m\geq 1$,

$$W^{s,m}(\mathcal{E}_h)=\{v\in L^m(\Omega): v|_E\in W^{s,m}(E)\ \forall E\in\mathcal{E}_h\},$$

and we denote it by $H^s(\mathcal{E}_h)$ when $m=2$. The usual Sobolev norm of H^s on $E\subset\mathbb{R}^d$ is denoted by $\|\cdot\|_{s,E}$. The L^2 inner product is denoted by $(\cdot,\cdot)_E$. If $E=\Omega$, then we simply write $(\cdot,\cdot)$. The norm associated with $H^s(\mathcal{E}_h)$ is the "broken" norm $|\!|\!|\cdot|\!|\!|_s^2=\sum_{E\in\mathcal{E}_h}\|\cdot\|_{s,E}^2$. For a,b real and Y Sobolev space, we define the space

$$L^k(a,b;Y)=\{w:\|w\|^k_{L^k(a,b;Y)}=\int_a^b\|w(t,\cdot)\|_Y^k<\infty\}.$$

We define for $w \in H^s(E)$, $s > 1/2$, the average $\{w\}$, the jump $[w]$ and the upwind w_* value. We assume below that $\boldsymbol{n}_k$ is outward to E_k^1.

$$\{w\} = \frac{1}{2}(w|_{E_k^1}) + \frac{1}{2}(w|_{E_k^2}), \qquad [w] = (w|_{E_k^1}) - (w|_{E_k^2}), \quad \forall \gamma_k = \partial E_k^1 \cap \partial E_k^2,$$
$$\{w\} = (w|_{E_k^1}), \qquad [w] = (w|_{E_k^1}), \quad \forall \gamma_k = \partial E_k^1 \cap \partial \Omega,$$

$$w_* = \begin{cases} w|_{E_k^1} & \text{if} \quad \boldsymbol{u}\cdot\boldsymbol{n}_k \geq 0, \\ w|_{E_k^2} & \text{if} \quad \boldsymbol{u}\cdot\boldsymbol{n}_k < 0. \end{cases}, \quad \forall \gamma_k = \partial E_k^1 \cap \partial E_k^2.$$

Let r be an integer. The finite element subspace consists of discontinuous piecewise polynomials:

$$\mathcal{D}_r(\mathcal{E}_h) = \{v : v|_E \in P_r(E) \quad \forall E \in \mathcal{E}_h\},$$

where $P_r(E)$ is a discrete space containing the set of polynomials of total degree less than or equal to r on E. We can construct a special interpolant in $\mathcal{D}_r(\mathcal{E}_h)$ that satisfies the optimal approximation properties:

LEMMA 2.1. *Let h be small enough. Let $c \in H^s(\mathcal{E}_h)$, for $s \geq 2$ and let $r \geq 2$. If, in addition $D(\boldsymbol{u}) \in (W^{1,\infty}(\mathcal{E}_h))^{d\times d}$, there exists an interpolant of c, $\tilde{c} \in \mathcal{D}_r(\mathcal{E}_h)$ such that for each E in $\mathcal{E}_h$ and each edge (or face) $e \in \partial E$ that is divided into disjoint open sets $\gamma^1, \ldots, \gamma^{s_e}$, the following properties hold:*

$$\int_{\gamma^j} D(\boldsymbol{u})\nabla(\tilde{c}|_E - c)\cdot\boldsymbol{n}_E = 0, \quad j = 1, \ldots, s_e, \tag{2.5}$$

where $\boldsymbol{n}_E$ is the outward unit normal to ∂E and s_e is a finite number. Furthermore, $\tilde{c}$ is optimally close to c:

$$|||\nabla^i(\tilde{c} - c)|||_0 \leq K\frac{h^{\mu-i}}{r^{s-\frac{3}{2}-\delta}}|||c|||_s, \quad i = 0, 1, 2, \tag{2.6}$$

where $\delta = 0$ if $i = 0, 1$, $\delta = \frac{1}{2}$ if $i = 2$, $\mu = \min(r+1, s)$ and K is independent of h and r.

REMARK 2.2. The proof of this lemma is given in [**T**] and it is a generalization to non conforming meshes of the approximation result proved in [**W**].

We also recall an inverse estimate for a polynomial χ defined on $E \in \mathcal{E}_h$. Let h_E be the diameter of E. There is a constant K independent of h and r such that, for $e \in \partial E$,

$$||\nabla\chi\cdot\boldsymbol{n}_k||_{0,e} \quad \leq \quad Krh_E^{-\frac{1}{2}}||\nabla\chi||_{0,E}. \tag{2.7}$$

A proof of this inverse estimate is recalled in [**W**].

