Inverse Problems for end damped strings

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int 1. Introduction

The displacement u of a string of unit length and density ρ^2, free at the left end, and in the presence of viscous damping at the right end, satisfies

$$\begin{aligned} \rho^2(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, &\quad 0 < x < 1,\ 0 < t, \\ u_x(0,t) = u_x(1,t) + u_t(1,t) = 0, &\quad 0 < t, \end{aligned} \qquad (1.1)_{\text{pde}}$$

upon being set in motion by the initial disturbance

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x),$$

assumed an element of the energy space $X = H^1(0,1) \times L^2(0,1)$ with inner product

$$\langle [u,v],[w,z] \rangle = \int_0^1 u'\overline{w}' + u\overline{w} + v\overline{z}\,dx.$$

We assume throughout that ρ is measurable and that

$$0 < \alpha \le \rho(x) \le \beta < \infty \quad \text{a.e. in} \quad (0,1). \qquad (1.2)_{\text{pwbnd}}$$

Let us observe, however, that the quantity

$$f(t) = u(1,t) + \int_0^1 \rho^2 u_t\,dx$$

remains constant along the trajectories. This is due to the lack of coercivity of the energy E in the space X. Given any initial data $[u_0, u_1]$ we may decompose it as $[u_0, u_1] = [\tilde{u}_0, u_1] + [f(0), 0]$ where $\tilde{u}_0 = u_0 - f(0)$. Then the solution $[u, u_t]$ of (1.1) can be written $[u, u_t] = [\tilde{u}, u_t] + [f(0), 0]$ where $[\tilde{u}, u_t]$ is the solution of (1.1) with initial data $[\tilde{u}_0, u_1]$ for which the corresponding quantity f vanishes, i.e., $f(t) = 0$ for all $t \ge 0$.

With this decomposition in mind, the large time behavior of all solutions in X is completely determined by the corresponding behavior of solutions that take their initial data from

$$V = \{[u,v] \in X : u(1) + \textstyle\int_0^1 \rho^2 v\,dx = 0\}.$$

Note that V is invariant under the flow given by (1.1) and is a closed subspace of X in which the norm induced by the energy,

$$\|[u,v]\|_V^2 \equiv \int_0^1 |u'|^2 + \rho^2|v|^2\,dx$$

is equivalent to the one induced by X.

As in our study of the internally damped string, [2], we strive to identify $\omega(\rho)$ with the spectral abscissa of the matrix differential operator obtained on expressing (1.1) as the first order system $U_t = AU$. Here $U = [u, u_t]$, and $A : D(A) \to V$, is given by

$$A = \begin{pmatrix} 0 & I \\ \frac{1}{\rho^2}\frac{d^2}{dx^2} & 0 \end{pmatrix} \qquad \begin{aligned} D(A) = \{[u,v] \in (H^2(0,1) \times H^1(0,1)) \cap V :& \\ u'(0) = u'(1) + v(1) = 0\}.& \end{aligned}$$

We shall see, assuming no more than (1.2), that A possesses a compact dissipative inverse on V. As a result $\sigma(A)$, the spectrum of A, is composed of at most a countable number of eigenvalues, each an element of the left half plane.

If $U = [y,z] \in D(A)$ is an eigenvector of A with eigenvalue λ then $z = \lambda y$ and $y'' = \lambda\rho^2 z$, or

$$y'' = \lambda^2\rho^2 y, \quad y'(0) = y'(1) + \lambda y(1) = 0. \tag{1.3}$$ode

When ρ is constant it follows that $y(x) = \cosh(\lambda\rho x)$ where λ is determined by the right end condition, i.e., by $\lambda\rho\sinh(\lambda\rho) + \lambda\cosh(\lambda\rho) = 0$, or

$$\lambda\rho\tanh(\lambda\rho) = -\lambda.$$

That $\lambda = 0$ is not permitted follows from the fact that the associated eigenvector $U = [1,0] \notin V$. Hence, the eigenvalues are

$$\begin{aligned} \lambda_n &= -\frac{1}{\rho}\tanh^{-1}\frac{1}{\rho} \\ &= -\frac{1}{2\rho}\log\left|\frac{\rho+1}{\rho-1}\right| + i\frac{\pi}{\rho}\begin{cases} n & \text{if } \rho > 1 \\ n+\frac{1}{2} & \text{if } \rho < 1 \end{cases} \qquad n \in \mathbf{Z}, \end{aligned} \tag{1.4}$$lcon

while the corresponding eigenvectors are

$$U_n(x) = \cosh(\lambda_n\rho x)[1, \lambda_n]. \tag{1.5}$$econ

ex 2. The Existence of Eigenvalues

The eigenvalues of A are the poles of the resolvent $\lambda \mapsto (A-\lambda)^{-1}$. To solve $(A - \lambda)[v_1, v_2] = [f_1, f_2]$ in V is to set $v_2 = \lambda v_1 + f_1$ and solve

$$\begin{aligned} v_1'' - \lambda^2\rho^2 v_1 &= \rho^2(\lambda f_1 + f_2) \\ v_1'(0) = 0, \quad v_1'(1) + \lambda v_1(1) + f_1(1) &= 0, \end{aligned} \tag{2.1}$$nhbp

subject to

$$v_1(1) = -\lambda \int_0^1 \rho^2 v_1 \, dx - \int_0^1 \rho^2 f_1 \, dx. \tag{2.2}_{\text{vcon}}$$

