Memo WAV.4: Beaded Strings with Symmetric Configurations

In these notes we elaborate the claim of Gantmacher & Krein that relates the spectrum of a beaded string to the spectra of two problems on the left half of the string.

▶ Symmetric beaded strings

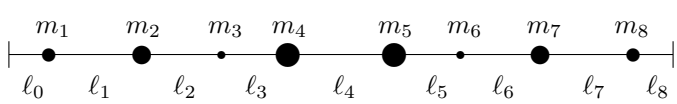
We consider massless strings of length 2ℓ loaded with $2n$ beads whose placement and masses obey a symmetric configuration. That is, we assume that

$$m_j = m_{2n-j+1}, \qquad j = 1, \ldots, n$$

and

$$\ell_j = \ell_{2n-j}, \quad j = 0, \ldots, n.$$

The illustration below shows an example with $n = 4$.



Let $y_j(t)$ denote the transverse displacement of the mass m_j at time t.

The formulas for the kinetic and potential energy of this string under constant tension σ take the form

$$T = \sum_{j=0}^{2n} \frac{m_j}{2} \dot{y}_j^2$$

and

$$V = \frac{\sigma}{2} \sum_{j=0}^{2n} \frac{1}{\ell_j} (y_{j+1} - y_j)^2,$$

where $y_0 = y_{2n+1} = 0 = \dot{y}_0 = \dot{y}_{n+1}$ since the ends are fixed.

We can expand V to obtain

$$V = \sum_{j=1}^{2n} \frac{\sigma}{2} \Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j} \Big) y_j^2 - 2 \sum_{j=1}^{2n-1} \frac{\sigma}{2} \Big(\frac{1}{\ell_j} \Big) y_j y_{j+1}.$$

Notice that T and V fit the template of a Sturm system:

$$T = \sum_{j=1}^{2n} c_j \dot{y}_j^2, \qquad V = \sum_{j=1}^{n} a_j y_j^2 - 2 \sum_{j=1}^{2n-1} b_j y_j y_{j+1}$$

with coefficients

$$a_j = \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big), \qquad b_j = \frac{\sigma}{2}\Big(\frac{1}{\ell_j}\Big), \qquad c_j = \frac{m_j}{2},$$

and from them build the matrices

$$M = \begin{pmatrix} c_1 & & & & & \\ & \ddots & & & & \\ & & c_n & & & \\ & & & c_{n+1} & & \\ & & & & \ddots & \\ & & & & & c_{2n} \end{pmatrix}$$

and

$$K = \begin{pmatrix} a_1 & -b_1 & & & & & & \\ -b_1 & a_2 & \ddots & & & & & \\ & \ddots & \ddots & -b_{n-1} & & & & \\ & & -b_{n-1} & a_n & -b_n & & & \\ & & & -b_n & a_{n+1} & -b_{n+1} & & \\ & & & & -b_{n+1} & \ddots & \ddots & \\ & & & & & \ddots & a_{2n-1} & -b_{2n-1} \\ & & & & & & -b_{2n-1} & a_{2n} \end{pmatrix}.$$

The Euler–Lagrange equations describe the evolution of this system according to the differential equation

$$M\ddot{y} + Ky = 0.$$

Substitute the ansatz

$$y(t) = u\sin(pt + \alpha) \tag{1}$$

into this differential equation and simplify to find that solutions of the form (1) exist provided

$$Ku = p^2 Mu.$$

We can convert this into a standard eigenvalue problem by premultiplying by $M^{-1/2}$:

$$M^{-1/2}KM^{-1/2}M^{1/2}u = p^2 M^{1/2}u.$$

The matrix $M^{-1/2}$ always exists and is trivial to compute:

$$M^{-1/2} = \operatorname{diag}(1/\sqrt{c_1}, \ldots, 1/\sqrt{c_{2n}}).$$

Relabeling variables

$$\mathcal{A} = M^{-1/2}KM^{-1/2}, \quad v = M^{1/2}u, \quad \lambda = p^2$$

we arrive at the standard eigenvalue problem

$$\mathcal{A}v = \lambda v.$$

Notice that $M^{-1/2} = (M^{-1/2})^T$, so $\mathcal{A}$ is a symmetric matrix. It inherits the structure of K,