3. Scheme

We introduce the bilinear form $b_{NS} : H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h) \to \mathbb{R}$, $s > 3/2$, and the linear form $L : L^2(\Omega) \to \mathbb{R}$:

$$\begin{aligned} b_{NS}(\boldsymbol{u}; w, v) &= \sum_{E\in\mathcal{E}_h} \int_E D(\boldsymbol{u})\nabla w \cdot \nabla v - \sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{u} w \cdot \nabla v \\ &\quad - \sum_{\gamma_k\in\Gamma} \int_{\gamma_k} \{D(\boldsymbol{u})\nabla w \cdot \boldsymbol{n}_k\}[v] + \sum_{\gamma_k\in\Gamma} \int_{\gamma_k} \{D(\boldsymbol{u})\nabla v \cdot \boldsymbol{n}_k\}[w] \\ &\quad + \sum_{\gamma_k\in\Gamma} \int_{\gamma_k} \boldsymbol{u}\cdot\boldsymbol{n}_k w_*[v] + \sum_{\gamma_k\in\Gamma_{\text{out}}} \int_{\gamma_k} \boldsymbol{u}\cdot\boldsymbol{n}_k wv. \end{aligned} \tag{3.1}$$

$$L(c; v) = \int_\Omega f(c)v - \sum_{\gamma_k\in\Gamma_{\text{in}}} \int_{\gamma_k} \boldsymbol{u}\cdot\boldsymbol{n}_k c_{\text{in}} v. \tag{3.2}$$

We can now give the weak formulation of the transport problem.

Lemma 3.1. *If c is the solution of* (2.1)-(2.4), *then c satisfies*

$$(\phi c_t, v) + b_{NS}(\boldsymbol{u}; c, v) = L(c; v), \qquad \forall v \in H^s(\mathcal{E}_h), s > 3/2. \tag{3.3}$$

Proof. Let $s > 3/2$ and let v be a test function in $H^s(\mathcal{E}_h)$. We multiply (2.1) by $v|_E$, and integrate by parts on one element $E \in \mathcal{E}_h$.

$$(\phi c_t, v)_E - \int_E (\boldsymbol{u}c - D(\boldsymbol{u})\nabla c)\cdot\nabla v + \int_{\partial E} (\boldsymbol{u}c - D(\boldsymbol{u})\nabla c)\cdot\boldsymbol{n}_E v = \int_E f(c)v.$$

Summing over all E and noting that both the concentration and its normal flux are continuous, we obtain:

$$\begin{aligned} &(\phi c_t, v) - \sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{u}c - D(\boldsymbol{u})\nabla c)\cdot\nabla v - \sum_{\gamma_k\in\Gamma\cup\Gamma_{\text{out}}} \int_{\gamma_k} \{D(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}_k\}[v] \\ &+ \sum_{\gamma_k\in\Gamma} \int_{\gamma_k} \boldsymbol{u}\cdot\boldsymbol{n}_k c[v] + \sum_{\gamma_k\in\Gamma_{\text{out}}} \int_{\gamma_k} \boldsymbol{u}\cdot\boldsymbol{n}_k cv + \sum_{\gamma_k\in\Gamma} \int_{\gamma_k} \{D(\boldsymbol{u})\nabla v\cdot\boldsymbol{n}_k\}[c] \\ &\qquad + \sum_{\gamma_k\in\Gamma_{\text{in}}} \int_{\gamma_k} (\boldsymbol{u}c - D(\boldsymbol{u})\nabla c)\cdot\boldsymbol{n}_k v = (f(c), v). \end{aligned}$$

Using the boundary condition and noting that $c_* = c$, we clearly have (3.3). □

The discontinuous Galerkin approximation C_{DG} in $L^2(0,T;\mathcal{D}_r(\mathcal{E}_h))$ satisfies the formulation:

$$(\phi\frac{\partial C_{\text{DG}}}{\partial t}, v) + b_{NS}(\boldsymbol{u}; C_{\text{DG}}, v) = L(C_{\text{DG}}, v), \quad t > 0, \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h), \tag{3.4}$$

$$(C_{\text{DG}}(0), v) = (c_0, v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{3.5}$$

Remark 3.2. It should be noted that the approximation of the concentration satisfies on each element E the following mass balance

$$\int_E \phi\frac{\partial C_{\text{DG}}}{\partial t} - \int_{\partial E} \{D(\boldsymbol{u})\nabla C_{\text{DG}}\}\cdot\boldsymbol{n}_E + \int_{\partial E} \boldsymbol{u}\cdot\boldsymbol{n}_E C_*^{\text{DG}} = \int_E f(C^{\text{DG}}).$$

This property is an unique feature of the discontinuous Galerkin methods.