We note that $\frac{1}{2}f_1(1)(1-x^2)$ satisfies the boundary conditions in (2.1) and so write $v_1(x) = \frac{1}{2}f_1(1)(1-x^2) + w(x)$ where w must now solve

$$\begin{aligned} w'' - \lambda^2\rho^2 w &= \rho^2(\lambda f_1 + f_2 + \tfrac{1}{2}\lambda^2 f_1(1)(1-x^2)) + f_1(1), \\ w'(0) &= w'(1) + \lambda w(1) = 0. \end{aligned} \tag{2.3}_{\text{red}}$$

We first show that this problem has a one parameter family of solutions when $\lambda = 0$ and that (2.2) selects a particular one. When $\lambda = 0$ we find

$$w'' = \rho^2 f_2 + f_1(1), \quad w'(0) = w'(1) = 0. \tag{2.4}_{\text{baby}}$$

The solvability condition, $\int_0^1 \rho^2 f_2 \, dx + f_1(1) = 0$, that stems from the Fredholm Alternative exactly coincides with one of the requirements for $[f_1, f_2] \in V$. As a result,

$$w(x) = a + \int_0^x (x-s)(\rho^2(s) f_2(s) + f_1(1)) \, ds, \quad a \in \mathbf{R}.$$

We choose a to satisfy (2.2), i.e., $w(1) = -\int_0^1 \rho^2 f_1 \, dx$, and so arrive at

$$v_1(x) = \int_0^x (x-s)\rho^2 f_2 \, ds - \int_0^1 (1-s)\rho^2 f_2 \, ds - \int_0^1 \rho^2 f_1 \, ds.$$

Noting that $v_2 = f_1$ when $\lambda = 0$, we summarize the above in

$$A^{-1}[f_1, f_2] = \left[\int_0^x (x-s)\rho^2 f_2 \, ds - \int_0^1 (1-s)\rho^2 f_2 \, ds - \int_0^1 \rho^2 f_1 \, ds, \; f_1 \right].$$

From the boundedness of ρ and the compactness of the imbedding of $H^1(0,1)$ in to $L^2(0,1)$ follows the compactness of A^{-1} on V. As a result, the spectrum of A is composed of at most a countable number of eigenvalues, $\{\lambda_k\}_k$.

We now return to (2.3), assume $\lambda \neq 0$, and characterize the λ_k as the zeros of a shooting function. In particular, we introduce $\phi(x, \lambda)$ and $\psi(x, \lambda)$, solutions of the respective initial and terminal value problems,

$$\phi'' - \lambda^2\rho^2\phi = 0, \quad \phi(0,\lambda) = 1, \; \phi'(0,\lambda) = 0, \tag{2.5}_{\text{init}}$$

$$\psi'' - \lambda^2\rho^2\psi = 0, \quad \psi(1,\lambda) = 1/\lambda, \; \psi'(1,\lambda) = -1. \tag{2.6}_{\text{term}}$$

We note that ϕ likewise satisfies the integral equation

$$\phi(x,\lambda) = 1 + \lambda^2 \int_0^x (x-s)\rho^2(s)\phi(s,\lambda) \, ds, \tag{2.7}_{\text{pint}}$$

and denote the corresponding shooting function by

$$Q(\lambda) \equiv \phi'(1,\lambda)/\lambda + \phi(1,\lambda). \qquad (2.8)_{\text{shoot}}$$

Clearly $\sigma(A)$ coincides with the set of zeros of Q. In the introduction we found that if $\rho \equiv 1$ then Q never vanishes and $\sigma(A) = \emptyset$. We now show that to be the only pathological case.

Theorem 2.1. *If ρ satisfies* (1.2) *and ρ is not identically one then $\sigma(A) \neq \emptyset$.*

Proof: Dym and McKean [3, §6.3(6)] demonstrate that Q is of exponential type $\int_0^1 \rho\,dx$, i.e.,

$$\int_0^1 \rho\,dx = \limsup_{R\uparrow\infty} R^{-1} \max_{\theta\in[0,2\pi)} \log|Q(Re^{i\theta})|.$$