$$\mathcal{A} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & & \\ \beta_1 & \alpha_2 & \ddots & & & & & \\ & \ddots & \ddots & \beta_{n-1} & & & & \\ & & \beta_{n-1} & \alpha_n & \beta_n & & & \\ & & & \beta_n & \alpha_{n+1} & \beta_{n+1} & & \\ & & & & \beta_{n+1} & \ddots & \ddots & \\ & & & & & \ddots & \alpha_{2n-1} & \beta_{2n-1} \\ & & & & & & \beta_{2n-1} & \alpha_{2n} \end{pmatrix}$$

with coefficients

$$\alpha_j = c_j a_j, \qquad \beta_j = -\sqrt{c_j c_{j+1}}\, b_j.$$

The symmetric configuration of the string allows us to simplify the entries of $\mathcal{A}$. Since $m_{2n-j+1} = m_j$, we have

$$c_{2n-j+1} = c_j = \frac{m_j}{2}, \qquad j = 1, \ldots, n,$$

and since $\ell_{2n-j} = \ell_j$,

$$b_{2n-j} = b_j = \frac{\sigma}{2\ell_j}, \qquad j = 1, \ldots, n-1$$

and

$$a_{2n-j+1} = a_j = \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big), \qquad j = 1, \ldots, n.$$

Likewise, we have

$$\alpha_{2n-j+1} = c_{2n-j+1} a_{2n-j+1} = c_j a_j = \alpha_j, \qquad j = 1, \ldots, n$$

and

$$\beta_{2n-j} = -\sqrt{c_{2n-j} c_{2n-j+1}}\, b_{2n-j} = -\sqrt{c_{j+1} c_j}\, b_j = \beta_j, \qquad j = 1, \ldots, n-1.$$

Notice that we have said nothing about $b_n = \sigma/2\ell_n$ or β_n, as ℓ_n corresponds to the central segment of the string.

Having identified the relationships between the coefficients corresponding to

each end of the symmtric string, we can write

$$\mathcal{A} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & & \\ \beta_1 & \alpha_2 & \ddots & & & & & \\ & \ddots & \ddots & \beta_{n-1} & & & & \\ & & \beta_{n-1} & \alpha_n & \beta_n & & & \\ & & & \beta_n & \alpha_n & \beta_{n-1} & & \\ & & & & \beta_{n-1} & \ddots & \ddots & \\ & & & & & \ddots & \alpha_2 & \beta_1 \\ & & & & & & \beta_1 & \alpha_1 \end{pmatrix}.$$

We can now partition this $2n \times 2n$ matrix into $n \times n$ blocks,

$$\mathcal{A} = \begin{pmatrix} A & B^* \\ B & JAJ \end{pmatrix},$$

where

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}, \qquad B = \beta_n e_1 e_n^*,$$

e_k denotes the kth column of the $k \times k$ identity matrix, and J denotes the *flip matrix* (or *anti-identity*)

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Now we are prepared to see how the spectrum for this symmetric string with $2n$ beads is equivalent to the union of the spectra of two n-bead configurations, one pinned at both ends, the other with right end free.

▶ Forward problem for the n-bead string

We shall now consider the behavior of the left half of the string subject to two different boundary conditions. We begin by recalling the general formulae for the kinetic and potential energy of a string of length ℓ with n beads:

$$T_\ell = \sum_{j=1}^{n} \frac{m_j}{2} \dot{y}_j^2$$

and

$$V_\ell = \frac{\sigma}{2} \sum_{j=0}^{n} \frac{1}{\ell_j} (y_{j+1} - y_j)^2.$$

This string differs slightly from the left half of our 2ℓ-length string with $2n$ beads, in that when we cut our string in half, the middle segment ℓ_n is cut in half. As a consequence, ℓ_n in the formulae for the standard n-bead string must be replaced by $\ell_n/2$ in our setting.