4. Continuous in Time Error estimates

In this section, we derive *a priori* error estimates for the semidiscrete scheme. These estimates are optimal in h and suboptimal in r for the energy norm, and they are suboptimal for the L^2 norm.

THEOREM 4.1. *Let c be solution of* (2.1)-(2.4). *If $c \in L^2(0,T;H^s(\mathcal{E}_h))$, $c_0 \in H^s(\mathcal{E}_h)$ and $c_t \in L^2(0,T;H^{s-1}(\mathcal{E}_h))$, then there exists a constant K independent of h and r such that*

$$\begin{aligned}\|c - C_{\mathrm{DG}}\|_{L^\infty(0,T;L^2(\Omega))} &+ \|D(\boldsymbol{u})^{1/2}\nabla(c - C_{\mathrm{DG}})\|_{L^2(0,T;L^2(\Omega))}\\ &\le K\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}\left(\|c\|_{L^2(0,T;H^s(\mathcal{E}_h))} + \|c_t\|_{L^2(0,T;H^{s-1}(\Omega))} + \|c_0\|_{H^s(\mathcal{E}_h)}\right),\end{aligned}$$

where $r \geq 2$ and $\mu = \min(r+1, s)$.

PROOF. Let $\tilde{c}$ be the interpolant of c defined in Lemma 2.1. Denote $\chi = C_{\mathrm{DG}} - \tilde{c}$ and $\xi = c - \tilde{c}$. Throughout the paper, we will denote K a generic constant with different values on different places, that is independent of h and r. The following error equation is satisfied for all v in $\mathcal{D}_r(\mathcal{E}_h)$.

$$(\phi\chi_t, v) + b_{NS}(\boldsymbol{u};\chi, v) = (\phi\xi_t, v) + b_{NS}(\boldsymbol{u};\xi,v) + (f(C_{\mathrm{DG}}) - f(c), v).$$

Now, by choosing $v = \chi$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi^{\frac{1}{2}}\chi\|_0^2 + b_{NS}(\boldsymbol{u};\chi,\chi) = (\phi\xi_t, \chi) + b_{NS}(\boldsymbol{u};\xi,\chi) + (f(C_{\mathrm{DG}}) - f(c), \chi).$$

We note that

$$\begin{aligned}b_{NS}(\boldsymbol{u};\chi,\chi) &= \sum_{E\in\mathcal{E}_h}\int_E D(\boldsymbol{u})\nabla\chi\nabla\chi - \sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\chi\cdot\nabla\chi\\ &+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi_*[\chi] + \sum_{\gamma_k\in\Gamma_{\mathrm{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2.\end{aligned}$$

Now using a technique found in [**G**], we integrate by parts the advection term, and use the fact that $\boldsymbol{u}$ is divergent free.

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\chi\cdot\nabla\chi &= \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\cdot\nabla\chi^2\\ &= \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_{\partial E}\boldsymbol{u}\cdot\boldsymbol{n}_E\chi^2\\ &= \frac{1}{2}\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k[\chi^2] + \frac{1}{2}\sum_{\gamma_k\in\partial\Omega}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2.\end{aligned}$$

Now, we combine the upwind terms:

$$
\begin{aligned}
&-\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\chi\cdot\nabla\chi+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi_*[\chi]+\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2\\
&=\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k(\chi_*[\chi]-\frac{1}{2}[\chi^2])-\frac{1}{2}\sum_{\gamma_k\in\partial\Omega}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2+\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2\\
&=\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k(\chi_*[\chi]-\{\chi\}[\chi])-\frac{1}{2}\sum_{\gamma_k\in\Gamma_{\text{in}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2+\frac{1}{2}\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\chi^2\\
&=\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|[\chi]^2+\frac{1}{2}\sum_{\gamma_k\in\Gamma_{\text{in}}}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|\chi^2+\frac{1}{2}\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|\chi^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi^{\frac{1}{2}}\chi\|_0^2+\sum_{E\in\mathcal{E}_h}\int_E D(\boldsymbol{u})\nabla\chi\cdot\nabla\chi+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|[\chi]^2\\
&+\frac{1}{2}\sum_{\gamma_k\in\partial\Omega}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|\chi^2=(\phi\xi_t,\chi)+b_{NS}(\boldsymbol{u};\xi,\chi)+(f(C_{\text{DG}})-f(c),\chi).
\end{aligned}
\tag{4.1}
$$

The first term in the right-hand side of (4.1) is bounded by Cauchy-Schwarz's inequality and the approximation result (2.6)

$$
\begin{aligned}
(\phi\xi_t,\chi) &\le \|\phi^{\frac{1}{2}}\xi_t\|_0\|\phi^{\frac{1}{2}}\chi\|_0\le\frac{1}{2}\|\phi^{\frac{1}{2}}\chi\|_0^2+\frac{1}{2}\|\xi_t\|_0^2\\
&\le \frac{1}{2}\|\phi^{\frac{1}{2}}\chi\|_0^2+K\frac{h^{2\mu-2}}{r^{2s-3}}\|c_t\|_{s-1}^2.
\end{aligned}
$$

