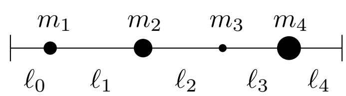
Memo WAV.5: Inverse Problem for the Beaded String

These notes describe how to recover the masses and lengths of a string with n beads given the eigenvalues of that string in two configurations: pinned at both ends, and pinned at the left end but free on the right. We closely follow Gantmacher & Krein (Supplement II).

▶ Construction of fundamental polynomials.

Consider a massless string of length ℓ with n beads with masses $m_1, \dots, m_n$ held at a tension σ.

The illustration below shows an example with $n = 4$.



Note that this set-up differs slightly from the left half of the symmetric string in Memo WAV.4, in that ℓ_4 in that case would be twice as long. We need to make sure we properly account for this difference.

Let $y_j(t)$ denote the transverse displacement of the mass m_j at time t.

The formulas for the kinetic and potential energy of this string under constant tension σ take the form

$$T = \sum_{j=0}^{n} \frac{m_j}{2} \dot{y}_j^2, \qquad V = \frac{\sigma}{2} \sum_{j=0}^{n} \frac{1}{\ell_j} (y_{j+1} - y_j)^2.$$

As we have seen repeatedly, we can write these formulae as

$$T = \sum_{j=1}^{n} c_j \dot{y}_j^2, \qquad V = \sum_{j=1}^{n} a_j y_j^2 - 2 \sum_{j=1}^{2n-1} b_j y_j y_{j+1}$$

with coefficients

$$a_j = \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big), \qquad b_j = \frac{\sigma}{2}\Big(\frac{1}{\ell_j}\Big), \qquad c_j = \frac{m_j}{2}.$$

The Euler–Lagrange equations give $M\ddot{y}(t) + Ky(t) = 0$, where

$$M = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}, \qquad K = \begin{pmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & \ddots & \\ & \ddots & \ddots & -b_{n-1} \\ & & -b_{n-1} & a_n \end{pmatrix}.$$

We look for solutions of the form $y(t) = u\sin(pt+\alpha)$, which leads to the generalized eigenvalue problem $Ku = \lambda Mu$. The jth row of this equation gives

$$-b_{j-1}u_{j-1} + a_j u_j - b_{j+1}u_{j+1} = \lambda c_j u_j,$$

with the convention that $b_0 = b_{n+1} = 0$. Substituting our formulae for the coefficients, we obtain

$$-\frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}}\Big)u_{j-1} + \frac{\sigma}{2}\Big(\frac{1}{\ell_{j-1}} + \frac{1}{\ell_j}\Big)u_j - \frac{\sigma}{2}\Big(\frac{1}{\ell_j}\Big)u_{j+1} = \lambda\frac{m_j}{2}u_j.$$

Replace all ℓ_k by ℓ_k/σ (or simply assume $\sigma = 1$) and rearrange to obtain

$$\frac{u_j - u_{j-1}}{\ell_{j-1}} + \frac{u_j - u_{j+1}}{\ell_j} = \lambda m_j u_j. \tag{1}$$

We always assume that the left end of the string is fixed, giving the boundary condition

$$u_0 = 0.$$

As we can scale the eigenvector u arbitrarily, we shall pick u_1 to be any (nonzero) value we wish. Properties of Sturm systems ensure that we cannot have $u_1 = 0$. For reasons that shall become clear, we will write

$$u_1 = R_0(\lambda)u_1, \qquad R_0(\lambda) = 1.$$

Now we can solve (1) with $j = 1$ to obtain a formula for u_2:

$$\begin{aligned} u_2 &= \Big(1 + \frac{\ell_1}{\ell_0} - \ell_1 m_1\lambda\Big)u_1 \\ &=: R_2(\lambda)u_1. \end{aligned}$$

Note that $R_2 \in \mathcal{P}_1$, where $\mathcal{P}_k$ denotes the space of polynomials of degree n or less. In general, we can use (1) to obtain

$$\begin{aligned} u_{j+1} &= \Big(1 + \frac{\ell_j}{\ell_{j-1}} - \ell_j m_j\lambda\Big)u_{j-1} \\ &= \Big(1 + \frac{\ell_j}{\ell_{j-1}} - \ell_j m_j\lambda\Big)R_{2j-4}(\lambda)u_1 \\ &=: R_{2j}(\lambda)u_1, \end{aligned}$$

where $R_{2j} \in \mathcal{P}_j$. We continue this approach until the last row, $j = n$, in which case

$$u_{n+1} = R_{2n}(\lambda)u_1.$$

If we have a string that is pinned at both ends then $u_0 = u_{n+1} = 0$. The homogeneous Dirichlet boundary condition on the right end of the string means that $u_{n+1} = 0$ for any eigenvalue of the pencil $Ku = \lambda Mu$, and so we must have that the eigenvalues λ are the n roots of $R_{2n} \in \mathcal{P}_n$.

