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Discontinuous Galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity

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Summary. We consider a finite-element-in-space, and quadrature-in-time-discretization of a compressible linear quasistatic viscoelasticity problem. The spatial discretization uses a discontinous Galerkin finite element method based on polynomials of degree r—termed DG(r)—and the time discretization uses a trapezoidal-rectangle rule approximation to the Volterra (history) integral. Both semi- and fully-discrete *a priori* error estimates are derived without recourse to Gronwall's inequality, and therefore the error bounds do not show exponential growth in time. Moreover, the convergence rates are optimal in both h and r providing that the finite element space contains a globally continuous interpolant to the exact solution (e.g. when using the standard $\mathbb{P}^k$ polynomial basis on simplicies, or tensor product polynomials, $\mathbb{Q}^k$, on quadrilaterals). When this is not the case (e.g. using $\mathbb{P}^k$ on quadrilaterals) the convergence rate is suboptimal in r but remains optimal in h. We also consider a reduction of the problem to standard linear elasticity where similarly optimal *a priori* error estimates are derived for the DG(r) approximation.

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1 Introduction

The stress tensor $\underline{\boldsymbol{\sigma}} = (\sigma_{ij})_{ij=1}^n$ at time $t \in \mathcal{J} := [0, T]$ at a point $\boldsymbol{x} = (x_i)_{i=1}^n$ in a compressible linear viscoelastic body, the interior of which occupies an open bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$), is given by the following linear functional of the strain tensor $\underline{\boldsymbol{\varepsilon}} = (\varepsilon_{ij})_{ij=1}^n$:

(1)
$$\sigma_{ij}(\boldsymbol{u}(\boldsymbol{x},t);t) = D_{ijkl}(\boldsymbol{x},0)\varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x},t)) - \int_0^t \frac{\partial D_{ijkl}}{\partial s}(\boldsymbol{x},t-s)\varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x},s))\,ds,$$

see, for example, Ferry [2], Findley *et al.* [3], Golden and Graham [5] or Lockett [8]. Here: $\boldsymbol{u} = (u_i)_{i=1}^n$ is the pointwise displacement; repeated indices imply summation; and, the strain tensor components are given by,

(2)
$$\varepsilon_{ij}(\boldsymbol{u}) := \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Also, $\underline{\boldsymbol{D}} = (D_{ijkl})_{ijkl=1}^n$ is a fourth order tensor of stress relaxation functions satisfying the following symmetries at $t = 0$ and $t = \infty$:

(3)
$$D_{ijkl} = D_{jikl}, \qquad D_{ijkl} = D_{ijlk} \qquad \text{and} \qquad D_{ijkl} = D_{klij}.$$

In fact the first two of these hold for all $t \geqslant 0$ but the third holds for all t only for isotropic materials. See [8, Equations (1.10) and (2.62)]. It is natural to assume that $\underline{\boldsymbol{D}}$ is positive definite at $t = 0$ in the sense that,

(4)
$$\gamma_{ij}\gamma_{kl}D_{ijkl}(0) > 0,$$

for all non-zero symmetric second order tensors $\underline{\boldsymbol{\gamma}}$. In this inequality we have omitted the argument $\boldsymbol{x}$, and will continue to do so below unless it is explicitly required. We also assume that $\underline{\boldsymbol{D}}(0)$ is piecewise constant in Ω and that the finite element mesh (described below) respects the discontinuities by placing element edges along them. For regularity we assume, at least, that each component of $\underline{\boldsymbol{D}}$ lies in $W_1^1(\mathcal{J}; L_\infty(\Omega))$ although this will be strengthened for the error estimates.

Let the body be acted on by a system of body forces $\boldsymbol{f} : \Omega \times \mathcal{J} \to \mathbb{R}^n$ and surface tractions $\boldsymbol{g} : \Gamma_N \times \mathcal{J} \to \mathbb{R}^n$, where $\Gamma_N \subset \partial\Omega$. Suppose also that on the remainder of the boundary, $\Gamma_D := \partial\Omega \setminus \Gamma_N$, the body is rigidly fixed in space. We assume that Γ_D is of strictly positive ($(n-1)$-dimensional Lebesgue) measure.

Under the assumption of quasistatic conditions the inertia of the body is neglected and Newton's second law gives the boundary value problem: find $\boldsymbol{u} : \Omega \times \mathcal{J} \to \mathbb{R}^n$ such that,

(5)
$$-\sigma_{ij,j}(\boldsymbol{u}) = f_i \qquad \text{in } \Omega \times \mathcal{J},$$

(6)
$$\boldsymbol{u} = \boldsymbol{u}_\Gamma \qquad \text{on } \Gamma_D \times \mathcal{J},$$

(7)
$$\sigma_{ij}(\boldsymbol{u})\nu_j = g_i \qquad \text{on } \Gamma_N \times \mathcal{J}.$$

Here $\boldsymbol{\nu} = (\nu_j)_{j=1}^n$ is the unit outward normal to $\partial\Omega$ and, throughout, repeated indices imply summation and the comma-subscript denotes differentiation.

Define the product spaces $\boldsymbol{H}^s(\Omega) := H^s(\Omega)^n$ and for the case where $\boldsymbol{u}_\Gamma = \boldsymbol{0}$ let

$$V := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0} \right\}. \tag{8}$$

Then, using (1) in (5) and eliminating the strain with (2), we arrive at an example of the abstract Volterra equation of the second kind:

$$A\boldsymbol{u}(t) = L(t) + \int_0^t B(t,s)\boldsymbol{u}(s)\, ds, \tag{9}$$

where we assume from Korn's inequality (see for example Friedrichs [4] or Horgan [6]) that A is a (self-adjoint) V-elliptic operator, with $B(t,s)$ similar (see [18]). If each $D_{ijkl} \in W_1^1(\mathcal{J}; L_\infty(\Omega))$ as well as, for example, $\boldsymbol{f} \in L_\infty(\mathcal{J}; V')$ and $\boldsymbol{g} \in L_\infty(\mathcal{J}; \partial V')$ (where ∂V is an appropriate space of traces), then the existence and uniqueness of a solution $\boldsymbol{u} \in L_\infty(\mathcal{J}; V)$ follows from the Riesz representation theorem and the theory of Volterra equations (e.g. the Picard iteration, [7]).

This model has been used in various contexts by engineers to model the response of mechanically loaded structures which contain polymeric (i.e. viscoelastic) damping components. The tensor $\underline{\boldsymbol{D}}$ is then usually taken to be piecewise constant in Ω. The practical engineering argument for neglecting the inertia term is based on the observation that if the external loads $\boldsymbol{f}$ and $\boldsymbol{g}$ are significant (and not oscillating near a natural frequency) then the internal dynamics of the material can be neglected.

It is our aim in this article to study a discontinuous-in-space Galerkin finite element approximation of this problem. In this we are building on earlier work by Shaw, Warby, Whiteman, Dawson and Wheeler [16] who discretized using "continuous Galerkin" in space and quadrature in time. This scheme was revisited in [15] in order to remove the large "Gronwall constant" in the error bounds.

The work in this paper is distinct in that it uses discontinuous piecewise polynomials of degree r in space (DG(r)). The time discretization is carried out by employing an explicit quadrature approximation to the history integral in (1). By "explicit" we mean that the operator $\int_0^{t_q} \cdot\, ds$ is replaced by a quadrature rule of the form $\Sigma_{p=0}^q \cdot\, \varpi_{qp}$, where the weights are chosen so that $\varpi_{qq} = 0$. Therefore the discrete scheme does not involve the current solution in the history term. Apart from being computationally simpler to implement (a straightforward modification of linear elasticity software) this also makes the error analysis easier (compare the results in [15]) and, in principle, allows for the inclusion of a *reduced time* constitutive nonlinearity in the history integral (see e.g. [14]) without creating any undue computational difficulties (i.e. a nonlinear system).

Rivière et al. in [12, 10, 11] have used DG(r) for spatial discretization for single phase flow problems in heterogeneous porous media. They introduced and analyzed the method, referred to as the Non-symmetric Interior Penalty Galerkin (NIPG) method, in the case of elliptic and parabolic problems. The NIPG methods are derived from the Interior Penalty method introduced by Wheeler in the seventies, [19], and the Discontinuous Galerkin methods introduced by Baumann, Oden and Babuska [9]. In the present work, we extend the NIPG methods to elasticity and viscoelasticity.

In this paper we show optimal bounds for the fully discrete scheme if we discretize using simplex elements and, moreover, as $h \to 0$ we can bypass the Gronwall inequality to obtain error constants uniformly bounded in time. It is not clear whether similarly sharp constants can be achieved for other types of element, but Gronwall's lemma can be invoked in these cases to demonstrate convergence. In any case, the error bounds are only optimal in the polynomial degree r for simplicies—see Remark 3.3. In our estimates below we consider linear elasticity discretized using general finite elements (Theorem 3.5) but restrict our attention to simplicial finite elements for viscoelasticity (Theorems 4.5, 5.5).

More recently Shaw and Whiteman in [17] have used a space-time finite element approximation with trial functions that are continuous piecewise linear in space and discontinuous piecewise constant or linear in time. The ultimate aim is to provide *a posteriori* error estimates and an adaptive solver. However, in all that follows the space mesh is time independent; we plan to consider adaptivity for the DG scheme at a later time.

The plan of the paper is as follows. In the next section we describe the weak formulation of the problem and then follow it in Section 3 with an analysis of a discrete formulation of the standard linear elasticity problem. Note that, as the elasticity problem does not contain time, this is an analysis in space only. This is a special case of the viscoelasticity problem described above. In Section 4 we formulate a semidiscrete approximation to the viscoelasticity problem and this leads on to an *a priori* error bound for the semidiscrete solution. This is followed by Section 5 where we describe the fully discrete scheme and derive an *a priori* error bound for the fully discrete solution. Numerical results are currently under development and these will be described elsewhere.

Finally in this section note that we can also write (1) as,

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}(t); t) = \underline{\boldsymbol{\sigma}}^{el}(\boldsymbol{u}(t)) - \underline{\boldsymbol{\sigma}}^{vs}(\boldsymbol{u}; t, t), \tag{10}$$

where the elastic and viscous stresses are defined as:

$$\sigma_{ij}^{el}(\boldsymbol{u}(t)) := D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{u}(t)), \tag{11}$$

$$\sigma_{ij}^{vs}(\boldsymbol{u}; \tau, t) := \int_0^{\tau} \frac{\partial D_{ijkl}}{\partial s}(t-s)\varepsilon_{kl}(\boldsymbol{u}(s))\, ds. \tag{12}$$

We also make use of the well known inequality,

$$ab \leqslant \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon} \qquad \forall a, b \in \mathbb{R} \text{ and } \forall \epsilon > 0. \tag{13}$$

2 Weak formulation and the DG(r) finite element space

The first step is to establish notation for the spatial discretization. Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω, where E_j is a triangle or a quadrilateral if $n = 2$, or a tetrahedron if $n = 3$. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \mathrm{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. Let $h = \max\{h_j : 1 \leqslant j \leqslant N_h\}$, the quasiuniformity requirement is that there exists $\tau > 0$ such that $h/h_j \leqslant \tau$ for all $j \in \{1, \ldots, N_h\}$. We denote the edges (faces for $n = 3$) of $\mathcal{E}_h$ by $\{e_1, e_2, \ldots, e_{P_h}, e_{P_{h+1}}, \ldots, e_{M_h}\}$ where $e_a \subset \Omega$ for $1 \leqslant a \leqslant P_h$, and $e_a \subset \partial\Omega$ for $P_h + 1 \leqslant a \leqslant M_h$. With each edge (or face) e_a, we associate a unit normal vector $\boldsymbol{\nu}^a$. For $a > P_h$, $\boldsymbol{\nu}^a$ is taken to be the unit outward vector normal to $\partial\Omega$.