The second term in the right-hand side of (4.1) can be rewriten as

$$
\begin{aligned}
b_{NS}(\boldsymbol{u};\xi,\chi) &= \sum_{E\in\mathcal{E}_h}\int_E D(\boldsymbol{u})\nabla\xi\nabla\chi-\sum_{E\in\mathcal{E}_h}\int_E\boldsymbol{u}\xi\cdot\nabla\chi\\
&\quad-\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\xi\cdot\boldsymbol{n}_k\}[\chi]+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\chi\cdot\boldsymbol{n}_k\}[\xi]\\
&\quad+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi_*[\chi]+\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi\chi.
\end{aligned}
\tag{4.2}
$$

We now proceed to bound each term in the right-hand side of (4.2). The first term can be bounded using Cauchy-Schwarz's inequality and approximation result (2.6)

$$
\begin{aligned}
\sum_{E\in\mathcal{E}_h}\int_E D(\boldsymbol{u})\nabla\xi\nabla\chi &\le \|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\xi\|_0\|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi\|_0\\
&\le \frac{1}{8}\|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi\|_0^2+K\frac{h^{2\mu-2}}{r^{2s-3}}\|c\|_s^2.
\end{aligned}
$$

The second term can be bounded using Cauchy-Schwarz's inequality and using the fact that $D(\boldsymbol{u})$ is symmetric positive definite.

$$\begin{aligned}
\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\xi\cdot\nabla\chi &\leq K\sum_{E\in\mathcal{E}_h}\|\xi\|_{0,E}\|\nabla\chi\|_{0,E} \\
&\leq K\sum_{E\in\mathcal{E}_h}\|\xi\|_{0,E}\|D(\boldsymbol{u})^{1/2}\nabla\chi\|_{0,E} \\
&\leq \frac{1}{8}\|D(\boldsymbol{u})^{1/2}\nabla\chi\|_0^2 + K\frac{h^{2\mu-2}}{r^{2s-3}}\|c\|_s^2.
\end{aligned}$$

Using the property (2.5) of the interpolant, we can rewrite the third term as

$$\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\xi\cdot\boldsymbol{n}_k\}[\chi] = \sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\xi\cdot\boldsymbol{n}_k\}([\chi]-a_k),$$

where a_k is a constant chosen as follows: we assume that $\gamma_k = \partial E_k^1 \cap \partial E_k^2$ where E_k^1 and E_k^2 are elements of $\mathcal{E}_h$; then we take $a_k = a_1 - a_2$ where $a_i = \frac{1}{|E_k^i|}\int_{E_k^i}\chi$. Then, based on a technique found in [**W**], we have

$$\|[\chi]-a_k\|_{0,\gamma_k} \leq Kh^{\frac{1}{2}}(\|\nabla\chi\|_{0,E_k^1} + \|\nabla\chi\|_{0,E_k^2}). \tag{4.3}$$

Therefore, combining (4.3) with the approximation result (2.6) yields

$$\begin{aligned}
\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\xi\cdot\boldsymbol{n}_k\}[\chi] &\leq \sum_{\gamma_k\in\Gamma}\|\{D(\boldsymbol{u})\nabla\xi\cdot\boldsymbol{n}_k\}\|_{0,\gamma_k}\|[\chi]-a_k\|_{0,\gamma_k} \\
&\leq K\frac{h^{\mu-1}}{r^{s-\frac{3}{2}}}|\!|\!|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi|\!|\!|_0\|c\|_s \\
&\leq \frac{1}{8}|\!|\!|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + K\frac{h^{2\mu-2}}{r^{2s-3}}\|c\|_s^2.
\end{aligned}$$

The fourth term can be bounded by the inverse estimate (2.7), a trace theorem and the approximation result (2.6)

$$\begin{aligned}
\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\chi\cdot\boldsymbol{n}_k\}[\xi] &\leq K\sum_{\gamma_k\in\Gamma}\|\{D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi\cdot\boldsymbol{n}_k\}\|_{0,\gamma_k}\|[\xi]\|_{0,\gamma_k} \\
&\leq \sum_{\gamma_k\in\Gamma} rh^{-\frac{1}{2}}\|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi\|_{0,E_k^{12}}\frac{h^{\mu-\frac{1}{2}}}{r^{s-\frac{3}{2}}}\|c\|_s \\
&\leq \frac{1}{8}|\!|\!|D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi|\!|\!|_0^2 + K\frac{h^{2\mu-2}}{r^{2s-5}}\|c\|_s^2.
\end{aligned}$$

