RICE

New Directions in the Application of Model Order Reduction

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- Collaborators: M. Heinkenschloss, K. Willcox
  S. Chaturantabut, R. Nong
- Support: AFOSR and NSF

MTNS 08 Blacksburg, VA July 2008

Impossible Calculations Made Possible with ROM

Experiments with many instances of same reduced model

Brief Intro to Gramian Based Model Reduction

Proper Orthogonal Decomposition (POD)

Balanced Reduction

Neural Modeling: Local Reduction ⇒ Many Interactions

Nonlinear MOR: Application of Empirical Interpolation (EIM)

Process/Design Variation: Monte-Carlo via ROM

LTI Model Reduction by Projection

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

Approximate $\mathbf{x} \in \mathcal{S}_V = \mathit{Range}(\mathbf{V})$, a k-diml. subspace
i.e. Put $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$, and then force

$$\mathbf{W}^T[\mathbf{V}\dot{\hat{\mathbf{x}}} \; - \; (\mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u})] = 0$$

$$\hat{\mathbf{y}} = \mathbf{C}\mathbf{V}\hat{\mathbf{x}}$$

If $\mathbf{W}^T\mathbf{V} = \mathbf{I}_k$, then the k dimensional reduced model is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

where $\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Moment Matching ↔ Krylov Subspace Projection

Based on Lanczos, Arnoldi, Rational Krylov methods

Padé via Lanczos (PVL)

Freund, Feldmann

Bai

Multipoint Rational Interpolation

Grimme

Gallivan, Grimme, Van Dooren

Recent: Optimal $\mathcal{H}_2$ approximation via interpolation
Gugercin, Antoulas, Beattie

Gramian Based Model Reduction

Proper Orthogonal Decomposition (POD)
Principal Component Analysis (PCA)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{y} = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$$

The Gramian

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau$$

Eigenvectors of $\mathcal{P}$

$$\mathcal{P} = \mathbf{V}\mathbf{S}^2\mathbf{V}^T$$

Orthogonal Basis

$$\mathbf{x}(t) = \mathbf{V}\mathbf{S}\mathbf{w}(t)$$

PCA or POD Reduced Basis

Low Rank Approximation

$$\mathbf{x} \approx \mathbf{V}_k \hat{\mathbf{x}}_k(t)$$

Galerkin condition – Global Basis

$$\dot{\hat{\mathbf{x}}}_k = \mathbf{V}_k^T \mathbf{f}(\mathbf{V}_k \hat{\mathbf{x}}_k(t), \mathbf{u}(t))$$

Global Approximation Error ? ($\mathcal{H}_2$ bound for LTI)

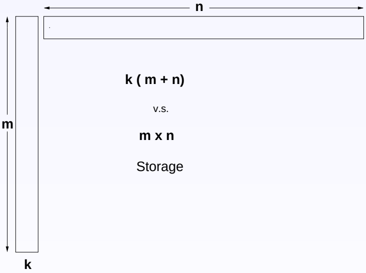
$$\|\mathbf{x} - \mathbf{V}_k \hat{\mathbf{x}}_k\|_2 \approx \sigma_{k+1}$$

Snapshot Approximation to $\mathcal{P}$

$$\mathcal{P} \approx \frac{1}{m} \sum_{j=1}^{m} \mathbf{x}(t_j)\mathbf{x}(t_j)^T = \mathbf{X}\mathbf{X}^T$$

Truncate SVD : $\mathbf{X} = \mathbf{V}\mathbf{S}\mathbf{U}^T \approx \mathbf{V}_k \mathbf{S}_k \mathbf{U}_k^T$

SVD Compression



Advantage of SVD Compression

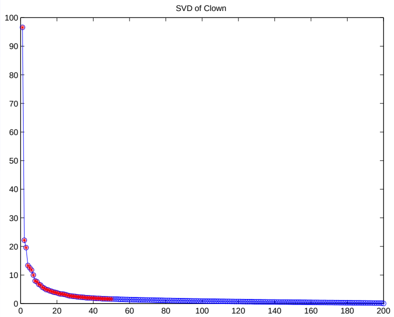
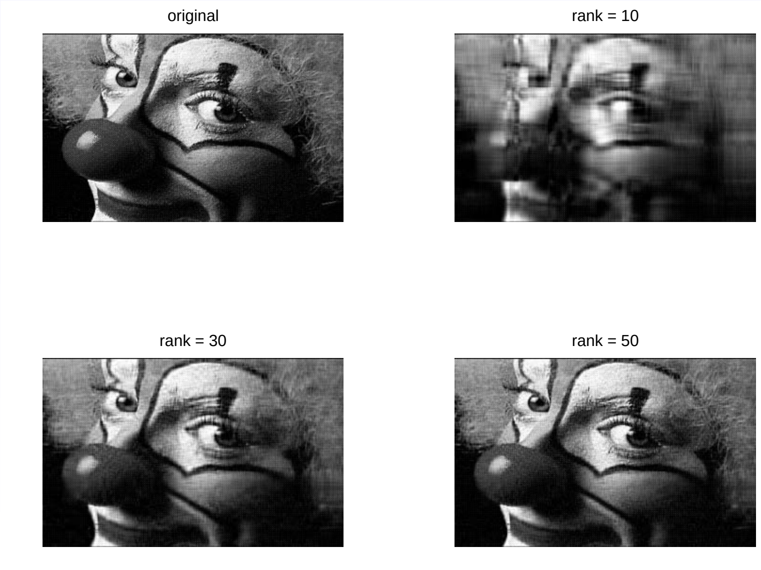


