VIGRE STRING SEMINAR · SPRING 2009 · WEEK 6 (FEB 17)

STOKES' PARADOX

Consider the Stokes flow problem

$$\begin{aligned} \Delta \operatorname{curl} \mathbf{v} &= 0 && \text{in } \mathbb{R}^3 \setminus \mathbf{C} \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \mathbb{R}^3 \setminus \mathbf{C} \\ \mathbf{v} &= 0 && \text{on } \partial \mathbf{C} \\ \mathbf{v} &\to \mathbf{u} && \text{as } r \to \infty \end{aligned}$$

where $\mathbf{C} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq \ell^2\}$ is the infinite cylinder of radius ℓ with the x_3 axis running through its center, and where $\mathbf{u} = (u, 0, 0)$ is a given constant velocity field. Stokes' paradox is that there is no solution to this problem.

To prove this, we appeal to the stream function for this flow. Since $\operatorname{div} \mathbf{v} = 0$, $\mathbf{v} = \operatorname{curl} \mathbf{w}$ for some vector field $\mathbf{w}$. That is,

$$(v_1, v_2, 0) = (\partial_{x_2} w_3 - \partial_{x_3} w_2, \partial_{x_3} w_1 - \partial_{x_1} w_3, \partial_{x_1} w_2 - \partial_{x_2} w_1).$$

By symmetry, $\mathbf{v}$ and $\mathbf{w}$ clearly should not depend on x_3. Letting $\psi(x_1, x_2) = w_3(x_1, x_2)$, we have

$$(v_1, v_2) = (\partial_{x_2} \psi, -\partial_{x_1} \psi).$$

ψ is the *stream function.* Its level curves are called *stream lines*, as they are the integral curves of the ODE $\mathbf{x}'(t) = \mathbf{v}(\mathbf{x}(t))$.

The Stokes flow problem above can be posed in terms of the stream function. Since $\operatorname{curl} \mathbf{v} = \operatorname{curl} \operatorname{curl} \mathbf{w} = \operatorname{grad}(\operatorname{div} \mathbf{w}) - \Delta \mathbf{w} = 0 - \Delta \mathbf{w} = (0, 0, \Delta \psi)$, the problem becomes

$$\begin{cases} \Delta^2 \psi = 0 & (r > \ell) \\ \dfrac{\partial}{\partial \theta} \psi = 0 & (r = \ell) \\ \dfrac{\partial}{\partial r} \psi = 0 & (r = \ell) \\ \left| \dfrac{\partial}{\partial r} \psi - u \sin \theta \right| \to 0 & \text{as } r \to \infty \\ \left| \dfrac{1}{r} \dfrac{\partial}{\partial \theta} \psi - u \cos \theta \right| \to 0 & \text{as } r \to \infty \end{cases} \tag{1}$$

Now, suppose ψ is a solution to (1). ψ admits a Fourier decomposition

$$\begin{aligned}\psi(r,\theta) &= \sum_{n\in\mathbb{Z}} \widehat{\psi}_n(r)e^{in\theta},\\ \widehat{\psi}_n(r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(r,\theta)e^{-in\theta}\,\mathrm{d}\theta.\end{aligned}$$

If there is any justice world, we should be able to carry the operator Δ^2 under the sum:

$$\Delta^2\psi(r,\theta) = \sum_{n\in\mathbb{Z}} \Delta^2\Big(\widehat{\psi}_n(r)e^{in\theta}\Big).$$

In polar coordinates, $\Delta^2 = \big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\big)^2$. After some work, we find

$$\begin{aligned}\Delta^2\Big(\widehat{\psi}_n(r)e^{in\theta}\Big) &= \left[\left(\partial_r^4 + \frac{2}{r}\partial_r^3 + \frac{1+2n^2}{r^2}\partial_r^2 + \frac{1+2n^2}{r^3}\partial_r + \frac{n^4-4n^2}{r^4}\right)\widehat{\psi}_n(r)\right]e^{in\theta}\\ &=: \big[\mathcal{L}_n\widehat{\psi}_n(r)\big]e^{in\theta}\end{aligned}$$

(I used Maple to make sure I made no mistakes.) Therefore,

$$0 = \Delta^2\psi(r,\theta) = \sum_{n\in\mathbb{Z}} \big[\mathcal{L}_n\widehat{\psi}_n(r)\big]e^{in\theta}.$$

But the Fourier coefficients of the zero function are all zero, and so for every integer n,

$$\mathcal{L}_n\widehat{\psi}_n(r) = 0, \quad r > \ell.$$

The general solution of this linear fourth order ODE is

$$\widehat{\psi}_{\pm n}(r) = \begin{cases} A_{\pm n}r^{-n} + B_{\pm n}r^{2-n} + C_{\pm n}r^n + D_{\pm n}r^{n+2} & \text{for } n \geq 2,\\ A_{\pm 1}r^{-1} + B_{\pm 1}r + C_{\pm 1}r\log r + D_{\pm 1}r^3 & \text{for } n = 1,\\ A_0 + B_0\log r + C_0r^2\log r + D_0r^2 & \text{for } n = 0.\end{cases}$$

(I used Maple to find these, but you can easily check that they are correct). Because of the conditions on ψ as $r\to\infty$, every C_n and D_n must be zero, leaving

$$\widehat{\psi}_{\pm n}(r) = \begin{cases} A_{\pm n}r^{-n} + B_{\pm n}r^{2-n} & \text{for } n \geq 2,\\ A_{\pm 1}r^{-1} + B_{\pm 1}r & \text{for } n = 1,\\ A_0 + B_0\log r & \text{for } n = 0.\end{cases}$$

The boundary conditions give

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \widehat{\psi}_n(\ell)ine^{in\theta} &= 0,\\ \sum_{n\in\mathbb{Z}} \widehat{\psi}_n'(\ell)e^{in\theta} &= 0.\end{aligned}$$

As a result,

$$\begin{aligned}A_{\pm n}\ell^{-n} + B_{\pm n}\ell^{2-n} &= 0 \quad \text{for } n \geq 2,\\ A_{\pm 1}\ell^{-1} + B_{\pm 1}\ell &= 0 \quad \text{for } n = 1,\\ 0 &= 0 \quad \text{for } n = 0,\end{aligned}$$

and

$$\begin{aligned} -nA_{\pm n}\ell^{-n-1} + (2-n)B_{\pm n}\ell^{1-n} &= 0 \quad \text{for } n \geq 2, \\ -A_{\pm 1}\ell^{-2} + B_{\pm 1} &= 0 \quad \text{for } n = 1, \\ B_0\ell^{-1} &= 0 \quad \text{for } n = 0. \end{aligned}$$

These equations are satisfied only when $A_n = B_n = 0$ for all n, except for A_0, which can be any constant. This means $\psi \equiv A_0$ is a constant function. But a constant function cannot possibly satisfy the asymptotic conditions as $r \to \infty$! And that is the paradox.