For real $s \geqslant 0$ define,

$$\mathcal{H}^s(\mathcal{E}_h) := \{v \in L^2(\Omega) : v|_{E_j} \in H^{s+\epsilon}(E_j)\ \forall E_j \in \mathcal{E}_h \text{ and some } \epsilon > 0\},$$
$$\boldsymbol{\mathcal{H}}^s(\mathcal{E}_h) := \{\boldsymbol{v} := (v_i)_{i=1}^n \in \boldsymbol{L}^2(\Omega) : v_i \in \mathcal{H}^s(\mathcal{E}_h),\ i = 1, \ldots, n\}.$$

Note the use of ϵ in the first definition. This is to guarantee that certain traces exist on the element edges e_a. The notation is designed to avoid clumsy expressions below and so, to keep things simple, when $s = 0$ we set $\mathcal{H}^0(\mathcal{E}_h) := H^0(\Omega) \equiv L_2(\Omega)$.

We now define the average and the jump for $\boldsymbol{w} \in \boldsymbol{\mathcal{H}}^s(\mathcal{E}_h)$ when $s \geqslant \frac{1}{2}$. For each of the interior edges $\{e_a\}_{a=1}^{P_h}$ suppose the neighbouring elements of e_a are E_a^1 and E_a^2 so that $e_a = \partial E_a^1 \cap \partial E_a^2$, and for a boundary edge suppose that E_a is the neighbouring element. We define the averaging operator $\{\cdot\}$ by,

$$\{\boldsymbol{w}\} := \begin{cases} \frac{1}{2}(\boldsymbol{w}|_{E_a^1})|_{e_a} + \frac{1}{2}(\boldsymbol{w}|_{E_a^2})|_{e_a} & \text{if } e_a \subset \Omega, \\ (\boldsymbol{w}|_{E_a})|_{e_a} & \text{if } e_a \subset \partial\Omega. \end{cases}$$

and the jump operator $[\cdot]$ by,

$$[\boldsymbol{w}] := \begin{cases} (\boldsymbol{w}|_{E_a^1})|_{e_a} - (\boldsymbol{w}|_{E_a^2})|_{e_a} & \text{if } e_a \subset \Omega, \\ (\boldsymbol{w}|_{E_a})|_{e_a} & \text{if } e_a \subset \partial\Omega. \end{cases}$$

The distinction between $[\cdot]$ and $-[\cdot]$ can be made because each edge e_a has a unit normal $\boldsymbol{\nu}^a$ associated with it. The "direction" in which the jump takes place is unimportant.

The usual Sobolev norm of $\boldsymbol{H}^m$ on $E \subset \mathbb{R}^n$ is denoted by $\|\cdot\|_{m,E}$. We define the following broken norms for m a positive integer:

$$|||w|||_m := \left(\sum_{j=1}^{N_h} \|w\|_{m,E_j}^2\right)^{\frac{1}{2}} \qquad \forall w \in \mathcal{H}^m(\mathcal{E}_h),$$

$$|||\boldsymbol{w}|||_m := \left(\sum_{i=1}^{n} |||w_i|||_m^2\right)^{\frac{1}{2}} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{H}}^m(\mathcal{E}_h).$$

Let r be a positive integer. The finite element subspace is taken to be,

$$\boldsymbol{\mathcal{D}}_r(\mathcal{E}_h) = \{\boldsymbol{v} : \boldsymbol{v}|_{E_j} \in (\mathbb{P}_r(E_j))^n \quad \forall j = 1, \ldots, N_h\},$$

where $\mathbb{P}_r(E_j)$ denotes the set of polynomials of (total) degree less than or equal to r on E_j.

Following Rivière *et al.* in [12] we assume the following hp approximation properties proved by Babuška and Suri in [1]. For every $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$ there exists a constant C, depending on s, τ, ρ but independent of ϕ, r, h, and a sequence $\{z_r^h\}_{r\geqslant 1}$ with each $z_r^h \in \mathbb{P}_r(E_j)$ such that, for any $0 \leqslant q \leqslant s$,

$$\|\phi - z_r^h\|_{q,E_j} \leqslant C\frac{h_j^{\mu-q}}{r^{s-q}}\|\phi\|_{s,E_j}, \qquad s \geqslant 0, \tag{14}$$

$$\|\phi - z_r^h\|_{0,\gamma_i} \leqslant C\frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}}\|\phi\|_{s,E_j}, \qquad s > \frac{1}{2}, \tag{15}$$

$$\|\phi - z_r^h\|_{1,\gamma_i} \leqslant C\frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}}\|\phi\|_{s,E_j}, \qquad s > \frac{3}{2}, \tag{16}$$

where $\mu = \min\{r+1, s\}$ and $\gamma_i \subset \partial E_j$. Clearly these can be extended to vector-valued functions also and, by summing (14), we obtain a global estimate: let $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}^s(\Omega)$ $(s \geqslant 0)$, then there exists $\boldsymbol{z}_r^h \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ such that,

$$|||\boldsymbol{\phi} - \boldsymbol{z}_r^h|||_q \leqslant C\frac{h^{\mu-q}}{r^{s-q}}|||\boldsymbol{\phi}|||_s, \qquad \text{for } 0 \leqslant q \leqslant s, \tag{17}$$

with μ as above and C independent of ϕ, r, h and $\mathcal{E}_h$. We also assume that for every edge (face in 3D) the following inverse estimate holds:

$$\|\underline{\boldsymbol{D}}^{\frac{1}{2}}(0)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|_{0,e_a} \leqslant C_0 h^{-\frac{1}{2}} r \|\underline{\boldsymbol{D}}^{\frac{1}{2}}(0)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|_{0,E} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h), \tag{18}$$

where e_a is an edge/face of the element E and C_0 is a positive constant independent of h and r and the normed quantity. Note that we are using our assumption that $\underline{\boldsymbol{D}}$ is piecewise constant here. Inequalities of this type are given, for example, by Schwab in [13, Theorem 4.76] and by Oden *et al.* in [9, Eq. (17)].

In general no element $\boldsymbol{u}$ of any $\mathcal{H}^s$ $(s \geqslant 0)$ can satisfy the equations (5)—(7) in the strong sense and so we now work toward a weak formulation that is suited to a DG(r) spatial discretization. In setting up this weak formulation we realize that we need to enhance the continuity of the discrete discontinuous solution across element edges (or faces) in order to be able to obtain error estimates. This is achieved by imposing a penalty term on each edge. This term is (see Wheeler, [19]),

$$
\begin{aligned}
J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{w}) = &\sum_{a=1}^{P_h} \frac{\delta_a r^2}{|e_a|^\beta} \int_{e_a} [\boldsymbol{v}]\cdot[\boldsymbol{w}]\,d\ell \\
&+ \sum_{e_a\in\Gamma_D} \frac{\delta_a r^2}{|e_a|^\beta} \int_{e_a} \boldsymbol{v}\cdot\boldsymbol{w}\,d\ell \qquad \forall \boldsymbol{w},\boldsymbol{v}\in \mathcal{H}^{\frac{1}{2}}(\mathcal{E}_h)
\end{aligned} \tag{19}
$$

where δ is a discrete positive function that takes the constant value δ_a on the edge or face e_a, $|e_a|$ denotes the measure of e_a and $\beta \geqslant (n-1)^{-1}$ is a real number. Motivated by the splitting (10) and the definitions (11) and (12), we now define the bilinear forms $A, B : \mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h)\times\mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h)\to\mathbb{R}$:

$$
A(\boldsymbol{w},\boldsymbol{v}) := \mathcal{A}(\boldsymbol{w},\boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{w},\boldsymbol{v}), \tag{20}
$$

$$
B(t;\boldsymbol{w},\boldsymbol{v}) := \mathcal{B}_1(t;\boldsymbol{w},\boldsymbol{v}) + \mathcal{B}_2(t;\boldsymbol{w},\boldsymbol{v}) \quad \text{or} \quad \mathcal{B}_1(t;\boldsymbol{w},\boldsymbol{v}). \tag{21}
$$

(Note that we have a choice for the second.) Here (for each given $t\in\mathcal{J}$) the bilinear forms $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 : \mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h)\times\mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h)\to\mathbb{R}$ are defined by:

$$
\begin{aligned}
\mathcal{A}(\boldsymbol{w},\boldsymbol{v}) := &\sum_{E\in\mathcal{E}_h}\int_E D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{w})\varepsilon_{ij}(\boldsymbol{v})\,dE \\
&- \sum_{a=1}^{P_h}\int_{e_a}\left\{D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{w})\nu_j^a\right\}[v_i]\,d\ell \\
&+ \sum_{a=1}^{P_h}\int_{e_a}\left\{D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{v})\nu_j^a\right\}[w_i]\,d\ell \\
&- \sum_{e_a\in\Gamma_D}\int_{e_a} D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{w})\nu_j^a v_i\,d\ell
\end{aligned}
$$

$$+\sum_{e_a\in\Gamma_D}\int_{e_a} D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{v})\nu_j^a w_i\,d\ell, \tag{22}$$

$$\mathcal{B}_1(t;\boldsymbol{w},\boldsymbol{v}) := \sum_{E\in\mathcal{E}_h}\int_E \sigma_{ij}^{vs}(t,t;\boldsymbol{w})\varepsilon_{ij}(\boldsymbol{v})\,dE - \sum_{a=1}^{P_h}\int_{e_a}\{\sigma_{ij}^{vs}(t,t;\boldsymbol{w})\nu_j^a\}[v_i]\,d\ell$$

$$-\sum_{e_a\in\Gamma_D}\int_{e_a}\sigma_{ij}^{vs}(t,t;\boldsymbol{w})\nu_j^a v_i\,d\ell, \tag{23}$$

$$\mathcal{B}_2(t;\boldsymbol{w},\boldsymbol{v}) := \sum_{a=1}^{P_h}\int_{e_a}\{\sigma_{ij}^{vs}(t,t;\boldsymbol{v})\nu_j^a\}[w_i]\,d\ell$$

$$+\sum_{e_a\in\Gamma_D}\int_{e_a}\sigma_{ij}^{vs}(t,t;\boldsymbol{v})\nu_j^a w_i\,d\ell, \tag{24}$$

where $\boldsymbol{\nu}^a$ is the unit normal vector associated to the edge e_a. The form $\mathcal{B}_1+\mathcal{B}_2$ will be used in the error analysis and so, in effect, we have two choices for the bilinear form B that can be used in computational implementations. We also define the following seminorm and norm,

$$|\boldsymbol{v}|_{\mathcal{A}} := (\mathcal{A}(\boldsymbol{v},\boldsymbol{v}))^{\frac{1}{2}}, \qquad |||\boldsymbol{v}|||_H := (A(\boldsymbol{v},\boldsymbol{v}))^{\frac{1}{2}} \qquad \forall \boldsymbol{v}\in\mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h). \tag{25}$$

Using (10), we can now give a weak formulation of the problem in the case $\boldsymbol{u}_\Gamma = \boldsymbol{0}$.

Lemma 2.1 (weak formulation) *In* (6) *take* $\boldsymbol{u}_\Gamma = \boldsymbol{0}$. *A weak solution* $\boldsymbol{u} : \mathcal{J} \to \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega)$ (*for some* $\epsilon > 0$) *to the problem* (5)–(7) *with* (1) *and* (2) *also satisfies,*

$$A(\boldsymbol{u}(t),\boldsymbol{v}) = L(t;\boldsymbol{v}) + B(t;\boldsymbol{u},\boldsymbol{v}), \qquad \forall \boldsymbol{v}\in\mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h), \tag{26}$$

Here $L(t;\boldsymbol{v})$ *is* (*a.e. in* $\mathcal{J}$) *a linear form which for sufficiently regular* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *is defined by,*

$$L(t;\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\Omega + \oint_{\Gamma_N}\boldsymbol{g}\cdot\boldsymbol{v}\,d\Gamma.$$

In the case of linear elasticity, we clearly have $B = 0$.