We shall consider two different configurations of our half-string of length ℓ. First, we shall fix both the right and left ends, giving

$$y_0 = y_{n+1} = 0.$$

We shall distinguish this case by adding hats to all related quantities. With these values for the end displacements and a rightmost segment of length $\ell_n/2$, the kinetic energy remains the same

$$\widehat{T} = \sum_{j=1}^{n} \frac{m_j}{2} \dot{y}_j^2$$

and the potential energy becomes

$$\widehat{V} = \sum_{j=1}^{n-1} \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big) y_j^2 + \frac{\sigma}{2}\Big(\frac{1}{\ell_{n-1}} + \frac{2}{\ell_n}\Big) y_n^2 - 2\sum_{j=1}^{n-1} \frac{\sigma}{2}\Big(\frac{1}{\ell_j}\Big) y_j y_{j+1}.$$

From these expressions we can construct the entries of the mass and stiffness matrices $\widehat{M}$ and $\widehat{K}$:

$$\begin{aligned}
\widehat{a}_j &= \frac{\sigma}{2}\left(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\right), && j = 1, \ldots, n-1, \\
\widehat{a}_n &= \frac{\sigma}{2}\Big(\frac{1}{\ell_{n-1}} + \frac{2}{\ell_n}\Big), && \\
\widehat{b}_j &= \frac{\sigma}{2}\left(\frac{1}{\ell_j}\right), && j = 1, \ldots, n-1, \\
\widehat{c}_j &= \frac{m_j}{2}, && j = 1, \ldots, n.
\end{aligned}$$

Next, suppose that the left end of our half-string is fixed, $y_0 = 0$, but the right end varies freely with y_n;

$$y_{n+1} = y_n.$$

We distinguish this case by adding bars to the relevant quantities. Now these conditions (with rightmost segment of length $\ell_n/2$) give the same kinetic energy

$$\overline{T} = \sum_{j=1}^{n} \frac{m_j}{2} \dot{y}_j^2$$

but potential energy

$$\overline{V} = \sum_{j=1}^{n-1} \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big) y_j^2 + \frac{\sigma}{2}\Big(\frac{1}{\ell_{n-1}}\Big) y_n^2 - 2\sum_{j=1}^{n-1} \frac{\sigma}{2}\Big(\frac{1}{\ell_j}\Big) y_j y_{j+1}.$$

These expressions lead to the matrix entries

$$\begin{aligned}
\overline{a}_j &= \frac{\sigma}{2}\left(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\right), && j = 1, \ldots, n-1, \\
\overline{a}_n &= \frac{\sigma}{2}\Big(\frac{1}{\ell_{n-1}}\Big), \\
\overline{b}_j &= \frac{\sigma}{2}\left(\frac{1}{\ell_j}\right), && j = 1, \ldots, n-1, \\
\overline{c}_j &= \frac{m_j}{2}, && j = 1, \ldots, n.
\end{aligned}$$

Notice that these changes to the ends have caused only modest alterations to our matrix coefficients:

$$\begin{aligned}
\overline{a}_j &= \widehat{a}_j = \overline{a}_j, && j = 1, \ldots, n-1, \\
\overline{b}_j &= \widehat{b}_j = \overline{b}_j, && j = 1, \ldots, n-1, \\
\overline{c}_j &= \widehat{c}_j = \overline{c}_j, && j = 1, \ldots, n;
\end{aligned}$$

only the values of a_n, $\widehat{a}_n$, and $\overline{a}_n$ differ.

▶ Connection between eigenvalues of full and half strings

The vibrations of the two half-length strings can be understood through the generalized eigenvalue problems

$$\widehat{K}\widehat{u} = \widehat{\lambda}\widehat{M}\widehat{u}, \qquad \overline{K}\overline{u} = \overline{\lambda}\overline{M}\overline{u},$$

which can be transformed into the standard eigenvalue problems

$$\widehat{A}\widehat{v} = \widehat{\lambda}\widehat{v}, \qquad \overline{A}\overline{v} = \overline{\lambda}\overline{v},$$

where

$$\widehat{A} = \widehat{M}^{-1/2}\widehat{K}\widehat{M}^{-1/2}, \qquad \overline{A} = \overline{M}^{-1/2}\overline{K}\overline{M}^{-1/2}.$$

Owing to the similarities in the a_j, b_j, and c_j values noted above, the matrices $\widehat{A}$ and $\overline{A}$ differ only in their (n, n) entry, and aside from these entries both matrices agree with A, the upper-left $n \times n$ block of the matrix $\mathcal{A}$ for the length 2ℓ string:

$$\widehat{A} = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \widehat{\alpha}_n \end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \overline{\alpha}_n \end{pmatrix},$$

where

$$\widehat{\alpha}_n = c_n\widehat{a}_n = c_n\Big(\frac{\sigma}{2}\Big)\Big(\frac{1}{\ell_{n-1}} + \frac{1}{\ell_n}\Big) = \alpha_n + c_n\Big(\frac{\sigma}{2}\Big)\Big(\frac{1}{\ell_n}\Big)$$

and

$$\overline{\alpha}_n = c_n \overline{a}_n = c_n \Big(\frac{\sigma}{2}\Big)\Big(\frac{1}{\ell_{n-1}}\Big) = \alpha_n - c_n \Big(\frac{\sigma}{2}\Big)\Big(\frac{1}{\ell_n}\Big).$$