If we can measure the n eigenvalues of this system, then we can build the polynomial R_{2n}, up to a scaling factor. Unfortunately, knowledge of R_{2n} is not enough to recover the values of the masses m_j and lengths ℓ_j.

We shall supplement this information with data from a second set of eigenvalues, measured from a string whose left end is fixed ($u_0 = 0$ again) but whose right end is allowed to slide freely according to the condition $u_{n+1} = u_n$. This 'zero slope' condition over the last segment of the string will prove to be very useful. It may not be easily to experimentally measure, but recall that the symmetric strings explained in Memo WAV.4 with fixed boundaries embed all the data that we require.

To proceed, we construct a second set of polynomials that can be used to determine the slope of the string. The slope between masses m_j and m_{j+1} is simply

$$\frac{u_{j+1} - u_j}{\ell_j} = \frac{R_{2j}(\lambda)u_1 - R_{2j-2}(\lambda)u_1}{\ell_j} =: R_{2j-1}(\lambda)u_1,$$

where

$$R_{2j-1}(\lambda) = \frac{1}{\lambda_j}\big(R_{2j}(\lambda) - R_{2j-2}(\lambda)\big) \tag{2}$$

is a polynomial of degree j or lower. The formula (1) gives a simple recurrence relating the even- and odd-index polynomials,

$$R_{2j-1}(\lambda) = R_{2j-3}(\lambda) + \lambda m_j R_{2j-2}(\lambda),$$

while the definition of the slope polynomials gives the recurrence

$$R_{2j}(\lambda) = \ell_j R_{2j-1}(\lambda) - R_{2j-2}(\lambda).$$

For the string with the zero-slope boundary condition on the right, we must have

$$R_{2n-1}(\lambda)u_1 = 0,$$

and hence λ must be a root of the polynomial $R_{2n-1}(\lambda)$. Thus, knowledge of the n eigenvalues of the string with zero-slope boundary condition on the right allows us to construct the polynomial R_{2n-1} up to a scaling factor.

Now we shall see how knowledge of the polynomials R_{2n-1} and R_{2n} can be used to determine material properties of the beaded string. using the recurrences for the odd- and even-index polynomials, we have

$$
\begin{aligned}
\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)} &= \frac{\ell_n R_{2n-1}(\lambda) + R_{2n-2}(\lambda)}{R_{2n-1}(\lambda)} \\
&= \ell_n + \frac{R_{2n-2}(\lambda)}{R_{2n-1}(\lambda)} \\
&= \ell_n + \cfrac{1}{\cfrac{R_{2n-1}(\lambda)}{R_{2n-2}(\lambda)}} \\
&= \ell_n + \cfrac{1}{\cfrac{m_n\lambda R_{2n-2}(\lambda) + R_{2n-3}(\lambda)}{R_{2n-2}(\lambda)}} \\
&= \ell_n + \cfrac{1}{m_n\lambda + \cfrac{1}{\cfrac{R_{2n-2}(\lambda)}{R_{2n-3}(\lambda)}}} \\
&= \ell_n + \cfrac{1}{m_n\lambda + \cfrac{1}{\cfrac{\ell_{n-1}R_{2n-3}(\lambda) + R_{2n-4}(\lambda)}{R_{2n-3}(\lambda)}}} \\
&\;\vdots \\
&= \ell_n + \cfrac{1}{m_n\lambda + \cfrac{1}{\ell_{n-1} + \cfrac{1}{m_{n-1}\lambda + \cdots + \cfrac{1}{\ell_1 + \cfrac{1}{m_1\lambda + \cfrac{1}{\ell_0}}}}}}.
\end{aligned}
$$

Notice that the rational function R_{2n}/R_{2n-1} thus has a natural decomposition that reveals the values for the masses and string lengths.