Image Compression - Feature Detection



POD in CFD

Extensive Literature

Karhunen-Loéve, L. Sirovich

Burns, King

Kunisch and Volkwein

Gunzburger

Many, many others

Incorporating Observations – Balancing

Lall, Marsden and Glavaski

K. Willcox and J. Peraire

POD vs. FEM

- Both are Galerkin Projection
- POD - Global Basis fns. vs FEM - Local Basis fns.
- FEM - Complex Behavior via Mesh Refinement/ Higher Order
  High Dimension - Sparse Matrices
- POD - Complex Behavior contained in Global Basis
  Low Dimension - Dense Matrices
- Caveat: POD is ad hoc: Must sample rich set of inputs and parameter settings
- Qx: How to automate parameter/input sampling for POD

POD for LTI systems

Impulse Response: $\mathcal{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \quad s \geq 0$

Input to State Map: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{B}$

Controllability Gramian:

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau = \int_o^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

State to Output Map: $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$

Observability Gramian:

$$\mathcal{Q} = \int_o^\infty e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}\tau} d\tau$$

Balanced Reduction (Moore 81)

Lyapunov Equations for system Gramians

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

With $\mathcal{P} = \mathcal{Q} = \mathbf{S}$: Want Gramians Diagonal and Equal

States Difficult to Reach are also Difficult to Observe

Reduced Model $\mathbf{A}_k = \mathbf{W}_k^T\mathbf{A}\mathbf{V}_k$, $\mathbf{B}_k = \mathbf{W}_k^T\mathbf{B}$, $\mathbf{C}_k = \mathbf{C}_k\mathbf{V}_k$

- $\mathcal{P}\mathbf{V}_k = \mathbf{W}_k\mathbf{S}_k$ $\qquad$ $\mathcal{Q}\mathbf{W}_k = \mathbf{V}_k\mathbf{S}_k$
- Reduced Model Gramians $\mathcal{P}_k = \mathbf{S}_k$ and $\mathcal{Q}_k = \mathbf{S}_k$.

Hankel Norm Error estimate (Glover 84)

Why Balanced Truncation?

- Hankel singular values $= \sqrt{\lambda(\mathcal{PQ})}$
- Model reduction $\mathcal{H}_\infty$ error (Glover)

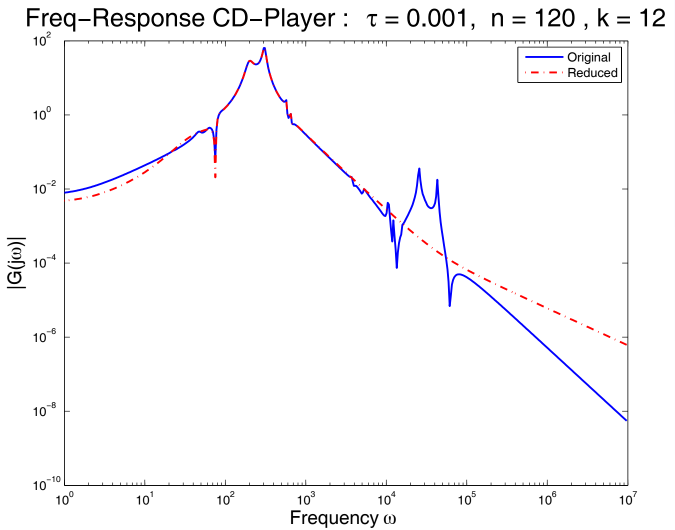
  $\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values}) \|u\|_2$
- Extends to MIMO
- Preserves Stability

Key Challenge

- Approximately solve large scale Lyapunov Equations
  in <u>Low Rank Factored Form</u>

CD Player Frequency Response

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$



CD Player Frequency Response

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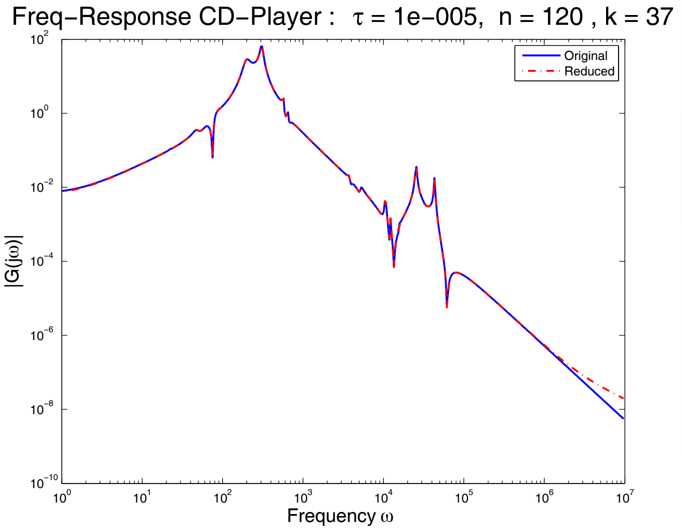
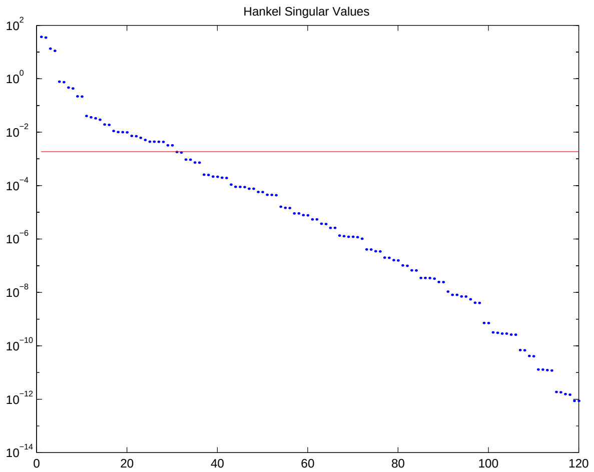