Proof. We take the $\boldsymbol{L}_2(E)$ scalar product of (5) with an arbitrary $\boldsymbol{v}|_E \in \boldsymbol{H}^{\frac{3}{2}}(E)$, for some $E\in\mathcal{E}_h$, and formally integrate by parts. This gives (dropping the time dependence for clarity),

$$\int_E \boldsymbol{f}\cdot\boldsymbol{v}\,dE = -\int_E \sigma_{ij,j}v_i\,dE = \int_E \sigma_{ij}v_{i,j}\,dE - \oint_{\partial E}\sigma_{ij}n_j^E v_i\,d\Gamma.$$

Using the symmetry of $\underline{\boldsymbol{\sigma}}(\boldsymbol{u})$ (and exploiting the summation convention) we get $\sigma_{ij}(\boldsymbol{u})v_{i,j} \equiv \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})$, and so summing over all $E \in \mathcal{E}_h$ gives,

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega = \sum_{E\in\mathcal{E}_h} \int_E \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dE - \sum_{E\in\mathcal{E}_h} \oint_{\partial E} \sigma_{ij}(\boldsymbol{u}) n_j^E v_i\, dE.$$

Splitting up the ∂E integrals then gives,

$$\sum_{E\in\mathcal{E}_h} \left(\int_E \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dE - \oint_{\partial E\cap\Omega} \sigma_{ij}(\boldsymbol{u}) n_j^E v_i\, dE - \oint_{\partial E\cap\Gamma_D} \sigma_{ij}(\boldsymbol{u}) n_j^E v_i\, dE \right)$$
$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma,$$

Now recombining the $\partial E \cap \Omega$ boundary integral summations and using the fact that $\boldsymbol{u} \in \boldsymbol{H}^{\frac{3}{2}+\epsilon}(E)$, we have,

$$\sum_{E\in\mathcal{E}_h} \int_E \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dE - \sum_{a=1}^{P_h} \int_{e_a} \{\sigma_{ij}(\boldsymbol{u})\nu_j^a\}[v_i]\, d\ell - \sum_{e_a\in\Gamma_D} \int_{e_a} \sigma_{ij}(\boldsymbol{u})\nu_j^a v_i\, d\ell$$
$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma,$$

Using (10), we obtain,

$$\sum_{E\in\mathcal{E}_h} \int_E \sigma_{ij}^{el}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dE - \sum_{a=1}^{P_h} \int_{e_a} \{\sigma_{ij}^{el}(\boldsymbol{u})\nu_j^a\}[v_i]\, d\ell - \sum_{e_a\in\Gamma_D} \int_{e_a} \sigma_{ij}^{el}(\boldsymbol{u})\nu_j^a v_i\, d\ell$$
$$= \sum_{E\in\mathcal{E}_h} \int_E \sigma_{ij}^{vs}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\, dE - \sum_{a=1}^{P_h} \int_{e_a} \{\sigma_{ij}^{vs}(\boldsymbol{u})\nu_j^a\}[v_i]\, d\ell$$
$$- \sum_{e_a\in\Gamma_D} \int_{e_a} \sigma_{ij}^{vs}(\boldsymbol{u})\nu_j^a v_i\, d\ell + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma.$$

We now add zero-valued terms to the left of this to get,

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{B}_1(t; \boldsymbol{u}, \boldsymbol{v}) + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma$$

and, since $\boldsymbol{u}$ is continuous, we can add zero-valued penalty terms,

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{B}_1(t; \boldsymbol{u}, \boldsymbol{v}) + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\, d\Gamma,$$

Since in this case $\mathcal{B}_2(t; \boldsymbol{u}, \boldsymbol{v})$ is zero, we could add it to the above equation. Thus for either of our two definitions of B we have,

$$A(\boldsymbol{u}(t), \boldsymbol{v}) = B(t; \boldsymbol{u}, \boldsymbol{v}) + \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\, d\Omega + \oint_{\Gamma_N} \boldsymbol{g}(t) \cdot \boldsymbol{v}\, d\Gamma,$$

as required by the lemma. □

Remark 2.2 We can deal with the case $\boldsymbol{u}_\Gamma \neq \boldsymbol{0}$ by adding

$$\sum_{e_a \in \Gamma_D} \int_{e_a} \sigma_{ij}(\boldsymbol{v}) \nu_j^a (\boldsymbol{u}_\Gamma)_i + \sum_{e_a \in \Gamma_D} \frac{\delta_a r^2}{|e_a|^\beta} \int_{e_a} \boldsymbol{v} \cdot \boldsymbol{u}_\Gamma\, d\ell$$

to $L(t; \boldsymbol{v})$. The Dirichlet condition will then be imposed weakly along Γ_D. The analysis derived in the rest of the paper remains the same (provided $\boldsymbol{u}_\Gamma$ can be imposed using the trial functions—otherwise we need interpolation error estimates).

Note that one advantage of this DG(r) scheme is that equilibrium is satisfied in a weak sense for each element $E \in \mathcal{E}_h$.

Lemma 2.3 *Equilibrium is satisfied weakly on each element in that,*

$$\int_E f_i(t)\, dE - \oint_{\partial E} \{\sigma_{ij}(\boldsymbol{u}; t) n_j^E\}\, d\Gamma = 0 \qquad \textit{for } i = 1, \ldots, n$$

on each $E \in \mathcal{E}_h$, *and where* $\boldsymbol{n}^E$ *is the unit outward normal to* ∂E.

Proof. In Lemma 2.1 choose $v_i = 1$ on E with the other components zero and $\boldsymbol{v} = \boldsymbol{0}$ on $\Omega \setminus E$. Now collect the remaining boundary integrals together in a manner consistent with how they were originally split up. □

This property is carried through in an approximate sense to the discrete scheme (see below in Lemma 3.2).

In the next section we reduce the viscoelasticity problem to the standard linear elasticity problem by removing all time dependencies—thereby setting $B(t; \cdot, \cdot) := 0$. We give the discrete DG(r) scheme for the linear elasticity problem and prove an *a priori* error estimate.

3 *A priori* error bound for linear elasticity

In this section we simplify the viscoelasticity problem somewhat by taking $\boldsymbol{f}$, $\boldsymbol{g}$ and $\underline{\boldsymbol{D}}$ to be time independent. The problem then becomes the standard linear elasticity problem (which is not time dependent). We prove an *a priori* error bound for the resulting DG(r) discretization.

In place of Lemma 2.1 we then have: find $\boldsymbol{u} \in \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega)$, for some $\epsilon > 0$, such that,

$$A(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h). \tag{27}$$

The DG(r) approximation to this problem is: find $\boldsymbol{U}_{el}^{DG} \in \mathcal{D}_r(\mathcal{E}_h)$ such that,

$$A(\boldsymbol{U}_{el}^{DG}, \boldsymbol{v}) = L(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathcal{D}_r(\mathcal{E}_h). \tag{28}$$

Since $\mathcal{D}_r(\mathcal{E}_h) \subset \mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h)$ the Galerkin orthogonality property is immediate:

$$A(\boldsymbol{u} - \boldsymbol{U}_{el}^{DG}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \mathcal{D}_r(\mathcal{E}_h). \tag{29}$$

In this section we again assume that $\boldsymbol{u}_\Gamma = \boldsymbol{0}$.

Lemma 3.1 *The solution $\boldsymbol{U}_{el}^{DG}$ of* (28) *exists and is unique.*

Proof. Since (28) is a finite dimensional problem we need only prove uniqueness. In (28) set $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$ and take $\boldsymbol{v} = \boldsymbol{U}_{el}^{DG}$. This implies that for each $E \in \mathcal{E}_h$,

$$\varepsilon_{ij}(\boldsymbol{U}_{el}^{DG}) = 0 \quad \forall i, j, \quad \text{and} \quad J_0^{\delta,\beta}(\boldsymbol{U}_{el}^{DG}, \boldsymbol{U}_{el}^{DG}) = 0.$$

The first of these means that the displacement $\boldsymbol{U}_{el}^{DG}$ is a rigid body motion: $\boldsymbol{U}_{el}^{DG} = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}$, for some constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The interior penalty term forces the continuity of $\boldsymbol{U}_{el}^{DG}$, and, with our assumption that $|\Gamma_D| > 0$, the boundary penalty ensures that $\boldsymbol{U}_{el}^{DG} = \boldsymbol{0}$. □

For the discrete scheme equilibrium is approximately satisfied in a weak sense for each element $E \in \mathcal{E}_h$.

Lemma 3.2 *Equilibrium is approximately satisfied weakly on each element in that,*

$$\begin{aligned}\int_E f_i \, dE &+ \oint_{\partial E} \{\sigma_{ij}(\boldsymbol{U}_{el}^{DG}) n_j^E\} d\Gamma \\ &= \sum_{e_a \in \partial E} \frac{\delta_a r^2}{|e_a|} \int_{e_a} [(\boldsymbol{U}_{el}^{DG})_i] d\ell \qquad \forall i = 1, \dots, n, \forall E \in \mathcal{E}_h\end{aligned}$$

Proof. The proof is analogous to that of Lemma 2.3 but now the penalty terms survive. □

Note that the error in equilibrium is *computable* in terms of the discrete solution and the penalty.

The *a priori* error estimate for the scheme will follow from the next lemma. We give the lemma separately because it will also be important later in the error analysis of the viscoelasticity problem. The lemma itself refers to whether or not we can construct a continuous interpolant, the background to this is explained in the following remark.

Remark 3.3 As discussed by Rivière *et al.* in [12], DG implementations are free to use only elementwise polynomials from $\mathbb{P}^r$ rather than $\mathbb{Q}^r$. If this is the case then by discretizing using simplicial elements (triangles/tetrahedra) we can build a continuous interpolant to $\boldsymbol{u}$ in the finite element space $\boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$. However, if the elements are not simplices (quadrilaterals/bricks/prisms) then, in general, a continuous interpolant cannot be found. This impacts only on the r-convergence rate: the h-convergence rate is optimal in either case.

Lemma 3.4 *For $\Omega \subset \mathbb{R}^n$, assume that for each edge e_a we have $|e_a| \leqslant C_e h^{n-1}$ for a constant $C_e > 0$. Then, under the assumptions of Lemma 2.1, assuming that there is a continuous interpolant $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\bar{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ of $\boldsymbol{u}$, and if $\boldsymbol{u} \in \mathcal{H}^m(\mathcal{E}_h) \cap \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega)$ for some $\epsilon > 0$, and $\beta \geqslant (n-1)^{-1}$ we have:*

$$|A(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{v})| \leqslant C \frac{h^{\mu-1}}{r^{m-1}} |\!|\!| \boldsymbol{u} |\!|\!|_m |\!|\!| \boldsymbol{v} |\!|\!|_H \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h),$$

where $\mu = \min\{r+1, m\}$, $r \geqslant 1$ and $m \geqslant 2$. On the other hand, if $\tilde{\boldsymbol{u}} \notin \boldsymbol{C}(\bar{\Omega})$ then with $\beta = (n-1)^{-1}$:

$$|A(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{v})| \leqslant C \frac{h^{\mu-1}}{r^{m-3/2}} |\!|\!| \boldsymbol{u} |\!|\!|_m |\!|\!| \boldsymbol{v} |\!|\!|_H \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h).$$

In the first estimate the constant $C > 0$ depends upon C_e, the penalties $\{\delta_a\}$, the constants in (14) *and* (16)*, and the tensor $\underline{\boldsymbol{D}}$. In the second estimate the constant depends also on* (15) *and* (18).