Since $c_n = c_{n+1}$ (as $m_n = m_{n+1}$), we have

$$\beta_n = -\sqrt{c_{n+1}c_n}\, b_n = -c_n \Big(\frac{\sigma}{2}\Big)\Big(\frac{1}{\ell_n}\Big),$$

we can write

$$\widehat{\alpha}_n = \alpha_n + \beta_n, \qquad \overline{\alpha}_n = \alpha_n - \beta_n,$$

from which it follows that

$$\widehat{A} = A + \beta_n e_n e_n^*, \qquad \overline{A} = A - \beta_n e_n e_n^*.$$

We are ready to make the connection to the length-2ℓ string. Recall that we could write

$$\mathcal{A} = \begin{pmatrix} A & B^* \\ B & JAJ \end{pmatrix},$$

where

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}, \qquad B = \beta_n e_1 e_n^*,$$

and J denotes the flip matrix

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Define the matrix

$$Q = \tfrac{\sqrt{2}}{2} \begin{pmatrix} I & J \\ I & -J \end{pmatrix},$$

which is unitary since

$$Q^*Q = \tfrac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix} \begin{pmatrix} I & J \\ I & -J \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 2I & 0 \\ 0 & 2J^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

We shall show that Q block-diagonalizes $\mathcal{A}$, revealing the matrices $\widehat{A}$ and $\overline{A}$ for the half-string. Perform a similarity transformation of $\mathcal{A}$ to obtain

$$Q\mathcal{A}Q^* = \begin{pmatrix} A + \frac{1}{2}(JB + B^*J) & \frac{1}{2}(JB - B^*J) \\ \frac{1}{2}(B^*J - JB) & A - \frac{1}{2}(JB + B^*J) \end{pmatrix},$$

which must have the same eigenvalues as $\mathcal{A}$. As $\mathcal{A}$ is symmetric, one can see from the left hand side that this matrix must be as well. Hence the off-diagonal blocks must be equal, yet in the above formula they differ by a sign. This

implies they must be zero. The purely algebraic demonstration of this fact is also interesting: Note that $Je_1 = e_n$, and hence

$$JB = J\beta_n e_1 e_n^* = \beta_n e_n e_n^*, \qquad B^* J = \beta_n e_n e_1^* J = \beta_n e_n e_n^*,$$

whose difference is clearly zero.

Similarly, we can compute

$$JB + B^* J = 2\beta_n e_n e_n^*,$$

and so it follows that

$$Q\mathcal{A}Q^* = \begin{pmatrix} A + \beta_n e_n e_n^* & 0 \\ 0 & A - \beta_n e_n e_n^* \end{pmatrix}.$$

Yet we have seen previously that

$$\widehat{A} = A + \beta_n e_n e_n^*, \qquad \overline{A} = A - \beta_n e_n e_n^*,$$

so we have showed that $\mathcal{A}$ is a unitarily similar to a block diagonal matrix containing the two problems for the half-string as the diagonal blocks. Thus the spectrum of $\mathcal{A}$ equals the union of the spectra of $\widehat{A}$ and $\overline{A}$. Moreover, we can see that the eigenvalues of $\widehat{A}$ must all be at least as large as the corresponding eigenvalues of A, while the eigenvalues of $\overline{A}$ can be no larger than the same eigenvalues of A.

It is also interesting to note that one can write

$$\mathcal{A} = \tfrac{1}{2} \begin{pmatrix} \widehat{A} + \overline{A} & (\widehat{A} - \overline{A})J \\ J(\widehat{A} - \overline{A}) & J(\widehat{A} + \overline{A})J \end{pmatrix}.$$

▶ Points remaining unaddressed

Why do the eigenvalues of $\widehat{A}$ and $\overline{A}$ nest? How can we be sure that $\overline{A}$ remains positive definite?

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