CD Player - Hankel Singular Values $\sqrt{\lambda(\mathcal{P}\mathcal{Q})}$

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$



Approximate Balancing

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

- Sparse Case: Iteratively Solve in Low Rank Factored Form,

$$\mathcal{P} \approx \mathbf{U}_k\mathbf{U}_k^T, \quad \mathcal{Q} \approx \mathbf{L}_k\mathbf{L}_k^T$$

$$[\mathbf{X}, \mathbf{S}, \mathbf{Y}] = \mathrm{svd}(\mathbf{U}_k^T\mathbf{L}_k)$$

$$\mathbf{W}_k = \mathbf{L}\mathbf{Y}_k\mathbf{S}_k^{-1/2} \text{ and } \mathbf{V}_k = \mathbf{U}\mathbf{X}_k\mathbf{S}_k^{-1/2}.$$

Now: $\mathcal{P}\mathbf{W}_k \approx \mathbf{V}_k\mathbf{S}_k$ and $\mathcal{Q}\mathbf{V}_k \approx \mathbf{W}_k\mathbf{S}_k$

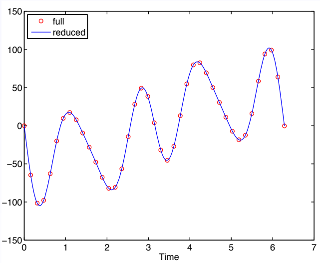
Recent Progress LTI MOR

- Low Rank Approximate Solutions to Lyapunov Eqns
  n = 1M Now Possible – Large Scale BTMOR Possible
- Optimal $\mathcal{H}_2$ reduction – IRKA
  Promising for Large no. Inputs
- Descriptor Systems
  e.g. Stykel (LAA 06) – general theory and approach

Balanced Truncation MOR of Oseen Eqn.

Semi-Discrete Oseen Equations: A Descriptor System

$$\mathbf{E}\frac{d}{dt}\mathbf{v}(t) = \mathbf{A}\mathbf{v}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathbf{C}\mathbf{v}(t) + \mathbf{D}\mathbf{u}(t)$$



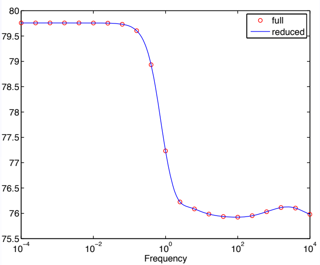


Figure: Time response (left) and frequency response (right) for the full order model (circles) and for the reduced order model (solid line).

Velocity Profile

full 22K dof | reduced 15 dof

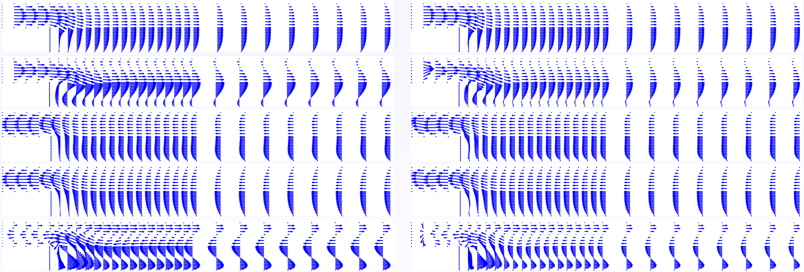


Figure: Velocities generated with the full order model (left column) and with the reduced order model (right column) at $t = 1.0996, 2.9845, 3.7699, 4.86965, 6.2832$ (top to bottom).

Heinkenschloss, Sun, S., SISSC 08

NOTICE

State Variables will be $\mathbf{y}$ for remainder

Reduced Order Neural Modeling

Steve Cox   Tony Kellems

LINEAR MODELS

Balanced Truncation   Optimal $\mathcal{H}_2$

Nan Xiao and Derrick Roos   Ryan Nong

Complex Model (Dim 160 K) $\rightarrow$ 20 variable ROM

NONLINEAR MODELS

Empirical Interpolation (EIM) - Patera

Saifon Chaturantabut   T. Kellems

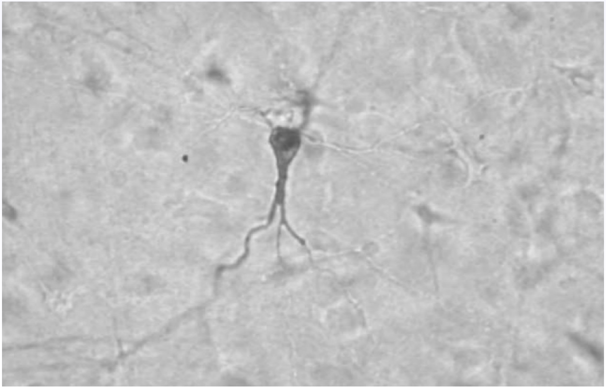
Nonlinear H-H Neuron Model (Dim 1198) $\rightarrow$ 30 variable ROM