Proof. As usual C will be a generic constant that is independent of h and r. Let $\tilde{\boldsymbol{u}} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ be an interpolant of $\boldsymbol{u}$ having optimal hp-approximation errors. Dropping the time dependence (so as to model linear elasticity) we set $D_{ijkl} := D_{ijkl}(0)$, for notational convenience, and then we have for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ that,

$$A(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{v}) \tag{30}$$

$$= \sum_{E \in \mathcal{E}_h} \int_E D_{ijkl} \varepsilon_{kl}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \varepsilon_{ij}(\boldsymbol{v}) dE \tag{31}$$

$$- \sum_{a=1}^{P_h} \int_{e_a} \left\{ D_{ijkl} \varepsilon_{kl}(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \nu_j^a \right\} [v_i] \, d\ell$$

$$+ \sum_{a=1}^{P_h} \int_{e_a} \left\{ D_{ijkl} \varepsilon_{kl}(\boldsymbol{v}) \nu_j^a \right\} [u_i - \tilde{u}_i] \, d\ell \tag{32}$$

$$
\begin{aligned}
&- \sum_{e_a \in \Gamma_D} \int_{e_a} D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a v_i \, d\ell \\
&+ \sum_{e_a \in \Gamma_D} \int_{e_a} D_{ijkl}\varepsilon_{kl}(\boldsymbol{v})\nu_j^a (u_i - \tilde{u}_i) \, d\ell \qquad (33) \\
&+ \sum_{a=1}^{P_h} \frac{\delta_a r^2}{|e_a|^\beta} \int_{e_a} [\boldsymbol{u}-\tilde{\boldsymbol{u}}] \cdot [\boldsymbol{v}] \, d\ell \\
&+ \sum_{e_a \in \Gamma_D} \frac{\delta_a r^2}{|e_a|^\beta} \int_{e_a} (\boldsymbol{u}-\tilde{\boldsymbol{u}}) \cdot \boldsymbol{v} \, d\ell. \qquad (34)
\end{aligned}
$$

The term in (31) can be bounded by using Cauchy-Schwarz inequality, (13) and the global approximation result in (17).

$$
\begin{aligned}
&\left| \sum_{E \in \mathcal{E}_h} \int_E D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\varepsilon_{ij}(\boldsymbol{v}) \, dE \right| \\
&\quad \leqslant \sum_{E \in \mathcal{E}_h} \left(\int_E D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\varepsilon_{ij}(\boldsymbol{u}-\tilde{\boldsymbol{u}}) \, dE \right)^{\frac{1}{2}} \left(\int_E D_{ijkl}\varepsilon_{kl}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) \, dE \right)^{\frac{1}{2}} \\
&\quad \leqslant C \frac{h^{\mu-1}}{r^{m-1}} |||\boldsymbol{u}|||_m |\boldsymbol{v}|_{\mathcal{A}}.
\end{aligned}
$$

To bound the first term in (32) we apply the Cauchy-Schwarz inequality and “multiply by one” to get,

$$
\begin{aligned}
&\left| \int_{e_a} \left\{ D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a \right\} [v_i] \, d\ell \right| \\
&\quad \leqslant \left(\frac{|e_a|^\beta}{\delta_a r^2} \right)^{\frac{1}{2}} \|\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a\}\|_{0,e_a} \left(\frac{\delta_a r^2}{|e_a|^\beta} \right)^{\frac{1}{2}} \|[v_i]\|_{0,e_a}.
\end{aligned}
$$

Summing over all internal edges then gives,

$$
\begin{aligned}
&\left| \sum_{a=1}^{P_h} \int_{e_a} \left\{ D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a \right\} [v_i] \, d\ell \right| \\
&\quad \leqslant \left(\sum_{a=1}^{P_h} \left(\frac{|e_a|^\beta}{\delta_a r^2} \right) \|\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a\}\|_{0,e_a}^2 \right)^{\frac{1}{2}} \left(\sum_{a=1}^{P_h} \left(\frac{\delta_a r^2}{|e_a|^\beta} \right) \|[v_i]\|_{0,e_a}^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

Using the triangle inequality on the averaging operator, $\{\cdot\}$, the approximation result in (16) and noting that $1 \leqslant 1/r^{-\frac{1}{2}}$ we get,

$$
\left(\sum_{a=1}^{P_h} \frac{|e_a|^\beta}{\delta_a r^2} \|\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a\}\|_{0,e_a}^2 \right)^{\frac{1}{2}} \leqslant C_i \frac{h^{\mu-3/2+\beta(n-1)/2}}{r^{m-1}} |||\boldsymbol{u}|||_m.
$$

Thus,

$$\left|\sum_{a=1}^{P_h}\int_{e_a}\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\nu_j^a\}[v_i]\,d\ell\right| \leqslant C\frac{h^{\mu-3/2+\beta(n-1)/2}}{r^{m-1}}|||\boldsymbol{u}|||_m\; J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}},$$

and the first term in (33) is bounded in the same way. In the case of triangles or tetrahedra we can choose $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\bar{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ and this gives,

$$[\boldsymbol{u}-\tilde{\boldsymbol{u}}] = \boldsymbol{0}$$

on every edge in the mesh. Hence the second terms in (32) and (33), along with both terms in (34) are zero (recall that it has been assumed that $\tilde{\boldsymbol{u}}|_{\Gamma_D} = \boldsymbol{u}_\Gamma = \boldsymbol{0}$).

Since $|||\boldsymbol{v}|||_H^2 = |\boldsymbol{v}|_{\mathcal{A}}^2 + J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})$ this proves the first estimate.

Now, in the general case where $\tilde{\boldsymbol{u}} \notin \boldsymbol{C}(\bar{\Omega})$ we bound the second term in (32) by considering the contribution from each interior edge. We assume that $e_a = \partial E_a^1 \cap \partial E_a^2$, where E_a^1 and E_a^2 are elements of $\mathcal{E}_h$ and denote $E_a^{12} = E_a^1 \cup E_a^2$. Then by the Cauchy-Schwarz and triangle inequalities, (15) and (18) we have,

$$\begin{aligned}
&\left|\int_{e_a}\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{v})\nu_j^a\}[u_i-\tilde{u}_i]\,d\ell\right| \\
&\quad\leqslant \|\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{v})\nu_j^a\}\|_{0,e_a}\|[u_i-\tilde{u}_i]\|_{0,e_a}, \\
&\quad\leqslant \left(C_0 r h^{-\frac{1}{2}}\|\underline{\boldsymbol{D}}^{\frac{1}{2}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|_{0,E_a^{12}}\right)\left(C(\underline{\boldsymbol{D}})\frac{h^{\mu-\frac{1}{2}}}{r^{m-\frac{1}{2}}}\|\boldsymbol{u}\|_{m,E_a^{12}}\right),
\end{aligned}$$

and so summing over all the edges yields:

$$\left|\sum_{a=1}^{P_h}\int_{e_a}\{D_{ijkl}\varepsilon_{kl}(\boldsymbol{v})\nu_j^a\}[u_i-\tilde{u}_i]\,d\ell\right| \leqslant C\frac{h^{\mu-1}}{r^{m-3/2}}|||\boldsymbol{u}|||_m|\boldsymbol{v}|_{\mathcal{A}}. \tag{35}$$

The terms in (33) are handled in exactly the same way as those in (32).

The penalty terms in (34) are bounded using (15) as follows,

$$\begin{aligned}
&\left|\sum_{a=1}^{P_h}\frac{\delta_a r^2}{|e_a|^\beta}\int_{e_a}[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\cdot[\boldsymbol{v}]\,d\ell + \sum_{e_a\in\Gamma_D}\frac{\delta_a r^2}{|e_a|^\beta}\int_{e_a}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{v}\,d\ell\right| \\
&\quad\leqslant C\frac{h^{\mu-1/2-\beta(n-1)/2}}{r^{m-3/2}}|||\boldsymbol{u}|||_m\; J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}}.
\end{aligned}$$

Thus, in the general case, by combining the bounds together, we get the second estimate for $\beta = (n-1)^{-1}$. □

We now give the *a priori* error estimate.

Theorem 3.5 *Under the assumptions of Lemmas* 2.1 *and* 3.4*, assuming that there is a continuous interpolant* $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\bar{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ *of* $\boldsymbol{u}$*, if* $\boldsymbol{u} \in \mathcal{H}^m(\mathcal{E}_h) \cap \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega)$ *for some* $\epsilon > 0$*, and* $\beta \geqslant (n-1)^{-1}$ *we have*:

$$|||\boldsymbol{U}_{el}^{DG} - \boldsymbol{u}|||_H \leqslant C\frac{h^{\mu-1}}{r^{m-1}}|||\boldsymbol{u}|||_m,$$

where $\mu = \min\{r+1, m\}$*,* $r \geqslant 1$ *and* $m \geqslant 2$*. This means that for a smooth solution the error has an exponential rate of convergence* $O((\frac{h}{r})^r)$*.*

In the more general case where $\tilde{\boldsymbol{u}} \notin \boldsymbol{C}(\bar{\Omega})$ (*a continuous interpolant cannot be built*)*, and for* $\beta = (n-1)^{-1}$*, then*:

$$|||\boldsymbol{U}_{el}^{DG} - \boldsymbol{u}|||_H \leqslant C\frac{h^{\mu-1}}{r^{m-\frac{3}{2}}}|||\boldsymbol{u}|||_m,$$

and convergence is optimal with respect to h but suboptimal with respect to r*. In the first estimate the constant* $C > 0$ *depends upon* C_e*, the penalties* $\{\delta_a\}$*, the constants in* (14)*,* (16) *and* (17)*, and the tensor* $\underline{\boldsymbol{D}}$*. In the second estimate the constant depends also on* (15) *and* (18)*.*

Proof. Let $\tilde{\boldsymbol{u}} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ be an interpolant of $\boldsymbol{u}$ having optimal hp-approximation errors and set $\boldsymbol{\chi} = \boldsymbol{U}_{el}^{DG} - \tilde{\boldsymbol{u}}$. Then from (29) and Lemma 3.4,

$$\begin{aligned}|||\boldsymbol{\chi}|||_H^2 &= A(\boldsymbol{\chi}, \boldsymbol{\chi}) = A(\boldsymbol{\chi}, \boldsymbol{\chi}) + A(\boldsymbol{u} - \boldsymbol{U}_{el}^{DG}, \boldsymbol{\chi}) = A(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{\chi})\\ &\leqslant \epsilon|||\boldsymbol{\chi}|||_H^2 + \frac{\tilde{C}(h, r)}{\epsilon}|||\boldsymbol{u}|||_m^2 \qquad \forall\epsilon > 0,\end{aligned}$$

and where $\tilde{C}(h, r)$ is given in Lemma 3.4. Choosing ϵ small enough, we obtain

$$|||\boldsymbol{\chi}|||_H^2 \leqslant \frac{\tilde{C}(h, r)}{\epsilon}|||\boldsymbol{u}|||_m^2.$$

The proof then follows from using the triangle inequality,

$$|||\boldsymbol{u} - \boldsymbol{U}_{el}^{DG}|||_H \leqslant |||\boldsymbol{u} - \tilde{\boldsymbol{u}}|||_H + |||\tilde{\boldsymbol{u}} - \boldsymbol{U}_{el}^{DG}|||_H$$

and the global approximation result (17). □

Thus, in the case where a continuous interpolant cannot be built (i.e. if we use $\mathbb{P}_k$ for quadrilaterals), the error estimate is suboptimal with respect to the degree of polynomial, but optimal with respect to the mesh size.

We now move on to a semidiscrete version of the viscoelasticity problem.

4 *A priori* error bound for semidiscrete viscoelasticity

In this section we form a semidiscrete approximation to the quasistatic linear viscoelasticity problem as described earlier in Lemma 2.1.

From now on we will often represent the tensor $\underline{\boldsymbol{D}}$ by a matrix $\boldsymbol{D}$ with corresponding vector representations, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, of the tensors $\underline{\boldsymbol{\sigma}}$ and $\underline{\boldsymbol{\varepsilon}}$. This is for the purpose of keeping the notation "clean" since we will then be able to write, for example, $\boldsymbol{\varepsilon} \cdot \boldsymbol{D}\boldsymbol{\varepsilon}$ in place of $D_{ijkl}\varepsilon_{kl}\varepsilon_{ij}$.

The following result is elementary, but convenient.