Finally, we bound the last terms in the right-hand side of (4.2)

$$\begin{aligned}
\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi_*[\chi] &\leq \sum_{\gamma_k\in\Gamma}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}[\chi]\|_{0,\gamma_k}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}\xi_*\|_{0,\gamma_k} \\
&\leq \frac{1}{4}\sum_{\gamma_k\in\Gamma}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}[\chi]\|_{0,\gamma_k}^2 + K\frac{h^{2\mu-1}}{r^{2s-3}}\|c\|_s^2.
\end{aligned}$$

Similarly

$$\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi\chi \leq \frac{1}{4}\sum_{\gamma_k\in\Gamma_{\text{out}}}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}\chi\|_{0,\gamma_k}^2 + K\frac{h^{2\mu-1}}{r^{2s-3}}\|c\|_s^2.$$

We now consider the third term in the right-hand side of (4.1). This term is bounded by Cauchy-Schwarz's inequality and by the Lipschitz property of f.

$$\begin{aligned}\int_\Omega (f(C_{\mathrm{DG}}) - f(c))\chi &\leq L\|C_{\mathrm{DG}} - c\|_{0,\Omega}\|\chi\|_{0,E}\\ &\leq K\|\chi\|_{0,\Omega}^2 + K\|\xi\|_{0,\Omega}^2\\ &\leq K\|\phi^{\frac{1}{2}}\chi\|_{0,\Omega}^2 + K\frac{h^{2\mu}}{r^{2s-3}}\|c\|_s^2.\end{aligned}$$

We then rewrite (4.1) by combining all the bounds derived above, and we integrate between 0 and t the resulting equation.

$$\begin{aligned}\|\phi^{\frac{1}{2}}\chi\|_0^2(t) - \|\phi^{\frac{1}{2}}\chi\|_0^2(0) + \frac{1}{2}\int_0^t \vert\!\vert\!\vert D(\boldsymbol{u})^{\frac{1}{2}}\nabla\chi\vert\!\vert\!\vert_0^2 + \frac{3}{4}\int_0^t \sum_{\gamma_k\in\Gamma}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|[\chi]^2&\\ + \frac{1}{4}\int_0^t \sum_{\gamma_k\in\partial\Omega}\int_{\gamma_k}|\boldsymbol{u}\cdot\boldsymbol{n}_k|\chi^2 \leq K\frac{h^{2\mu-2}}{r^{2s-5}}\int_0^t \|c\|_s^2 + K\int_0^t \|\phi^{\frac{1}{2}}\chi\|_0^2&\\ + K\frac{h^{2\mu-2}}{r^{2s-3}}\int_0^t \|c_t\|_{s-1}^2.&\end{aligned}$$

Noting that $\|\phi^{1/2}\chi\|_0^2(0) \leq Kh^{2\mu}/r^{2s-3}\vert\!\vert\!\vert c_0\vert\!\vert\!\vert_s^2$, using Gronwall's lemma and taking supremum over all t, we conclude that

$$\begin{aligned}&\|\phi^{\frac{1}{2}}\chi\|_{L^\infty(0,T;L^2(\Omega))} + \|D(\boldsymbol{u})^{1/2}\nabla\chi\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad\leq K\frac{h^{\mu-1}}{r^{s-\frac{5}{2}}}(\|c\|_{L^2(0,T;H^s(\mathcal{E}_h))} + \vert\!\vert\!\vert c_0\vert\!\vert\!\vert_s + \|c_t\|_{L^2(0,T;H^{s-1}(\mathcal{E}_h))}).\end{aligned}$$

The theorem is finally obtained by using triangle inequality and approximation results. □

5. Fully discrete analysis

In this section, we define and analyse a family of fully discrete formulations of the continuous problem, that is parametrized by θ: the case $\theta = 0$ corresponds to Crank-Nicolson discretization and the case $\theta = 1$, Backward-Euler in time discretization. We first introduce some standard notation. Let $\Delta t > 0$ denote the time step and let $t^n = n\Delta t$ for $n = 0, \ldots, N$. Let $v^n = v(t^n)$ and let $v^{n,\theta} = \frac{1}{2}(1+\theta)v^{n+1} + \frac{1}{2}(1-\theta)v^n$. The discrete approximations of the concentration satisfy:

$$\frac{1}{\Delta t}(\phi(C_{\mathrm{DG}}^{n+1} - C_{\mathrm{DG}}^n), v) + b_{NS}(\boldsymbol{u}; C_{\mathrm{DG}}^{n,\theta}, v) = L(C_{\mathrm{DG}}^{n,\theta}; v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h), \tag{5.1}$$

$$(C_{\mathrm{DG}}^0, v) = (c_0, v), \quad \forall v \in \mathcal{D}_r(\mathcal{E}_h). \tag{5.2}$$

We recall the following lemma that is a straigthforward application of Taylor expansion.