Complex nonlinear behavior well approximated

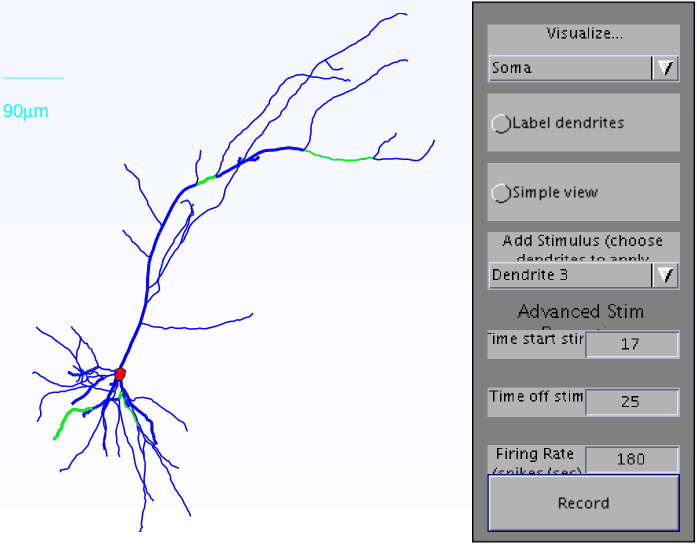
Neuron Image AR-1-20-04-A

Image from Neuromart
J.O. Martinez, Rice-Baylor Archive of Neuronal Morphology,
http://www.caam.rice.edu/ cox/neuromart/, accessed 29 July 08

“Developed” using Neurolucida and NeuroExplorer



Neuron Cell


90µm
Visualize...
Soma
Label dendrites
Simple view
Add Stimulus (choose
Dendrite 3
Advanced Stim
ime start stir
17
Time off stim
25
Firing Rate
180
Record

Hodgkin-Huxley Neuron Model

Full Non-Linear Model

$I_{j,syn}$ is the synaptic input into branch j

$$\begin{aligned} \frac{a_j}{2R_i}\partial_{xx}v_j = C_m\partial_t v_j &+ G_{Na}m_j^3 h_j(v_j - E_{Na}) \\ &+ G_K n_j^4(v_j - E_K) + G_l(v_j - E_l) + I_{j,syn}(x,t) \end{aligned}$$

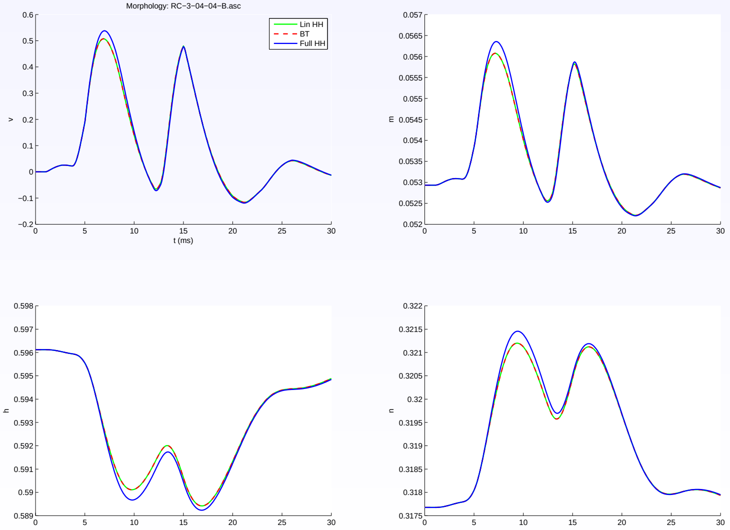
Kinetics of the potassium (n) and sodium (h, m) channels

$$\begin{aligned} \partial_t m_j &= \alpha_m(v_j)(1 - m_j) - \beta_m(v_j)m_j \\ \partial_t h_j &= \alpha_h(v_j)(1 - h_j) - \beta_h(v_j)h_j \\ \partial_t n_j &= \alpha_n(v_j)(1 - n_j) - \beta_n(v_j)n_j. \end{aligned}$$

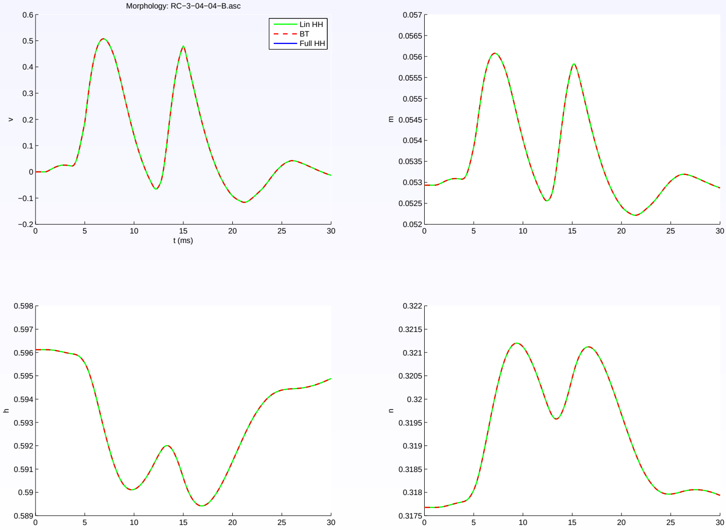
Ultimate Goal for Neural Modeling

- Ultimate goal is to simulate a few-Million neuron system over a minute of brain-time: Feasibility Demonstrated
- Currently limited to a 10K neuron system over a few brain-seconds without new technology
- Single Cell Simulation Time Reduction: 100 - 1000 times
- ROM computation time: seconds - few minutes
- 20 - 30 neuron types sufficient for Cortex
- Parallel computing required - under development(Kellems)

