Boyko-Pivovarchik, Damped two-beaded String

We suppose we have five measured eigenvalues

$$\lambda_2 = 2+i,\ \lambda_1 = 1+i,\ \lambda_0 = i,\ \lambda_{-1} = -1+i,\ \lambda_{-2} = -2+i$$

and prescribed lengths $\hat{\ell} = 1$ and $\ell = 3$ and we proceed to build the two two-beaded damped strings with this spectra.
We work our way through the proof, beginning with (5.2)

$$\Phi(\lambda) \equiv \prod_{k=-2}^{2} \left(1 - \frac{\lambda}{\lambda_k}\right)$$

we find

$$P(\lambda^2) \equiv \frac{\Phi(\lambda)+\Phi(-\lambda)}{2} = (1/2)(\lambda^2)^2 - (5/2)\lambda^2 + 1$$

and

$$Q(\lambda^2) \equiv \frac{\Phi(\lambda)-\Phi(-\lambda)}{2i\lambda} = (1/10)(\lambda^2)^2 - (3/2)\lambda^2 + 12/5$$

The damping coefficient comes immediately via

$$\alpha = Q(0)(1/\hat{\ell} + 1/(\ell - \hat{\ell})) = 18/5.$$

The masses and lengths will stem from the partial fraction expansion (e.g., via `residue` in MATLAB) of the quotient

$$\frac{P(z)}{Q(z)} = \frac{A_1}{z-\nu_1^2} + \frac{A_2}{z-\nu_2^2} + B$$

where

$$A_1 = 48.3319,\ A_2 = 1.6681,\ \nu_1^2 = 13.1789,\ \nu_2^2 = 1.8211,\ B = 5.$$

We arrive at two substrings by writing $B = B_1 + B_2$ and defining

$$\frac{P_1(z)}{Q_1(z)} = \frac{A_1}{z-\nu_1^2} + B_1 \quad \text{and} \quad \frac{P_2(z)}{Q_2(z)} = \frac{A_2}{z-\nu_2^2} + B_2$$

and then inverting to find

$$\frac{Q_1(z)}{\alpha P_1(z)} = \ell_1 + \frac{1}{-m_1 z + 1/\ell_0}$$

and

$$\frac{Q_2(z)}{\alpha P_2(z)} = \tilde{\ell}_1 + \frac{1}{-\tilde{m}_1 z + 1/\tilde{\ell}_0}$$

We first note that once we make the left-right choice the partition of B is fixed. In particular,

$$B_1 = \frac{A_1}{\nu_1^2} + \frac{1}{\alpha\hat{\ell}} = 3.9451$$

Now, via brute force, we find

$$m_1 = \alpha B_1^2/A_1 = 1.1593, \quad \ell_1 = 1/(\alpha B_1) = 0.0704, \quad \ell_0 = \hat{\ell} - \ell_1 = 0.9206$$

and

$$\tilde{m}_1 = \alpha B_2^2/A_2 = 2.4014, \quad \tilde{\ell}_1 = 1/(\alpha B_2) = 0.2633, \quad \tilde{\ell}_0 = (\ell - \hat{\ell}) - \tilde{\ell}_1 = 1.7367.$$

Now we check our reconstruction via Hunter's eigensolver. We execute

```
[x,e]=bp4eig(1.1593,2.4014,[.9206 .0704],[1.7367 .2633],18/5)
```

and find `-i*diag(e)` to be

$$2.0068 + 1.0004i \quad -2.0068 + 1.0004i \quad 0.9906 + 0.9888i \quad -0.9906 + 0.9888i \quad 0 + 1.0224i$$

as measured.
We now make the only other choice, i.e.,

$$\frac{Q_2(z)}{\alpha P_2(z)} = \ell_1 + \frac{1}{-m_1 z + 1/\ell_0}$$

and

$$\frac{Q_1(z)}{\alpha P_1(z)} = \tilde{\ell}_1 + \frac{1}{-\tilde{m}_1 z + 1/\tilde{\ell}_0}$$

It follows that,

$$B_2 = \frac{A_2}{\nu_2^2} + \frac{1}{\alpha\hat{\ell}} = 1.1938$$

Now, via brute force, we find

$$m_1 = \alpha B_2^2/A_2 = 3.0755, \quad \ell_1 = 1/(\alpha B_2) = 0.2327, \quad \ell_0 = \hat{\ell} - \ell_1 = 0.7673$$

and

$$\tilde{m}_1 = \alpha B_1^2/A_1 = 1.0791, \quad \tilde{\ell}_1 = 1/(\alpha B_1) = 0.0730, \quad \tilde{\ell}_0 = (\ell - \hat{\ell}) - \tilde{\ell}_1 = 1.9270.$$

Now we check our reconstruction via Hunter's eigensolver. We execute

```
[x,e]=bp4eig(3.0755,1.0791,[.7673 .2327],[1.9270 0.0730],18/5)
```

`and find -i*diag(e) to be`

$$2.0002 + 0.9998i \quad -2.0002 + 0.9998i \quad 1.0000 + 0.9998i \quad -1.0000 + 0.9998i \quad 0 + 0.9998i$$

`as measured.`