LEMMA 5.1.

$$\frac{1}{\Delta t}(c^{n+1} - c^n) = c_t(t^{n,\theta}) + \Delta t\rho_{n,\theta}, \tag{5.3}$$

where

$$\|\rho_{n,\theta}\|_0 \leq \left\{\begin{array}{ll} K_1\|c_{tt}\|_{L^\infty(t^n,t^{n+1};H^1)} & \text{if} \quad \theta > 0\\ K_2\Delta t\|c_{ttt}\|_{L^\infty(t^n,t^{n+1};H^1)} & \text{if} \quad \theta = 0\end{array}\right. \tag{5.4}$$

PROOF. The proof is given in [**U**]. □

We now state the convergence results for the θ formulations.

THEOREM 5.2. *Assume that* c *is the solution of (2.1)-(2.4), that* c *belongs to* $L^\infty(0,T;H^s(\mathcal{E}_h))$*,* c_t *to* $L^\infty(0,T;H^{s-1}(\mathcal{E}_h))$ *and that* $c_{tt} \in L^\infty(0,T;H^1)$ *for* $\theta \in (0,1]$*. If* $\theta = 0$*, assume that* $c_{ttt} \in L^\infty(0,T;H^1)$*. Then, if* $\theta > 0$*, we have*

$$\begin{aligned}\|c - C_{\mathrm{DG}}\|_{l^\infty(L^2(\Omega))} &+ K\Delta t^{1/2}\Big(\sum_n \|D(\boldsymbol{u})^{1/2}\nabla(c - C_{\mathrm{DG}})^{n,\theta}\|^2_{l^2(L^2(\mathcal{E}_h))}\Big)^{1/2}\\ &\leq K\frac{h^{\mu-1}}{r^{s-5/2}}(\|c\|_{L^\infty(0,T;H^s(\mathcal{E}_h))} + \|c_t\|_{L^\infty(0,T;H^{s-1}(\mathcal{E}_h))})\\ &\qquad + K\Delta t(\|c_t\|_{L^\infty(0,T;L^2(\Omega))} + \|c_{tt}\|_{L^\infty(0,T;H^1(\Omega))}).\end{aligned}$$

In the case where $\theta = 0$*, we obtain*

$$\begin{aligned}\|c - C_{\mathrm{DG}}\|_{l^\infty(L^2(\Omega))} &+ K\Delta t^{1/2}\|D(\boldsymbol{u})^{1/2}\nabla(c - C_{\mathrm{DG}})^{n,\theta}\|_{l^2(L^2(\mathcal{E}_h))}\\ &\leq K\frac{h^{\mu-1}}{r^{s-5/2}}(\|c\|_{L^\infty(0,T;H^s(\mathcal{E}_h))} + \|c_t\|_{L^\infty(0,T;H^{s-1}(\mathcal{E}_h))})\\ &\qquad + K\Delta t^2\|c_{ttt}\|_{L^\infty(0,T;H^1(\Omega))}.\end{aligned}$$

Here, $\mu = \min(r+1, s)$ *and* $r \geq 2$.

PROOF. From (3.3) and (5.3), we can write that the solution to (2.1)-(2.4) satisfies for each n and for all $v \in \mathcal{D}_r(\mathcal{E}_h)$:

$$\frac{1}{\Delta t}(\phi(c^{n+1} - c^n), v) + b_{NS}(\boldsymbol{u}; c^{n,\theta}, v) = ([f(c)]^{n,\theta}, v) + \Delta t(\phi\rho_{n,\theta}, v). \tag{5.5}$$

As in the continuous case, let denote $\chi^n = C^n_{\mathrm{DG}} - \tilde{c}^n$ and let $\xi^n = c^n - \tilde{c}^n$, where $\tilde{c}$ is the interpolant of c defined in Lemma 2.1. The following error equation is then obtained from (5.1) and (5.5).

$$\begin{aligned}\frac{1}{\Delta t}(\phi(\chi^{n+1} - \chi^n), v) + b_{NS}(\boldsymbol{u}; \chi^{n,\theta}, v) &= (f(C^{n,\theta}_{\mathrm{DG}}) - [f(c)]^{n,\theta}, v)\\ &+ \frac{1}{\Delta t}(\phi(\xi^{n+1} - \xi^n), v) + b_{NS}(\boldsymbol{u}; \xi^{n,\theta}, v) + \Delta t(\phi\rho_{n,\theta}, v).\end{aligned} \tag{5.6}$$