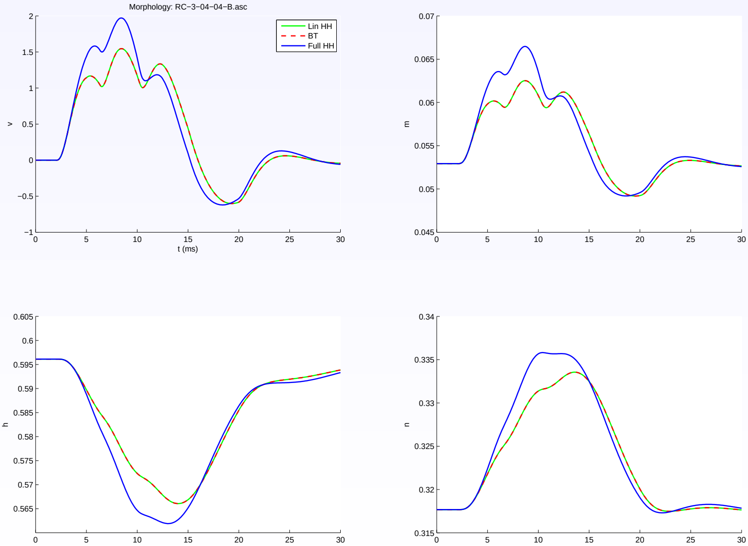
Cell Response - Lin and Non-Lin



Cell Response - Linear



Cell Response - Near Threshold



Optimal $\mathcal{H}_2$ Methods: IRKA

PROBLEM:

Many inputs $\Rightarrow$ Controllability Gramian Not Low Rank

Kellems and Nong

Using Variant of IRKA

Gugercin, Antoulas, Beattie (2008)

Reduced 160K Neuron Model to a ROM of order 20.

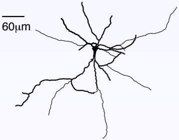
Solve times to construct ROM are under 2 mins
High End Workstation(Sun Ultra 20) in MATLAB

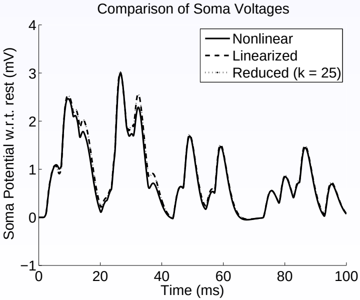
Parameter Study Experiment 158 hrs (full) $\rightarrow$ 3.4 hrs (ROM)

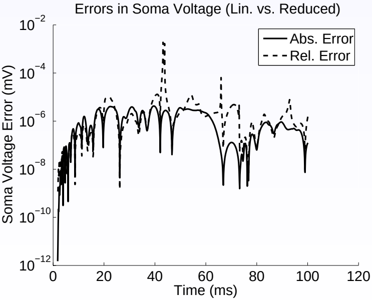
ROM Results on Realistic Neuron

AR-1-20-04-A (Rosenkranz lab)

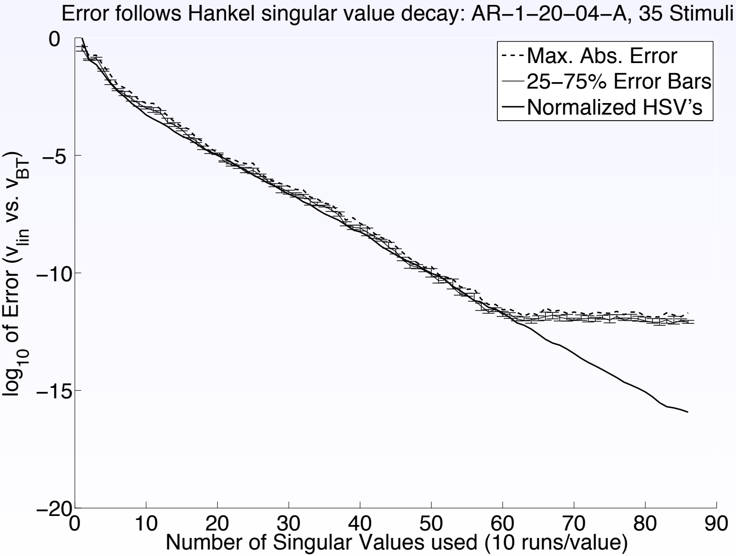
Full system size 6726 Reduced system size 25







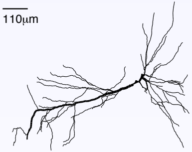
Error vs HS-Values: AR-1-20-04-A

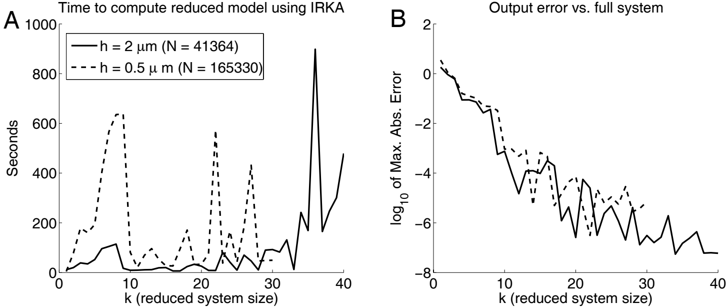


Highly Branched Neuron

n408 (Pyapali et al., 1998)