Lemma 4.1 *For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$,*

$$\left| \boldsymbol{x} \cdot \frac{\partial^n \boldsymbol{D}(t)}{\partial t^n} \boldsymbol{y} \right| \leqslant \phi^{(n)}(t) \| \boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{x} \|_{\mathbb{E}} \| \boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{y} \|_{\mathbb{E}},$$

where $\| \cdot \|_{\mathbb{E}}$ denotes the Euclidean norm and,

$$\phi^{(n)}(t) := \left\| \boldsymbol{D}^{-\frac{1}{2}}(0) \frac{\partial^n \boldsymbol{D}(t)}{\partial t^n} \boldsymbol{D}^{-\frac{1}{2}}(0) \right\|_{L_\infty(\Omega;\mathbb{E})}.$$

Below we shall often write $\phi := \phi^{(1)}$ for clarity.

In this section we apply the DG(r) method in space to the weak form of the viscoelasticity problem (26). For each t, this results in the variational problem: find $\boldsymbol{U}_{vs}^{DG} \in \mathcal{D}_r(\mathcal{E}_h)$ such that,

$$A(\boldsymbol{U}_{vs}^{DG}(t), \boldsymbol{v}) = L(t; \boldsymbol{v}) + B(t; \boldsymbol{U}_{vs}^{DG}, \boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathcal{D}_r(\mathcal{E}_h). \tag{36}$$

We note that because of (21) we obtain two DG formulations.

For each $t \in [0, T]$ let $\tilde{\boldsymbol{u}} \in \mathcal{D}_r(\mathcal{E}_h)$ be an interpolant of $\boldsymbol{u}$ having optimal hp-approximation errors, and set,

$$\boldsymbol{\eta} := \boldsymbol{u} - \tilde{\boldsymbol{u}} \in \mathcal{H}^{\frac{3}{2}}(\mathcal{E}_h) \qquad \text{and} \qquad \boldsymbol{\chi} := \tilde{\boldsymbol{u}} - \boldsymbol{U}_{vs}^{DG} \in \mathcal{D}_r(\mathcal{E}_h).$$

Now, subtracting (36) from (26) we obtain the orthogonality property,

$$A(\boldsymbol{u}(t) - \boldsymbol{U}_{vs}^{DG}(t), \boldsymbol{v}) = B(t; \boldsymbol{u} - \boldsymbol{U}_{vs}^{DG}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathcal{D}_r(\mathcal{E}_h), \tag{37}$$

and, by choosing $\boldsymbol{v} = \boldsymbol{\chi}(t)$, we obtain the following error equation,

$$A(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t)) = -A(\boldsymbol{\eta}(t), \boldsymbol{\chi}(t)) + B(t; \boldsymbol{\eta}, \boldsymbol{\chi}(t)) + B(t; \boldsymbol{\chi}, \boldsymbol{\chi}(t)). \tag{38}$$

The first term on the right-hand side of (38) can be bounded using Lemma 3.4 and so our first goal is to bound the other two terms. We accomplish this in the following two lemmas. The first, Lemma 4.2, shows that the middle term also contains approximation errors while the second, Lemma 4.3, forms the basis of a sharp Gronwall-type inequality. These preliminary estimates will then yield the error estimate which we give below in Theorem 4.5.

Lemma 4.2 *Assume that* $\tilde{\boldsymbol{u}}(t) \in \boldsymbol{C}(\overline{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ *for each* t*, and that for each edge* $|e_a| \leqslant C_e h^{n-1}$*. Then, if* $\beta \geqslant (n-1)^{-1}$*, there is a* $C > 0$ *such that,*

$$|B(t; \boldsymbol{\eta}, \boldsymbol{\chi}(t))| \leqslant C \frac{h^{\mu-1}}{r^{m-1}} \int_0^t \phi(t-s) \|\boldsymbol{u}(s)\|_m \, ds \; \|\boldsymbol{\chi}(t)\|_H,$$

where ϕ *is the function in Lemma 4.1 and* $\mu = \min\{r+1, m\}$*. The constant* C *depends only upon* C_e*,* $\underline{\boldsymbol{D}}(0)^{\frac{1}{2}}$ *and the constants in the interpolation estimates* (17)*,* (16)*. The estimate holds for either of the choices* $B = \mathcal{B}_1$ *or* $B = \mathcal{B}_2$ *in* (21).

Proof. For the case $B = \mathcal{B}_1$ in (21) we work from (23) and use (12). First,

$$\begin{aligned}
&\left| \int_0^t \sum_{E\in\mathcal{E}_h} \int_E \frac{\partial D_{ijkl}(t-s)}{\partial s} \varepsilon_{kl}(\boldsymbol{\eta}(s)) \varepsilon_{ij}(\boldsymbol{\chi}(t)) \, dE \, ds \right| \\
&\quad \leqslant \int_0^t \sum_{E\in\mathcal{E}_h} \int_E \phi(t-s) \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}(s))\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}(t))\|_{\mathbb{E}} \, dE \, ds, \\
&\quad \leqslant \int_0^t \phi(t-s) \left(\sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\varepsilon}(\boldsymbol{\eta}(s)) \cdot \boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}(s)) \, dE \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\varepsilon}(\boldsymbol{\chi}(t)) \cdot \boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}(t)) \, dE \right)^{\frac{1}{2}} ds \\
&\quad \leqslant C(\underline{\boldsymbol{D}}(0)^{\frac{1}{2}}) \int_0^t \phi(t-s) \|\boldsymbol{\eta}(s)\|_1 \, ds \, |\boldsymbol{\chi}(t)|_A, \\
&\quad \leqslant C \frac{h^{\mu-1}}{r^{m-1}} \int_0^t \phi(t-s) \|\boldsymbol{u}(s)\|_m \, ds \, |\boldsymbol{\chi}(t)|_A,
\end{aligned}$$

where $\mu = \min\{r+1, m\}$ and we used the interpolation error estimate (17) and Lemma 4.1. Secondly, for the interior edge terms we suppose that the edge e_a is shared by the elements E_1 and E_2 and let $\boldsymbol{\eta}_i$ denote the restriction of $\boldsymbol{\eta}|_{E_i}$ to e_a, for $i = 1, 2$. Then, using the triangle inequality on the averaging operator $\{\cdot\}$, we have,

$$\begin{aligned}
&\left| \int_0^t \int_{e_a} \left\{ \frac{\partial D_{ijkl}(t-s)}{\partial s} \varepsilon_{kl}(\boldsymbol{\eta}(s)) \nu_j^a \right\} [\chi_i(t)] \, d\ell \, ds \right| \\
&\quad \leqslant C(\boldsymbol{D}(0)^{\frac{1}{2}}) \int_0^t \int_{e_a} \phi(t-s) \Big(\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(s))\|_{\mathbb{E}} \\
&\qquad + \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}_2(s))\|_{\mathbb{E}} \Big) \|[\boldsymbol{\chi}(t)]\|_{\mathbb{E}} \, d\ell \, ds,
\end{aligned}$$

$$\leqslant C \int_0^t \phi(t-s) \Big(\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(s))\|_{0,e_a} + \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}_2(s))\|_{0,e_a} \Big) \|[\boldsymbol{\chi}(t)]\|_{0,e_a}\, ds.$$

Summing over all edges we then "multiply by one" in the above to obtain,

$$\left| \sum_{a=1}^{P_h} \int_0^t \int_{e_a} \left\{ \frac{\partial D_{ijkl}(t-s)}{\partial s} \varepsilon_{kl}(\boldsymbol{\eta}(s)) \nu_j^a \right\} [\chi_i(t)]\, d\ell\, ds \right|$$
$$\leqslant C \sum_{i=1,2} \int_0^t \phi(t-s) \sum_{a=1}^{P_h} \left(\frac{|e_a|^\beta}{\delta_a r^2} \right)^{\frac{1}{2}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\eta}_i(s))\|_{0,e_a} \left(\frac{\delta_a r^2}{|e_a|^\beta} \right)^{\frac{1}{2}} \times \|[\boldsymbol{\chi}(t)]\|_{0,e_a}\, ds,$$
$$\leqslant C \frac{h^{\mu-\frac{3}{2}+\frac{\beta}{2}(n-1)}}{r^{m-\frac{1}{2}}} \int_0^t \phi(t-s) |||\boldsymbol{u}(s)|||_m\, ds\; J_0^{\delta,\beta}(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t))^{\frac{1}{2}},$$

where we used (16) and (19). Observe now that the remaining term in $\mathcal{B}_1$ can be estimated in exactly the same way as the term above and that, since $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\overline{\Omega})$, both terms in $\mathcal{B}_2$ are zero. Therefore, noting that $r^{\frac{1}{2}} \geqslant 1$, and combining the above with our first estimate then completes the proof. □

Lemma 4.3 *Assume that* $|e_a| \leqslant C_e h^{n-1}$ *and* $\beta \geqslant (n-1)^{-1}$. *Then, there is a* $\mathcal{C} > 0$ *such that,*

$$|B(t; \boldsymbol{\chi}, \boldsymbol{\chi}(t))| \leqslant \left(1 + \mathcal{C} h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right) \int_0^t \phi(t-s) |||\boldsymbol{\chi}(s)|||_H\, ds\; |||\boldsymbol{\chi}(t)|||_H$$

where ϕ *is the function in Lemma 4.1 and* $\mu = \min\{r+1, m\}$. *The constant* $\mathcal{C}$ *depends only upon* C_e, $\underline{\boldsymbol{D}}(0)^{\frac{1}{2}}$ *and the constants in the estimates* (17), (16) *and* (18). *The estimate holds for either of the choices* $B = \mathcal{B}_1$ *or* $B = \mathcal{B}_2$ *in* (21).

Proof. We first consider the case $B = \mathcal{B}_1$ in (21) and so need to estimate the terms in (23). For the first term we have,

$$\left| \int_0^t \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial D_{ijkl}(t-s)}{\partial s} \varepsilon_{kl}(\boldsymbol{\chi}(s)) \varepsilon_{ij}(\boldsymbol{\chi}(t))\, dE\, ds \right|$$
$$\leqslant \int_0^t \phi(t-s) \sum_{E \in \mathcal{E}_h} \int_E \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}(s))\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}(t))\|_{\mathbb{E}}\, dE\, ds$$
$$\leqslant \int_0^t \phi(t-s) |\boldsymbol{\chi}(s)|_{\mathcal{A}}\, ds\; |\boldsymbol{\chi}(t)|_{\mathcal{A}}.$$

For the interior edge summation term in (23) we suppose that the edge e_a is shared by the elements E_1 and E_2 and let $\boldsymbol{\chi}_i$ denote the restriction of $\boldsymbol{\chi}|_{E_i}$ to e_a, for $i = 1, 2$. We then have to begin with that,

$$\left|\int_0^t \int_{e_a} \left\{\frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{\chi}(s))\nu_j^a\right\}[\chi_i(t)]\,d\ell\,ds\right|$$
$$\leqslant C(\boldsymbol{D}(0)^{\frac{1}{2}})\int_0^t \phi(t-s)\int_{e_a}\Big(\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}_1(s))\|_{\mathbb{E}}$$
$$+\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}_2(s))\|_{\mathbb{E}}\Big)\|[\boldsymbol{\chi}(t)]\|_{\mathbb{E}}\,d\ell\,ds,$$

where we used the triangle inequality on the averaging operator. Summing over all interior edges, "multiplying by one", and then using the inverse estimate (18), we then obtain,

$$\left|\sum_{a=1}^{P_h}\int_0^t \int_{e_a} \left\{\frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{\chi}(s))\nu_j^a\right\}[\chi_i(t)]\,d\ell\,ds\right|$$
$$\leqslant C(\boldsymbol{D}(0)^{\frac{1}{2}})\sum_{\substack{a=1\\ i=1,2}}^{P_h}\int_0^t \phi(t-s)\int_{e_a}\left(\frac{|e_a|^\beta}{\delta_a r^2}\right)^{\frac{1}{2}}\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{\chi}_i(s))\|_{\mathbb{E}}$$
$$\times\left(\frac{\delta_a r^2}{|e_a|^\beta}\right)^{\frac{1}{2}}\|[\boldsymbol{\chi}(t)]\|_{\mathbb{E}}\,d\ell\,ds,$$
$$\leqslant C_0 C h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\int_0^t \phi(t-s)|\boldsymbol{\chi}(s)|_{\mathcal{A}}\,ds\left(\sum_{a=1}^{P_h}\frac{\delta_a r^2}{|e_a|^\beta}\int_{e_a}|[\boldsymbol{\chi}(t)]|^2\,d\ell\right)^{\frac{1}{2}}.$$

The remaining "$e_a \in \Gamma_D$" term can be estimated in a similar way to obtain,

$$\left|\sum_{e_a\in\Gamma_D}\int_0^t \int_{e_a}\frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{\chi}(s))\nu_j^a\chi_i(t)\,d\ell\,ds\right|$$
$$\leqslant C_0 C(\underline{\boldsymbol{D}}(0)^{\frac{1}{2}})h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\int_0^t \phi(t-s)|\boldsymbol{\chi}(s)|_{\mathcal{A}}\,ds$$
$$\times\left(\sum_{e_a\in\Gamma_D}\frac{\delta_a r^2}{|e_a|^\beta}\int_{e_a}|\boldsymbol{\chi}(t)|^2\,d\ell\right)^{\frac{1}{2}}.$$

This completes the proof for the case $B = \mathcal{B}_1$.