Choose $v = \chi^{n,\theta}$ and note that $(\phi(\chi^{n+1} - \chi^n), \chi^{n,\theta}) \geq K(\|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2)$:

$$\begin{aligned}\frac{K}{\Delta t}(\|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2) + b_{NS}(\boldsymbol{u}; \chi^{n,\theta}, \chi^{n,\theta}) &\leq (f(C^{n,\theta}_{\mathrm{DG}}) - [f(c)]^{n,\theta}, \chi^{n,\theta})\\ +\frac{1}{\Delta t}(\phi(\xi^{n+1} - \xi^n), \chi^{n,\theta}) &+ b_{NS}(\boldsymbol{u}; \xi^{n,\theta}, \chi^{n,\theta}) + \Delta t(\phi\rho_{n,\theta}, \chi^{n,\theta})\\ &\leq T_1 + \cdots + T_4.\end{aligned}$$

We bound T_1 by Cauchy-Schwarz inequality, the Lipschitz property of f and (2.6).

$$\begin{aligned}|T_1| &= |((f(C^{n,\theta}_{\mathrm{DG}}) - f(c^{n,\theta}), \chi^{n,\theta}) + (f(c^{n,\theta}) - [f(c)]^{n,\theta}, \chi^{n,\theta})|\\ &\leq K\|C^{n,\theta}_{\mathrm{DG}} - c^{n,\theta}\|_0\|\chi^{n,\theta}\|_0 + \|f(c^{n,\theta}) - [f(c)]^{n,\theta}\|_0\|\chi^{n,\theta}\|_0\\ &\leq K\|\chi^{n,\theta}\|_0^2 + K\|\xi^{n,\theta}\|_0^2 + K(1-\theta^2)^2\|c^{n+1} - c^n\|_0^2\\ &\leq K\|\chi^{n,\theta}\|_0^2 + K\frac{h^{2\mu}}{r^{2s-3}}(\|c^n\|_s^2 + \|c^{n+1}\|_s^2) + K(1-\theta^2)^2\Delta t^2 \sup_{t^n \leq t \leq t^{n+1}} \|c_t\|_0^2.\end{aligned}$$

T_2 can be bounded by Cauchy-Schwarz's inequality, by a Taylor expansion, and by the approximation result (2.6).

$$\begin{aligned} |T_2| &\leq K\frac{1}{\Delta t}\|\xi^{n+1}-\xi^n\|_0\|\chi^{n,\theta}\|_0 \\ &\leq K(\|\chi^n\|_0^2+\|\chi^{n+1}\|_0^2)+\frac{h^{2\mu-2}}{r^{2s-3}}\sup_{t^n\leq t\leq t^{n+1}}\|c_t\|_{s-1}^2. \end{aligned}$$

We rewrite T_3 as follows:

$$\begin{aligned} b_{NS}(\boldsymbol{u};\xi^{n,\theta},\chi^{n,\theta}) &= \sum_{E\in\mathcal{E}_h}\int_E D(\boldsymbol{u})\nabla\xi^{n,\theta}\cdot\nabla\chi^{n,\theta}-\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{u}\xi^{n,\theta}\cdot\nabla\chi^{n,\theta} \\ &-\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\xi^{n,\theta}\cdot\boldsymbol{n}_k\}[\chi^{n,\theta}]+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\{D(\boldsymbol{u})\nabla\chi^{n,\theta}\cdot\boldsymbol{n}_k\}[\xi^{n,\theta}] \\ &+\sum_{\gamma_k\in\Gamma}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi_*^{n,\theta}[\chi^{n,\theta}]+\sum_{\gamma_k\in\Gamma_{\text{out}}}\int_{\gamma_k}\boldsymbol{u}\cdot\boldsymbol{n}_k\xi^{n,\theta}\chi^{n,\theta}. \end{aligned} \tag{5.7}$$

Following the techniques used in the proof of the continuous in time error estimate, we can bound each term of the right-hand side of (5.7). The final bound is written below.

$$\begin{aligned} |T_3| &\leq \frac{1}{2}|\!|\!|D^{1/2}(\boldsymbol{u})\nabla\chi^{n,\theta}|\!|\!|_0^2+K\frac{h^{2\mu-2}}{r^{2s-5}}(\|c^{n+1}\|_s^2+\|c^n\|_s^2) \\ &+\frac{1}{4}\sum_{\gamma_k\in\Gamma}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}[\chi^{n,\theta}]\|_{0,\gamma_k}^2+\frac{1}{4}\sum_{\gamma_k\in\Gamma_{\text{out}}}\||\boldsymbol{u}\cdot\boldsymbol{n}_k|^{\frac{1}{2}}\chi^{n,\theta}\|_{0,\gamma_k}^2. \end{aligned}$$

We now easily bound T_4.