Full system size	41,364	165,330
Reduced system size	20	20





Model Reduction of Nonlinear Terms

Saifon Chaturantabut

Implemenation of EIM method of Patera et.al. (2004)

Test Case: FitzHugh-Nagumo equations

$$\begin{aligned} \varepsilon v_t(x,t) &= \varepsilon^2 v_{xx}(x,t) + f(v(x,t)) - w(x,t) \\ w_t(x,t) &= \beta v(x,t) - \gamma w(x,t), \end{aligned}$$

$$f(v) = v(v-0.1)(1-v)$$

IC's and BC's

$$\begin{aligned} v(x,0) = &\quad 0, \quad w(x,0) = 0 \qquad x \in [0,L] \\ v_x(0,t) = &\quad -i_0(t), \quad v_x(L,t) = 0 \qquad t \geq 0 \end{aligned}$$

After FEM discretization,

$$\mathbf{E}\mathbf{y}_t = \mathbf{A}\mathbf{y} + \mathbf{g}(t) + \mathcal{N}(\mathbf{y}), \quad \mathbf{y}(0) = 0$$

Problem with Direct POD

If we apply the POD basis directly to construct a discretized system, the original system of order N:

$$\mathbf{E}\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(t) + \mathcal{N}(\mathbf{y}(t))$$

become a system of order $k \ll N$:

$$\tilde{\mathbf{E}}\frac{d}{dt}\tilde{\mathbf{y}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{y}}(t) + \tilde{\mathbf{g}}(t) + \tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)),$$

where the **nonlinear term :**

$$\boxed{\tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)) = \underbrace{\mathbf{U}^T}_{k\times N}\underbrace{\mathcal{N}(\mathbf{U}\tilde{\mathbf{y}}(t))}_{N\times 1}}$$

$\Rightarrow$ Computational Complexity still depends on N!!

Nonlinear Approximation via EIM

WANT:

$$\boxed{\tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)) \leftarrow \underbrace{\mathbf{C}}_{k \times n_m} \underbrace{\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t))}_{n_m \times 1}} \dashrightarrow \boxed{k, n_m \ll N}$$

Complexity k Independent of N

EIM Steps

$$\mathbf{E}\mathbf{y}_t = \mathbf{A}\mathbf{y} + \mathbf{g}(t) + \mathcal{N}(\mathbf{y}), \quad \mathbf{y}(0) = 0$$

1) Run trajectory and collect snapshots $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots \mathbf{y}_m]$ perhaps with several different inputs and parameter values.

2) Truncate SVD of snapshots to get a POD basis for Trajectory

3) Collect nonlinear snapshots $\mathbf{s}_j = f(\mathbf{y}_j) = [f(y_{1,j}), f(y_{2,j}), \ldots, f(y_{N,j})]^T$

4) Truncate SVD of nonlinear snapshots to get another POD basis for the nonlinear term

5) Select EIM interpolation points and approximate nonlinear term via collocation in the non-linear POD basis

6) Construct the nonlinear ROM from the reduced linear and nonlinear terms

Nonlinear Approximation via EIM Contd.

HOW:

$$\mathbf{C}\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t)) = \underbrace{\left(\mathbf{U}^T \int_\Omega [\Psi(x)]^T \mathcal{Q}(x)\mathrm{d}x\right)}_{\mathbf{C}} \underbrace{\left(\mathcal{Q}_{\mathbf{z}}^{-1} f_{\mathbf{z}}(t)\right)}_{\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t))}$$

where

- $f_{\mathbf{z}}(t) = [f(\Phi(z_1)\tilde{\mathbf{y}}(t)), f(\Phi(z_2)\tilde{\mathbf{y}}(t)), \ldots f(\Phi(z_{n_m})\tilde{\mathbf{y}}(t))]^T$
- $\{z_j\}$ Empirical Interpolation Points

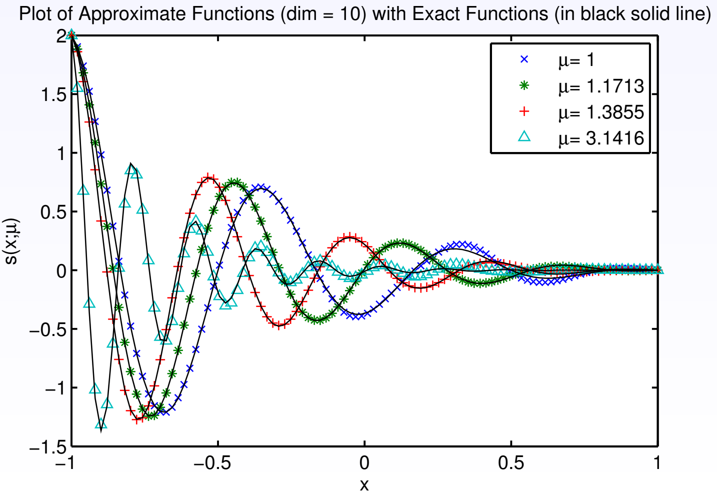
and where

- $\Psi(x) = [\psi_1(x), \psi_2(x), \ldots \psi_N(x)]$ - FEM basis
- $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots \phi_k(x)]$ - POD basis
  - From Snapshots $\Psi(x)\mathbf{y}(t_\ell)$
- $\mathcal{Q}(x) = [q_1(x), q_2(x), \ldots q_{n_m}(x)]$ - Nonlinear POD basis
  - From Snapshots $\mathbf{s}_\ell = f(\Psi(x)\mathbf{y}(t_\ell))$