For $B = \mathcal{B}_2$ we need to also estimate the terms in (24). This is straightforward because precisely the same arguments as used for the "edge terms" above also apply, but with the time variables s and t interchanged. That is:

$$|\mathcal{B}_2(t;\boldsymbol{\chi},\boldsymbol{\chi}(t))| \leqslant C h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\int_0^t \phi(t-s)J_0^{\delta,\beta}(\boldsymbol{\chi}(s),\boldsymbol{\chi}(s))^{\frac{1}{2}}\,ds\;|\boldsymbol{\chi}(t)|_{\mathcal{A}}.$$

Thus, for $B = \mathcal{B}_i$, $i = 1$ or 2, we obtain,

$$|B(t; \boldsymbol{\chi}, \boldsymbol{\chi}(t))| \leqslant \left(1 + Ch^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right) \int_0^t \phi(t-s) \|\boldsymbol{u}(s)\|_H \, ds \, \|\boldsymbol{\chi}(t)\|_H.$$

This completes the proof. □

We will consider the class of compressible linear viscoelastic solids for which the following assumption is physically realistic.

Assumption 4.4 (fading memory) *The function ϕ in Lemma 4.1 can, in $\mathcal{J}$, be written as $\phi(t) = -\varphi'(t)$ where, for some $\varphi_0 \in (0, 1]$, the* generic stress relaxation function $\varphi : [0, \infty) \to (\varphi_0, 1]$ *belongs to $L_\infty(\mathbb{R}_+) \cap W_1^1(\mathbb{R}_+)$ and satisfies $\varphi(0) = 1$, $\varphi'(t) \leqslant 0$ and $\varphi''(t) \geqslant 0$.*

This assumption is realistic and, in particular, allows for a much sharper analysis for the time dependence than does the usual Gronwall lemma. We refer to [18] for further details, and simply note here that a prototype for φ is,

$$\varphi(t) = \varphi_0 + \sum_{i=1}^{N} \varphi_i e^{-\alpha_i t}, \tag{39}$$

with $\varphi_0 > 0$, $\varphi_i, \alpha_i \geqslant 0$ and the normalization $\varphi(0) = 1$.

With this assumption we have

$$\|\phi\|_{L_1(0,t)} = 1 - \varphi(t) \leqslant 1 - \varphi_0 \qquad \text{for every } t \in \mathcal{J}.$$

In the following theorem we extend the *a priori* error estimates of Theorem 3.5 to the semidiscrete problem (36). The assumption made above on the behaviour of ϕ allows us to use a similar technique to those in [15,18] whereby we bypass the usual Gronwall inequality and obtain error estimates with sharper constants. (Of course the error estimate developed below can be proven under much more general assumptions by using the Gronwall lemma—see e.g. [16]. The constant will then be exponentially large in T.)

Theorem 4.5 *Under the assumptions of Lemmas 2.1, 3.4, 4.2 and 4.3, if $\boldsymbol{u} \in L_p(\mathcal{J}; \mathcal{H}^m(\mathcal{E}_h) \cap \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega))$, $\beta > (n-1)^{-1}$ and if, for each t, there exists a continuous interpolant $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\overline{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ of $\boldsymbol{u}$, then there is a constant $C > 0$ independent of h, r and T such that:*

$$\|\boldsymbol{u} - \boldsymbol{U}_{vs}^{DG}\|_{L_p(\mathcal{J};H)} \leqslant C \frac{h^{\mu-1}}{r^{m-1}} \left(\frac{2 + \varphi_0^2}{\varphi_0^2}\right) \|\boldsymbol{u}\|_{L_p(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))},$$

for all h small enough so that $\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}} \leqslant \varphi_0$ where $\mathcal{C}$ is that time independent *constant in Lemma 4.3. Also: $\mu = \min(r+1, m)$, $r \geqslant 1$ and $m \geqslant 2$ and C is a mild* time independent *constant that depends upon the interpolation estimates* (16) *and* (17). *Note also that this result is optimal in both h and r.*

Proof. Using (38) and putting together Lemmas 3.4, 4.2 and 4.3 we obtain,

$$\|\!|\boldsymbol{\chi}(t)|\!\|_H^2 \leqslant C\frac{h^{\mu-1}}{r^{m-1}}\left(\|\!|\boldsymbol{u}(t)|\!\|_m + \int_0^t \phi(t-s)\|\!|\boldsymbol{u}(s)|\!\|_m\,ds\right)\|\!|\boldsymbol{\chi}(t)|\!\|_H$$
$$+\left(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right)\int_0^t \phi(t-s)\|\!|\boldsymbol{\chi}(s)|\!\|_H\,ds\ \|\!|\boldsymbol{\chi}(t)|\!\|_H.$$

From this it follows that

$$\|\!|\boldsymbol{\chi}(t)|\!\|_H \leqslant \mathcal{F}(h,r,\phi,u;t) + \left(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right)\int_0^t \phi(t-s)\|\!|\boldsymbol{\chi}(s)|\!\|_H\,ds,$$

where $\mathcal{F}$ is abbreviation for the term containing the approximation errors. This now implies (using Hölder's inequality for convolutions wherein $\|f * g\|_{L_p(0,T)} \leqslant \|f\|_{L_1(0,T)}\|g\|_{L_p(0,T)}$ – see [18]),

$$\|\!|\boldsymbol{\chi}|\!\|_{L_p(\mathcal{J};H)} \leqslant \|\mathcal{F}\|_{L_p(\mathcal{J})} + \left(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right)\|\phi\|_{L_1(\mathcal{J})}\|\!|\boldsymbol{\chi}|\!\|_{L_p(\mathcal{J};H)}.$$

Since $\|\phi\|_{L_1(0,t)} \leqslant 1-\varphi_0$ we can rearrange this to get,

$$\left(1-\left(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right)(1-\varphi_0)\right)\|\!|\boldsymbol{\chi}(t)|\!\|_{L_p(\mathcal{J};H)} \leqslant \|\mathcal{F}\|_{L_p(\mathcal{J})},$$

and then our "smallness assumption" on h means that this can be written as,

$$(1-(1+\varphi_0))(1-\varphi_0))\,\|\!|\boldsymbol{\chi}(t)|\!\|_{L_p(\mathcal{J};H)} \leqslant \|\mathcal{F}\|_{L_p(\mathcal{J})}.$$
$$\Longrightarrow \|\!|\boldsymbol{\chi}(t)|\!\|_{L_p(\mathcal{J};H)} \leqslant \frac{1}{\varphi_0^2}\|\mathcal{F}\|_{L_p(\mathcal{J})}.$$

Now, we have that,

$$\|\mathcal{F}\|_{L_p(\mathcal{J})} \leqslant C\frac{h^{\mu-1}}{r^{m-1}}(1+\|\phi\|_{L_1(\mathcal{J})})\|\!|\boldsymbol{u}|\!\|_{L_p(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))},$$

and the triangle inequality gives,

$$\|\!|\boldsymbol{u}-\boldsymbol{U}_{vs}^{DG}|\!\|_{L_p(\mathcal{J};H)} \leqslant \|\!|\boldsymbol{\eta}|\!\|_{L_p(\mathcal{J};H)} + \|\!|\boldsymbol{\chi}|\!\|_{L_p(\mathcal{J};H)}$$
$$\leqslant C\frac{h^{\mu-1}}{r^{m-1}}\left(1+\left(\frac{1+\|\phi\|_{L_1(0,t)}}{\varphi_0^2}\right)\right)\|\!|\boldsymbol{u}|\!\|_{L_p(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}.$$

This completes the proof. □

In closing we remark that viscoelastic "fluids" (in the sense of Golden and Graham, [5]) do not satisfy the assumption on ϕ given above in Assumption 4.4. However, the results in [18] suggest that the theorem will still hold in such cases except that $C = O(T)$.

In the next section we form a fully discrete approximation to the viscoelasticity problem by discretizing in time also.

5 *A priori* error estimate for fully discrete viscoelasticity

A natural way of constructing a fully discrete approximation of the weak problem (26) is to define the constant time step, $k := T/N$, for some positive integer N, set $t_q := qk$, and then replace the time integral in (12) with a numerical quadrature rule. This is the approach we take in this section where we produce a numerical scheme and then quote an optimal *a priori* error estimate for simplicial finite element meshes (which allow us to build a continuous $\mathcal{D}_r(\mathcal{E}_h)$ interpolant, $\tilde{\boldsymbol{u}}(t)$ to $\boldsymbol{u}(t)$).

The quadrature scheme is defined as follows: for a generic integrand, g, we approximate as follows:

$$\int_0^{t_q} g(s)\,ds \approx k \sum_{p=0}^{q-1} \varpi_{qp} g(t_p), \qquad \text{for } q = 0, 1, \dots, N,$$

where, here and below, empty sums are set to zero. The $\{\varpi_{qp}\}$ are (positive) weights, determined as follows. Over (t_{q-1}, t_q) we integrate using the backward (i.e. left-end-point integrand evaluation) rectangle rule and then over $(0, t_{q-1})$, when $q > 1$, we use the standard trapezoidal rule. Hence, the sequences of weights, for each t_q, are given by:

$$\begin{aligned}
q = 0 &: \{\varpi_{qp}\}_{p=0}^{q-1} = \varnothing,\\
q = 1 &: \{\varpi_{qp}\}_{p=0}^{q-1} = \{1\},\\
q = 2 &: \{\varpi_{qp}\}_{p=0}^{q-1} = \{\tfrac{1}{2}, \tfrac{3}{2}\},\\
q = 3 &: \{\varpi_{qp}\}_{p=0}^{q-1} = \{\tfrac{1}{2}, 1, \tfrac{3}{2}\},\\
q > 3 &: \{\varpi_{qp}\}_{p=0}^{q-1} = \{\tfrac{1}{2}, 1, \dots, 1, \tfrac{3}{2}\}.
\end{aligned}$$

Note that the right-hand value of the integrand, $g(t_q)$, is not required in the approximation.