$$|T_4|\leq K\Delta t\|\rho_{n,\theta}\|_0\|\chi^{n,\theta}\|_0\leq K(\|\chi^{n+1}\|_0^2+\|\chi^n\|_0^2)+K\Delta t^2\|\rho_{n,\theta}\|_0^2.$$

Combining all the bounds above, multiplying by Δt and summing over $n=0,\dots,\tilde{n}$, with $\tilde{n}\leq N$, yields:

$$\begin{aligned} &\|\chi^{\tilde{n}+1}\|_0^2-\|\chi^0\|_0^2+K\Delta t\sum_{n=0}^{\tilde{n}}|\!|\!|D(\boldsymbol{u})^{1/2}\nabla\chi^{n,\theta}|\!|\!|_0^2\leq K\Delta t\sum_{n=0}^{\tilde{n}}\|\chi^n\|_0^2 \\ &\quad+K\Delta t^3\sum_{n=0}^{N}\|\rho_{n,\theta}\|_0^2+K\Delta t\frac{h^{2\mu-2}}{r^{2s-5}}\sum_{n=0}^{N}\|c^n\|_s^2+K\frac{h^{2\mu-2}}{r^{2s-3}}\sup_{0\leq t\leq T}\|c_t\|_{s-1}^2 \\ &\quad+K\Delta t^2(1-\theta^2)^2\sup_{0\leq t\leq T}\|c_t\|_0^2. \end{aligned}$$

If Δt is sufficiently small, we obtain by Gronwall's lemma:

$$\begin{aligned} &\|\chi^{\tilde{n}+1}\|_0^2+K\Delta t\sum_{n=0}^{\tilde{n}}|\!|\!|D(\boldsymbol{u})^{1/2}\nabla\chi^{n,\theta}|\!|\!|_0^2\leq\|\chi^0\|_0^2+K\Delta t^3\sum_{n=0}^{N}\|\rho_{n,\theta}\|_0^2 \\ &+K\frac{h^{2\mu-2}}{r^{2s-5}}\|c\|_{L^\infty(0,T;H^s(\mathcal{E}_h))}^2+K\frac{h^{2\mu-2}}{r^{2s-3}}\sup_{0\leq t\leq T}\|c_t\|_{s-1}^2+K\Delta t^2(1-\theta^2)^2\sup_{0\leq t\leq T}\|c_t\|_0^2. \end{aligned}$$

But,

$$\|\chi^0\|_0\leq\|C_{\text{DG}}(0)-c_0\|_0+\|c_0-\tilde{c}(0)\|\leq K\frac{h^\mu}{r^{s-3/2}}\|c_0\|_s.$$

Therefore, if $\theta > 0$, we can conclude:

$$\begin{aligned}\|\chi\|^2_{l^\infty(L^2(\Omega))} &+ K\Delta t \sum_{n=0}^{N} |\!|\!| D(\boldsymbol{u})^{1/2}\nabla\chi^{n,\theta} |\!|\!|_0^2 \\ &\le K\frac{h^{2\mu-2}}{r^{2s-5}}(\|c\|^2_{L^\infty(0,T;H^s(\mathcal{E}_h))} + \|c_t\|^2_{L^\infty(0,T;H^{s-1}(\mathcal{E}_h))}) \\ &\qquad + K\Delta t^2(\|c_t\|^2_{L^\infty(0,T;L^2(\Omega))} + \|c_{tt}\|^2_{L^\infty(0,T;H^1(\Omega))}).\end{aligned}$$

In the case where $\theta = 1$, then the scheme is of second order in time:

$$\begin{aligned}&\|\chi\|^2_{l^\infty(L^2(\Omega))} + K\Delta t \sum_{n=0}^{N} |\!|\!| D(\boldsymbol{u})^{1/2}\nabla\chi^{n,\theta} |\!|\!|_0^2 \\ &\le K\frac{h^{2\mu-2}}{r^{2s-5}}(\|c\|^2_{L^\infty(0,T;H^s(\mathcal{E}_h))} + \|c_t\|^2_{L^\infty(0,T;H^{s-1}(\mathcal{E}_h))}) + K\Delta t^4\|c_{ttt}\|^2_{L^\infty(0,T;H^1(\Omega))}.\end{aligned}$$

The final estimates are then obtained by the triangle inequality and the approximation properties. □

6. Conclusion

In this work, we have introduced and analyzed schemes for solving the transport problem, on nonconforming meshes. The analysis presented here holds for an approximation of degree at least two. In the case of piecewise constants and pure convection, the scheme reduces to the finite volume method and is known to converge. Numerical experiments show the convergence of the scheme for linears as wells. Optimal convergence rates for linears can be obtained if one adds penalty terms to the bilinear form.

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