EIM: Numerical Example

$$f(y;\mu) = (1-y)cos(3\pi\mu(y+1))e^{-(1+y)\mu},$$

6 POD basis fns from 50 snapshots $\mu \in [1,\pi]$ uniform



EIM Reduction of FitzHugh-Nagumo Fiber

click figure for movie

EIM Reduction of FitzHugh-Nagumo Fiber

Nonlinear Reduction $N = 1024 \rightarrow k = 40$

click figure for movie

EIM for Hodgkin-Huxley Equations

Kellems

After FEM or FD discretization HH-equations yield ODE system

$$\begin{bmatrix} a(x)C_m\mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} v_t \\ m_t \\ h_t \\ n_t \end{bmatrix} = \begin{bmatrix} \frac{1}{2R_i}\mathbf{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ m \\ h \\ n \end{bmatrix} + \begin{bmatrix} g_0(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{N}^v(v, m, h, n) \\ \mathcal{N}^m(v, m) \\ \mathcal{N}^h(v, h) \\ \mathcal{N}^n(v, n) \end{bmatrix}$$

succinctly written as $\mathbf{E}\mathbf{y}_t = \mathbf{A}\mathbf{y} + \mathbf{g}(t) + \mathcal{N}(\mathbf{y})$

Implementation Issues

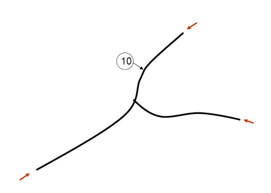
1) Choose input stimuli to generate snapshots over full wave

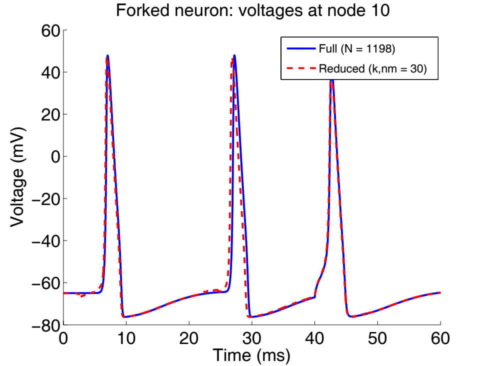
2) NL-snapshots generated for nonlinear term $\mathcal{N}^v(v, m, h, n)$

3) Terms $\mathcal{N}^m(v, m), \mathcal{N}^h(v, h), \mathcal{N}^n(v, n)$ evaluated only at EIM points

EIM Reduction of Hodgkin-Huxley Fiber

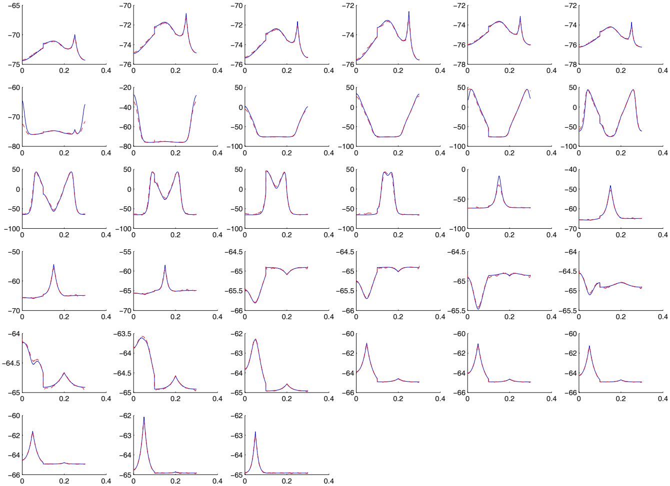
Three Inputs



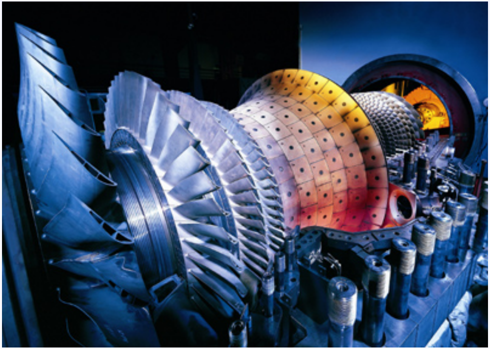


EIM Reduction of HH : 3 inputs

Voltage Profile at Various Times (Full vs ROM)



Blade Variation → Turbine Reliability/Performance



Siemens press picture

Small variations among blades ⇒ large impact on forced response

Determines high-cycle fatigue properties

Parametric ROM for Geometric Shape Variations

T. Bui-Thanh, , K. Willcox, and O. Ghattas 2007

Efficient reduced order models for probabilistic analysis of the effects of blade geometry

2D problem governed by the Euler equations

Three orders of magnitude reduction in number of states

Accurately reproduce CFD Monte Carlo simulation results at fraction of computational cost.