▶ Recovering material parameters from polynomials

We are faced with the problem of determining the $2n+1$ material parameters, two polynomials in $\mathcal{P}_n$. Suppose we measure the eigenvalues of the string with fixed ends

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and the eigenvalues of the string with a fixed left end and free right end

$$\widehat{\lambda}_1 < \widehat{\lambda}_2 < \cdots < \widehat{\lambda}_n.$$

We can distinguish which of the $2n$ eigenvalues of the symmetric string belong to each of these configurations because we must have

$$\widehat{\lambda}_1 < \lambda_1 < \widehat{\lambda}_2 < \lambda_2 < \cdots \widehat{\lambda}_n < \lambda_n.$$

Now form the polynomials

$$A(\lambda) := A_0 \prod_{j=1}^{n} (\lambda + \lambda_j), \qquad B(\lambda) := B_0 \prod_{j=1}^{n} (\lambda + \widehat{\lambda}_j)$$

for arbitrary positive constants A_0 and B_0. Since polynomials with identical roots must agree up to a constant, we have $A(\lambda) = \alpha R_{2n}(-\lambda)$ and $B(\lambda) = \beta R_{2n-1}(-\lambda)$ for some α and β.

Now write $A(\lambda)$ and $B(\lambda)$ out by coefficients:

$$\begin{aligned} A(\lambda) &= A_n + A_{n-1}\lambda + \cdots + A_0\lambda^n \\ B(\lambda) &= B_n + B_{n-1}\lambda + \cdots + B_0\lambda^n. \end{aligned}$$

Given the eigenvalues of our string, then we can multiply out to obtain these constants: these coefficients are thus known quantities.

It turns out (see Gantmacher & Krein) that we can always write

$$\frac{A(\lambda)}{B(\lambda)} = a + \cfrac{1}{b\lambda + \cfrac{B^{(1)}(\lambda)}{A^{(1)}(\lambda)}} \tag{3}$$

for some polynomials $A^{(1)}, B^{(1)} \in P_{n-1}$. Algebraic manipulation yields the equivalent form

$$\frac{A(\lambda)}{B(\lambda)} = \frac{a\big(B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda)\big) + A^{(1)}(\lambda)}{B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda)}.$$

Up to a scaling, we can thus take

$$B(\lambda) = B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda),$$

and then

$$A(\lambda) = aB(\lambda) + A^{(1)}(\lambda).$$

Now as $\lambda \to \infty$, we have

$$\frac{A(\lambda)}{B(\lambda)} \to \frac{A_0}{B_0}$$

using the coefficient forms of $A(\lambda)/B(\lambda)$, while using the expression (3) gives

$$\frac{A(\lambda)}{B(\lambda)} \to a.$$

Hence we make the identification

$$a = \frac{A_0}{B_0},$$

which we can compute since we know A_0 and B_0.

Now write the polynomials $A^{(1)}$ and $B^{(1)}$ in coefficient form as

$$\begin{aligned} A^{(1)}(\lambda) &= A_0^{(1)}\lambda^{n-1} + \cdots + A_{n-1}^{(1)} \\ B^{(1)}(\lambda) &= B_0^{(1)}\lambda^{n-1} + \cdots + A_{n-1}^{(1)}. \end{aligned}$$

We would like to be able to determine $A^{(1)}$ and $B^{(1)}$. Since we have a, we can use

$$A(\lambda) = aB(\lambda) + A^{(1)}(\lambda)$$

to solve for $A^{(1)}$:

$$A^{(1)}(\lambda) = A(\lambda) - aB(\lambda).$$

The leading coefficient of $A^{(1)}$ then takes the form

$$A_0^{(1)} = A_1 - aB_1,$$

which can also be simply computed if we note that

$$A_1 = \sum_{j=1}^{n} \lambda_j, \qquad B_1 = \sum_{j=1}^{n} \widehat{\lambda}_j.$$

It follows that

$$A_0^{(1)} = A_0 \sum_{j=1}^{n} (\lambda_j - \widehat{\lambda}_j).$$

Now we can use $B(\lambda) = B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda)$ to obtain

$$bA_0^{(1)} = B_0.$$

Now we have a recipe to find all the quantities in (3), and this procedure can be repeated to obtain a decomposition that matches the one that we obtained for the ratio R_{2n}/R_{2n-1}, thus recovering the material properties of the symmetric string.

1. Pick A_0 (nonzero scaling factor).
2. Compute $A_0^{(1)} = A_0 \sum_{j=1}^{n} (\lambda_j - \widehat{\lambda}_j)$.
3. Compute a from $A_1 - aB_1 = A_0^{(1)}$ or $a = A_0/B_0$.
4. Compute b from $bA_0^{(1)} = B_0$.
5. Use $A^{(1)}(\lambda) = A(\lambda) - aB(\lambda)$ to compute $A^{(1)}$.
6. Use $B^{(1)}(\lambda) + b\lambda A^{(1)}(\lambda) = B(\lambda)$ to compute $B^{(1)}$.