Replacing the time integral in (12) by this quadrature rule leads to a time discretized viscous stress:

$$\varsigma_{ij}^{vs}(\boldsymbol{w}; t_q, t_m) := k \sum_{p=0}^{q-1} \varpi_{qp} \left.\frac{\partial D_{ijkl}(t_m - s)}{\partial s}\right|_{s=t_p} \varepsilon_{kl}(\boldsymbol{w}(t_p)),$$
$$\text{for} \qquad 0 \leqslant m \leqslant q.$$

(Note the clash of notation where k is being used as a time step and a dummy tensor subscript, no confusion should arise here.) We then define discrete versions of $\mathcal{B}_1$ and $\mathcal{B}_2$, in (23) and (24), as,

$$\mathcal{B}_1^q(\boldsymbol{w}, \boldsymbol{v}) := \sum_{E\in\mathcal{E}_h} \int_E \varsigma_{ij}^{vs}(t_q, t_q; \boldsymbol{w})\varepsilon_{ij}(\boldsymbol{v})\, dE$$
$$- \sum_{a=1}^{P_h} \int_{e_a} \{\varsigma_{ij}^{vs}(t_q, t_q; \boldsymbol{w})\nu_j^a\}[v_i]\, d\ell$$
$$- \sum_{e_a\in\Gamma_D} \int_{e_a} \varsigma_{ij}^{vs}(t_q, t_q; \boldsymbol{w})\nu_j^a v_i\, d\ell, \tag{40}$$

$$\mathcal{B}_2^q(\boldsymbol{w}, \boldsymbol{v}) := \sum_{a=1}^{P_h} \int_{e_a} \{\varsigma_{ij}^{vs}(t_q, t_q; \boldsymbol{v})\nu_j^a\}[w_i]\, d\ell$$
$$+ \sum_{e_a\in\Gamma_D} \int_{e_a} \varsigma_{ij}^{vs}(t_q, t_q; \boldsymbol{v})\nu_j^a w_i\, d\ell. \tag{41}$$

The analogue of (21) is then,

$$B^q(\boldsymbol{w}, \boldsymbol{v}) := \mathcal{B}_1^q(\boldsymbol{w}, \boldsymbol{v}) + \mathcal{B}_2^q(\boldsymbol{w}, \boldsymbol{v}) \quad \text{or} \quad \mathcal{B}_1^q(\boldsymbol{w}, \boldsymbol{v}). \tag{42}$$

We now define our fully discrete approximation to (26) as the problem: find a piecewise linear (with respect to the $\{t_q\}$) time-continuous function $\boldsymbol{U}^{DG}\colon \mathcal{J} \to \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ such that, for each $q = 0, 1, \ldots, N$ in turn, $\boldsymbol{U}_q^{DG} := \boldsymbol{U}^{DG}(t_q)$ satisfies,

$$A(\boldsymbol{U}_q^{DG}, \boldsymbol{v}) = L(t_q; \boldsymbol{v}) + B^q(\boldsymbol{U}^{DG}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h). \tag{43}$$

Here, of course, $\boldsymbol{U}^{DG}(t)$ is the approximation to $\boldsymbol{u}(t)$ and $\boldsymbol{U}^{DG}$ is uniquely defined by linear interpolation within the time intervals. Note that existence and uniqueness of a solution at each discrete time is ensured by Lemma 3.1 after noting that our quadrature rule simply produces a linear elasticity problem at each time level. Also, Lemma 3.2 continues to hold for the discrete stress tensor $\underline{\varsigma}$.

Remark 5.1 Although we are allowing high-order approximation in space we are only considering a fixed order time approximation. The reason for using the second-order trapezoid/rectangle rule for the time discretization is to produce a fully discrete scheme that does not involve the solution at the current time level on the right. That is, in (43), $\boldsymbol{U}_q^{DG}$ appears only on the left. In this case the viscoelasticity algorithm can be generated by simply modifying the load vector in already existing elliptic solver software. Of course, higher order quadrature rules could be used but this would necessitate the current solution making a contribution to the history integral. These terms would then have to brought over to the left and would result in a modified stiffness matrix. One would also need (e.g. for Simpson's rule) a means of generating "starting values".

Before getting to the *a priori* error estimate we need some preliminary results based on the following error splitting. For each t, let $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\bar{\Omega}) \cap \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ be an interpolant of $\boldsymbol{u}$ having optimal hp-approximation errors, and set,

$$\boldsymbol{\eta} := \boldsymbol{u} - \tilde{\boldsymbol{u}} \in \boldsymbol{\mathcal{H}}^{\frac{3}{2}}(\mathcal{E}_h) \qquad \text{and} \qquad \boldsymbol{\zeta} := \tilde{\boldsymbol{u}} - \boldsymbol{U}^{DG} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h).$$

Now, subtract (43) from (26) to get,

$$\begin{aligned} A(\boldsymbol{u}(t_q) - \boldsymbol{U}_q^{DG}, \boldsymbol{v}) &= B(t_q; \boldsymbol{u}, \boldsymbol{v}) - B^q(\boldsymbol{U}^{DG}, \boldsymbol{v}), \\ &= B^q(\boldsymbol{\zeta}, \boldsymbol{v}) + B(t_q; \boldsymbol{\eta}, \boldsymbol{v}) \\ &\quad + \big(B(t_q; \tilde{\boldsymbol{u}}, \boldsymbol{v}) - B^q(\tilde{\boldsymbol{u}}, \boldsymbol{v})\big) \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h). \end{aligned} \tag{44}$$

Note that by choosing the error splitting in this way we need only estimate the quadrature error relative to the interpolant, $\tilde{\boldsymbol{u}}$, rather than $\boldsymbol{u}$. This enables us to use (18) in the following estimate for the quadrature error.

Lemma 5.2 (quadrature error) *Assume that $\beta > (n-1)^{-1}$, and that, for some $C_e > 0$, each edge satisfies $|e_a| \leqslant C_e h^{n-1}$. Then, for $q = 0, 1, 2, \dots, N$ and either choice in* (42) *we have,*

$$\big|B(t_q; \tilde{\boldsymbol{u}}, \boldsymbol{v}) - B^q(\tilde{\boldsymbol{u}}, \boldsymbol{v})\big| \leqslant CT\Upsilon(T)k^2 |||\boldsymbol{u}|||_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))} |||\boldsymbol{v}|||_H \;\; \forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_r(\mathcal{E}_h).$$

Here, in the notation of Lemma 4.1,

$$\Upsilon(T) := \max\{\|\phi^{(i)}\|_{W^1_\infty(\mathcal{J})} : i = 1, 2\},$$

and C is a positive constant depending on C_e, $\underline{\boldsymbol{D}}(0)^{\frac{1}{2}}$ and the constants in (17) *and* (18).

Proof. For a generic integrand, g, standard estimates give,

$$\left| \int_0^{t_q} g(\xi)\, d\xi - k \sum_{p=0}^{q-1} \varpi_{qp} g(t_p) \right| \leqslant CTk^2 \|g'\|_{W^1_\infty(0,t_q)},$$

where C is a time independent constant. Therefore, working from the definition (21), and using (23) and (24), we now have to estimate each integrand in turn.

For the first integrand in (23) we use Lemma 4.1 to get,

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial s} \sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{D}'(t_q - s)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(s))\, dE \right\|_{W^1_\infty(0,t_q)} \\
&= \left\| \sum_{E\in\mathcal{E}_h} \int_E \boldsymbol{\varepsilon}(\boldsymbol{v})\cdot \boldsymbol{D}'(t_q - s)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_s(s)) - \boldsymbol{\varepsilon}(\boldsymbol{v})\cdot \boldsymbol{D}''(t_q - s)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(s))\, dE \right\|_{W^1_\infty(0,t_q)}, \\
&\leqslant \left\| \sum_{E\in\mathcal{E}_h} \int_E \phi^{(1)}(t_q - s)\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_s(s))\|_{\mathbb{E}} \right. \\
&\qquad \left. +\phi^{(2)}(t_q - s)\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(s))\|_{\mathbb{E}}\, dE \right\|_{W^1_\infty(0,t_q)}, \\
&\leqslant \left\| \phi^{(1)}(t_q - s)|\boldsymbol{v}|_{\mathcal{A}} |\tilde{\boldsymbol{u}}(s)|_{\mathcal{A}} + \phi^{(2)}(t_q - s)|\boldsymbol{v}|_{\mathcal{A}}|\tilde{\boldsymbol{u}}_s(s)|_{\mathcal{A}} \right\|_{W^1_\infty(0,t_q)}, \\
&\leqslant \Upsilon(T) \left\| |\tilde{\boldsymbol{u}}(s)|_{\mathcal{A}} + |\tilde{\boldsymbol{u}}_s(s)|_{\mathcal{A}} \right\|_{W^1_\infty(0,t_q)} |\boldsymbol{v}|_{\mathcal{A}}, \\
&\leqslant 2\Upsilon(T) \left\| |\tilde{\boldsymbol{u}}(s)|_{\mathcal{A}} \right\|_{W^2_\infty(\mathcal{J})} |\boldsymbol{v}|_{\mathcal{A}}.
\end{aligned}
$$

For the second integrand in (23) we use the triangle inequality on the averaging operator, $\{\cdot\}$, along with Lemma 4.1 and the inverse estimate (18) to get,

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial s} \sum_{a=1}^{P_h} \int_{e_a} \left\{ \frac{\partial D_{ijkl}(t_q - s)}{\partial s} \varepsilon_{kl}(\tilde{\boldsymbol{u}}(s)) \nu_j^a \right\} [v_i]\, d\ell \right\|_{W^1_\infty(0,t_q)} \\
&\quad = \left\| \sum_{a=1}^{P_h} \int_{e_a} \left\{ D''_{ijkl}(t_q - s)\varepsilon_{kl}(\tilde{\boldsymbol{u}}(s))\nu_j^a \right\} [v_i] \right. \\
&\qquad \left. - \left\{ D'_{ijkl}(t_q - s)\varepsilon_{kl}(\tilde{\boldsymbol{u}}_s(s))\nu_j^a \right\} [v_i]\, d\ell \right\|_{W^1_\infty(0,t_q)}, \\
&\quad \leqslant \left\| \sum_{a=1}^{P_h} \int_{e_a} \phi^{(2)}(t_q - s)\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(s))\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\|_{\mathbb{E}} \|[\boldsymbol{v}]\|_{\mathbb{E}} \right. \\
&\qquad \left. + \phi^{(1)}(t_q - s)\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_s(s))\|_{\mathbb{E}} \|\boldsymbol{D}^{\frac{1}{2}}(0)\|_{\mathbb{E}} \|[\boldsymbol{v}]\|_{\mathbb{E}}\, d\ell \right\|_{W^1_\infty(0,t_q)}, \\
&\quad \leqslant \|\boldsymbol{D}^{\frac{1}{2}}(0)\|_{L_\infty(\Omega;\mathbb{E})} \left\| \phi^{(2)}(t_q - s) \left(\sum_{a=1}^{P_h} \frac{|e_a|^\beta}{\delta_a r^2} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}(s))\|^2_{0,e_a} \right)^{\frac{1}{2}} \right. \\
&\qquad \left. + \phi^{(1)}(t_q - s) \left(\sum_{a=1}^{P_h} \frac{|e_a|^\beta}{\delta_a r^2} \|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_s(s))\|^2_{0,e_a} \right)^{\frac{1}{2}} \right\|_{W^1_\infty(0,t_q)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\sum_{a=1}^{P_h}\frac{\delta_a r^2}{|e_a|^\beta}\|[\boldsymbol{v}]\|_{0,e_a}^2\right)^{\frac{1}{2}},\\
&\leqslant C_e C C_0 h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\|\boldsymbol{D}^{\frac{1}{2}}(0)\|_{L_\infty(\Omega;\mathbb{E})}\\
&\quad\times\left\|\phi^{(2)}(t_q-s)|\tilde{\boldsymbol{u}}(s)|_{\mathcal{A}}+\phi^{(1)}(t_q-s)|\tilde{\boldsymbol{u}}_s(s)|_{\mathcal{A}}\right\|_{W^1_\infty(0,t_q)} J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}},\\
&\leqslant C\|\boldsymbol{D}^{\frac{1}{2}}(0)\|_{L_\infty(\Omega;\mathbb{E})}\Upsilon(T)h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\left\||\tilde{\boldsymbol{u}}|_{\mathcal{A}}\right\|_{W^2_\infty(\mathcal{J})} J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})^{\frac{1}{2}}.
\end{aligned}
$$