Results impossible without ROM

Euler Equations $\longrightarrow$ DG Discretization

Mathematical Model governed by 2D Euler Equations

DG + BC's $\Rightarrow$

$$\mathbf{E}(\mathbf{w})\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} + \mathbf{R}(\mathbf{y}, \mathbf{u}; \mathbf{w}) = 0$$

$\mathbf{R}$ nonlinear function of $\mathbf{y}, \mathbf{u}; \mathbf{w}$

$\mathbf{u} \in \mathbb{R}^m$ m-external forcing inputs via BC's

$\mathbf{w}$ - random vector associated with geometric variability

Parametrically Dependent Linear Model

Linearize about steady state $\mathbf{y}_{ss}(g(\mathbf{w}))$ corresponding to geometry $g(\mathbf{w})$
$\mathbf{y} = \mathbf{y}_{ss} + \tilde{\mathbf{y}}$

$$\begin{aligned} \mathbf{E}(\mathbf{w})\frac{\mathrm{d}\tilde{\mathbf{y}}}{\mathrm{d}t} &= \mathbf{A}(\mathbf{w})\tilde{\mathbf{y}} + \mathbf{B}(\mathbf{w})\mathbf{u} \\ \mathbf{z} &= \mathbf{C}(\tilde{\mathbf{y}}) \end{aligned}$$

Linearize $\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$:

$$\text{e.g.} \quad \mathbf{A}(\mathbf{w}) \approx \mathbf{A}_0 + \mathbf{A}_1 w_1 + \mathbf{A}_2 w_2 + \ldots \mathbf{A}_m w_m$$

One time off-line cost: nominal geometry base

Construct POD Basis → ROM

Adaptive Sampling Method

Starting with existing POD basis Φ, construct ROM
Then iterate:

1) Select new parameter value $\mathbf{w}_j$ via

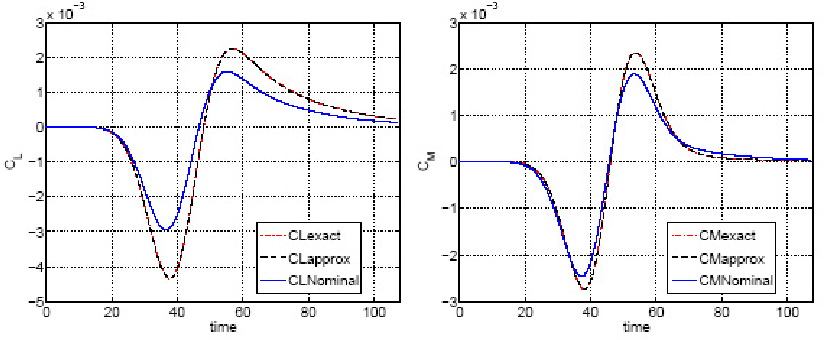
$$\mathbf{w}_j = argmax_{\mathbf{w}} \|\mathbf{z}(\mathbf{w}) - \mathbf{z}_r(\mathbf{w})\| \text{s.t. PDE Constraints}$$

2) Collect additional snapshots $\mathbf{y}(t; \mathbf{w}_j)$ for $\mathbf{w} = \mathbf{w}_j$

3) Update POD basis Φ

5) Project $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i$ via e.g. $\hat{\mathbf{A}}_i = \Phi^T \mathbf{A}_i \Phi$

ROM vs Full Linear Response

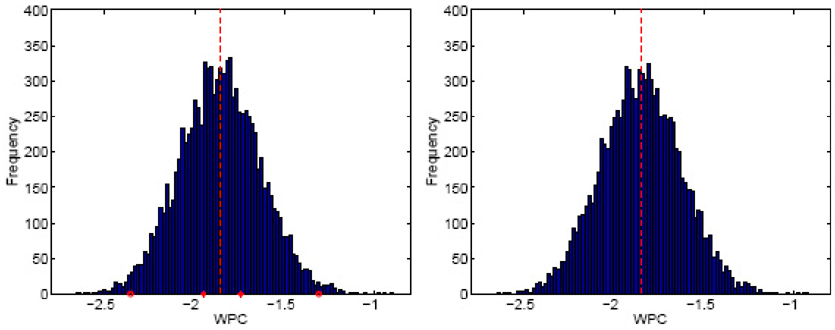


(a) (b)

Outputs:

$$\mathbf{z}(1) = C_L \text{ Lift Coefficient}$$
$$\mathbf{z}(2) = C_M \text{ Moment Coefficient}$$

ROM Preservation of Statistics



(a) Full WPC for Blade 1

(b) Reduced WPC for Blade 1

Linearized CFD (left)

ROM predictions of work per cycle (WPC)

Monte Carlo simulation results for 10,000 blade geometries

Prediction and Cost

	CFD (Full)	Reduced
Model Size	103,008	290
Number nonzeros	2,846,056	84,100
Offline cost	—	10.92 hrs
Online cost	515.61 hrs	1.10 hrs
Blade 1 WPC mean	-1.8583	-1.8515
Blade 1 WPC variance	0.0503	0.0506
Blade 2 WPC mean	-1.8599	-1.8583
Blade 2 WPC variance	0.0136	0.0138

10 random parameters

Computations on 64 bit PC with 3.2GH Pentium Processor

WPC = integral of the blade motion times the lift force over one unsteady cycle

Summary

Gramian Based Model Reduction: POD, Balanced Reduction

Optimal $\mathcal{H}_2$ Reduction via IRKA

Neural Modeling - Single Cell ROM $\Rightarrow$ Many Interactions
Example of important class of problems
(including Monte Carlo of Stochastic Systems)

Nonlinear MOR via EIM
Demonstrated effective reduction and feasibility

Process Variation - Blade Geometry Effects
Example of important future application of MOR