Precisely the same arguments can be applied to the third integrand in (23) and we note that both integrands in (24) are zero. Hence the lemma is proven when $\tilde{\boldsymbol{u}}$ appears on the right rather than $\boldsymbol{u}$. To address this we use (17) with $\mu = q = s = 1$ and get,

$$
|\tilde{\boldsymbol{u}}|_{\mathcal{A}} = |(\tilde{\boldsymbol{u}}-\boldsymbol{u})+\boldsymbol{u}|_{\mathcal{A}} \leqslant C(\underline{\boldsymbol{D}}(0)^{\frac{1}{2}})(|||(\tilde{\boldsymbol{u}}-\boldsymbol{u})|||_1 + |||\boldsymbol{u}|||_1) \leqslant C|||\boldsymbol{u}|||_1.
$$

Noting that this argument also works for $\tilde{\boldsymbol{u}}_t$ and $\tilde{\boldsymbol{u}}_{tt}$ then completes the proof. □

Choosing $\boldsymbol{v} = \boldsymbol{\zeta}_q$ in (44), and using Lemma 5.2 now yields,

$$
\begin{aligned}
|||\boldsymbol{\zeta}_q|||_H^2 \leqslant\; & CT\Upsilon(T)k^2|||\boldsymbol{u}|||_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))}|||\boldsymbol{\zeta}_q|||_H + |A(\boldsymbol{\eta}(t_q),\boldsymbol{\zeta}_q)|\\
& + |B(t_q;\boldsymbol{\eta},\boldsymbol{\zeta}_q)| + |B^q(\boldsymbol{\zeta},\boldsymbol{\zeta}_q)|.
\end{aligned} \tag{45}
$$

The second and third terms on the right-hand side of (45) can be bounded using Lemmas 3.4 and 4.2 – with $\boldsymbol{\chi}$ replaced by $\boldsymbol{\zeta}_q$. Our goal is to bound the last term. This term generates a Volterra inequality, and, in order to bypass the Gronwall lemma, we need the following result for the quadrature rule.

Lemma 5.3 *Let $\{\theta_p\}_{p=0}^{q-1}$ be a sequence of non-negative real numbers and set $\hat{\theta}_q := \max\{\theta_p\colon 0 \leqslant p \leqslant q\}$. Then, for ϕ of the form given in Assumption 4.4,*

$$
0 \leqslant k\sum_{p=0}^{q-1}\varpi_{qp}\phi(t_q-t_p)\theta_p \leqslant (1-\varphi_{0k})\hat{\theta}_{q-1},
$$

for $q = 0, 1, 2, \ldots, N$, with empty sums set to zero and where $\varphi_{0k} := \varphi_0 + k\varphi'(k)$.

Proof. The left hand inequality is obvious from the definitions so we need only prove the upper bound. We consider the summation as made up of a "trapezoidal rule" component and a "rectangle rule" component and take each in turn.

For the trapezoidal summation we use Assumption 4.4 to obtain firstly,

$$\begin{aligned}\sum_{p=1}^{q-1}\frac{k}{2}\big(\phi(t_q-t_p)+\phi(t_q-t_{p-1})\big) &\leqslant k\sum_{p=1}^{q-1}\max\{-\varphi'(t_q-t_p),\,-\varphi'(t_q-t_{p-1})\}\\ &= -\sum_{p=1}^{q-1}k\varphi'(t_q-t_p)\\ &\leqslant -\sum_{p=1}^{q-1}\int_{t_p}^{t_{p+1}}\varphi'(t_q-s)\,ds,\\ &= 1-\varphi(t_{q-1}).\end{aligned}$$

For the rectangle summation we have $k\phi(t_q-t_{q-1}) = -k\varphi'(k)$ and hence,

$$k\sum_{p=0}^{q-1}\varpi_{qp}\phi(t_q-t_p)\theta_p \leqslant \hat{\theta}_{q-1}\big(1-(\varphi_0+k\varphi'(k))\big),$$

where we noted that $\varphi(t_{q-1}) \geqslant \varphi_0$. □

As the last step in our preparation we need an analogue of Lemma 4.3.

Lemma 5.4 *Let Lemma 4.3 hold. Then,*

$$|B^q(\boldsymbol{\zeta},\boldsymbol{\zeta}_q)| \leqslant \Big(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\Big)\,k\sum_{p=0}^{q-1}\varpi_{qp}\phi(t_q-t_p)|\!|\!|\boldsymbol{\zeta}_p|\!|\!|_H|\!|\!|\boldsymbol{\zeta}_q|\!|\!|_H,$$

under the same conditions and assumptions.

We can now give an *a priori* error estimate that is optimal in both h and r.

Theorem 5.5 *Under the assumptions of Lemmas 2.1, 4.3, 3.4, 5.2 and Assumption 4.4, along with the requirement that* $\underline{\boldsymbol{D}} \in W^3_\infty(\mathcal{J};\underline{\boldsymbol{L}}_\infty(\Omega))$ *then, if* $\boldsymbol{u} \in W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega)) \cap L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h)\cap \boldsymbol{H}^{\frac{3}{2}+\epsilon}(\Omega))$, $\beta > (n-1)^{-1}$, *and for each edge* $|e_a| \leqslant C_e h^{n-1}$, *and if, for each* t, *there exists a continuous interpolant* $\tilde{\boldsymbol{u}} \in \boldsymbol{C}(\bar{\Omega})\cap\boldsymbol{\mathcal{D}}_r(\mathcal{E}_h)$ *of* $\boldsymbol{u}$, *there is a constant* $C > 0$ *independent of* h, r, *and* T *such that for* h *bounded above:*

$$\begin{aligned}|\!|\!|\boldsymbol{u}-\boldsymbol{U}^{DG}|\!|\!|_{\ell_\infty(\{t_q\}_{q=0}^N;H)} \leqslant C\Bigg(&\frac{T\Upsilon(T)k^2}{\varphi_{0k}^2}|\!|\!|\boldsymbol{u}|\!|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))}\\ &+\frac{h^{\mu-1}}{r^{m-1}}\left(\frac{2+\varphi_{0k}^2}{\varphi_{0k}^2}\right)|\!|\!|\boldsymbol{u}|\!|\!|_{L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}\Bigg)\end{aligned}$$

for all h small enough so that $\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}} \leqslant \varphi_{0k}$ *where* $\mathcal{C}$ *is that* time independent *constant in Lemma 4.3,* φ_{0k} *is defined by Lemma 5.3 and* $\Upsilon(T)$ *is defined by Lemma 5.2. Also:* $\mu = \min\{r+1, m\}$, $r \geqslant 1$ *and* $m \geqslant 2$ *and* C *is a mild* time independent *constant that depends upon* C_e, $\underline{\boldsymbol{D}}(0)^{\frac{1}{2}}$ *and the interpolation constants in* (16), (17) *and* (18).

Proof. From (45) and Lemmas 3.4, 4.2 and 5.4 we get,

$$
\begin{aligned}
\|\!|\boldsymbol{\zeta}_q\|\!|_H \leqslant{}& \left(CT\Upsilon(T)k^2 \|\!|\boldsymbol{u}\|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))} + C\frac{h^{\mu-1}}{r^{m-1}}\|\!|\boldsymbol{u}\|\!|_{L_\infty(\mathcal{J}:\mathcal{H}^m(\mathcal{E}_h))}\right) \\
&+C\frac{h^{\mu-1}}{r^{m-1}}\int_0^{t_q}\phi(t_q-s)\|\!|\boldsymbol{u}(s)\|\!|_m\,ds \\
&+\left(1+\mathcal{C}h^{\frac{\beta}{2}(n-1)-\frac{1}{2}}\right)k\sum_{p=0}^{q-1}\varpi_{qp}\phi(t_q-t_p)\|\!|\boldsymbol{\zeta}_p\|\!|_H .
\end{aligned}
$$

Using Lemma 5.3 and our bound on h we obtain from this,

$$
\begin{aligned}
\max_{0\leqslant p\leqslant q}\|\!|\boldsymbol{\zeta}_p\|\!|_H \leqslant{}& C\Big(T\Upsilon(T)k^2\|\!|\boldsymbol{u}\|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))} \\
&+\frac{h^{\mu-1}}{r^{m-1}}\left(1+\|\phi\|_{L_1(\mathcal{J})}\right)\|\!|\boldsymbol{u}\|\!|_{L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}\Big) \\
&+(1+\varphi_{0k})(1-\varphi_{0k})\max_{0\leqslant p\leqslant q}\|\!|\boldsymbol{\zeta}_p\|\!|_H,
\end{aligned}
$$

from which it follows that,

$$
\begin{aligned}
&\max_{0\leqslant p\leqslant q}\|\!|\boldsymbol{\zeta}_p\|\!|_H \\
&\leqslant C\left(\frac{T\Upsilon(T)k^2\|\!|\boldsymbol{u}\|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))}}{\varphi_{0k}^2}+\frac{h^{\mu-1}}{r^{m-1}}\left(\frac{2-\varphi_0}{\varphi_{0k}^2}\right)\|\!|\boldsymbol{u}\|\!|_{L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}\right).
\end{aligned}
$$

Using the triangle inequality, (17) now gives,

$$
\begin{aligned}
\max_{0\leqslant p\leqslant q}\|\!|\boldsymbol{u}(t_p)-\boldsymbol{U}^{DG}(t_p)\|\!|_H \leqslant{}& \|\!|\boldsymbol{\eta}\|\!|_{L_\infty(\mathcal{J};H)}+\max_{0\leqslant p\leqslant q}\|\!|\boldsymbol{\zeta}_p\|\!|_H, \\
\leqslant{}& C\Bigg(\frac{T\Upsilon(T)k^2\|\!|\boldsymbol{u}\|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))}}{\varphi_{0k}^2} \\
&+\frac{h^{\mu-1}}{r^{m-1}}\left(1+\frac{2-\varphi_0}{\varphi_{0k}^2}\right)\|\!|\boldsymbol{u}\|\!|_{L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}\Bigg), \\
\leqslant{}& C\Bigg(\frac{T\Upsilon(T)k^2\|\!|\boldsymbol{u}\|\!|_{W^2_\infty(\mathcal{J};\mathcal{H}^1(\Omega))}}{\varphi_{0k}^2} \\
&+\frac{h^{\mu-1}}{r^{m-1}}\left(\frac{2+\varphi_{0k}^2}{\varphi_{0k}^2}\right)\|\!|\boldsymbol{u}\|\!|_{L_\infty(\mathcal{J};\mathcal{H}^m(\mathcal{E}_h))}\Bigg).
\end{aligned}
$$

Since q was arbitrary, this completes the proof. □

Remark 5.6 The constant C in Theorem 5.5 is time independent, as is the coefficient in the spatial discretization error term. Moreover, if $\underline{\boldsymbol{D}}$ behaves, in time, as the Prony series, (39), (which is a reasonable assumption for compressible linear viscoelastic solids—see [18]) then,

$$\Upsilon(T) := \max\{\|\phi^{(i)}\|_{W^1_\infty(\mathcal{J})} : i = 1, 2\} = \max\left\{\sum_{i=1}^{N} \alpha_i \varphi_i, \sum_{i=1}^{N} \alpha_i^2 \varphi_i\right\}$$

and is bounded independent of T. Therefore, the *a priori* error estimate allows for long-time integration with essentially no accumulation of space discretization error and, at worst, only a linear growth in time discretization error.

6 Conclusions

In this paper we have presented *a priori* error estimates for DG(r) finite element approximations of: linear elasticity; time-continuous (i.e. semidiscrete) quasistatic linear viscoelasticity; and, a fully discrete scheme for quasistatic linear viscoelasticity. The estimates are optimal in terms of h-convergence but only optimal in the polynomial degree, r, when the finite element space contains a continuous interpolant to the solution of the continuous problem. Software is currently under development and numerical results will be reported elsewhere at a later date.

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