It then follows from Hadamard's Factorization Theorem that

$$Q(\lambda) = e^{a\lambda}\prod_n (1-\lambda/\lambda_n)e^{\lambda/\lambda_n}, \qquad (2.9)_{\text{hada}}$$

where $\{\lambda_n\}_n$ is the zero set of Q and a is a complex constant. If this zero set is empty then the product defaults to one and $Q(\lambda) = e^{a\lambda}$. As $Q(\lambda)$ is real for real λ it follows that a is real and in fact $a = \int_0^1 \rho\,dx$. We now deduce the restraints this places on ϕ. We follow Kac and Krein [7, §2] and develop ϕ, as the solution of (2.7), in powers of λ^2,

$$\phi(x,\lambda) = \sum_{n=0}^{\infty} \phi_n(x)\lambda^{2n}, \quad \phi_0(x) \equiv 1, \quad \phi_{j+1}(x) = \int_0^x (x-s)\rho^2\phi_j\,ds. \qquad (2.10)_{\text{ps}}$$

It follows that $\lambda \mapsto \phi(1,\lambda)$ and $\lambda \mapsto \phi'(1,\lambda)/\lambda$ are power series in λ^2 and λ respectively. As $Q(\lambda) = e^{a\lambda} = \cosh(a\lambda) + \sinh(a\lambda)$ we find the explicit representations

$$\phi(1,\lambda) = \cosh(a\lambda) \quad \text{and} \quad \phi'(1,\lambda) = \lambda\sinh(a\lambda). \qquad (2.11)_{\text{dsh}}$$

We recognize the former as the shooting function for the Neumann–Dirichlet problem

$$\eta'' = \nu^2\rho^2\eta, \quad \eta'(0) = \eta(1) = 0,$$

and the latter as the shooting function for the Neumann–Neumann problem

$$\zeta'' = \chi^2\rho^2\zeta, \qquad \zeta'(0) = \zeta'(1) = 0.$$

In particular, from (2.11) it follows that

$$\nu_n = \frac{i\pi}{2a}(2n-1) \quad \text{and} \quad \chi_n = \frac{i\pi}{a}(n-1), \quad n = 1,2,\ldots$$

On recalling the well known fact, see, e.g., [3, §6.6], that two such spectra uniquely determine a bounded ρ we immediately conclude that ρ is identically a. Hence, it suffices to restrict ourselves to constant ρ. From the introduction we now recall that $Q(\lambda) = e^{\rho\lambda}$ if and only if $\rho = 1$. ∎

If $\lambda_n \in \sigma(A)$ we note that as a consequence of the uniqueness of $\phi(1, \lambda_n)$ that each eigenvector of A corresponding to λ_n must be a scalar multiple of $U_n(x) \equiv \phi(x, \lambda_n)[1, \lambda_n]$. In other words, the geometric multiplicity of each eigenvalue is one. As a result, the algebraic multiplicity of an eigenvalue is its order as pole of $(A - \lambda)^{-1}$. We now associate this order with Q.

If $\lambda \notin \sigma(A)$ then (2.3) has a unique solution that is consistent with (2.2). In particular,

$$w(x) = G(\lambda)\{\rho^2(\lambda f_1 + f_2 + \tfrac{1}{2}\lambda^2 f_1(1)(1 - x^2)) + f_1(1)\},$$

where $G(\lambda)$ is the Green's operator

$$\begin{aligned} G(\lambda)\eta(x) &= \int_0^1 g(x, \xi, \lambda)\eta(\xi)\, d\xi, \quad \text{where} \\ g(x, \xi, \lambda) &= \begin{cases} \frac{\psi(x,\lambda)\phi(\xi,\lambda)}{Q(\lambda)} & \text{if } 0 \le \xi \le x \le 1 \\ \frac{\psi(\xi,\lambda)\phi(x,\lambda)}{Q(\lambda)} & \text{if } 0 \le x \le \xi \le 1. \end{cases} \end{aligned} \tag{2.12$_{\text{green}}$}$$

Hence the algebraic multiplicity of $\lambda_n \in \sigma(A)$ is its order as a zero of Q.

Some preliminary information is obtained upon taking the $L^2(0, 1)$ inner product of (2.5) at $\lambda = \lambda_n$ with $\phi_n(x) \equiv \phi(x, \lambda_n)$. One finds

$$\lambda_n^2 \int_0^1 \rho^2 |\phi_n|^2\, dx + \lambda_n |\phi_n(1)|^2 + \int_0^1 |\phi_n'|^2\, dx = 0,$$

and therefore

$$\lambda_n = \frac{-|\phi_n(1)|^2 \pm \sqrt{|\phi_n(1)|^4 - 4\int_0^1 \rho^2 |\phi_n|^2\, dx \int_0^1 |\phi_n'|^2\, dx}}{2\int_0^1 \rho^2 |\phi_n|^2\, dx}.$$

It follows that $\Re\lambda_n \le 0$, in fact $\Re\lambda_n < 0$, for equality would force $\phi_n(1) = \phi_n'(1) = 0$ and hence $\phi_n \equiv 0$. Dym and McKean [3, §6.3] also show that

$$\int_{-\infty}^{\infty} \frac{\log_+ |Q(-ix)|}{1 + x^2}\, dx < \infty,$$

from which it follows, see, e.g., Levin [14, §V.4, Theorem 11], that there exist C_1 and C_2 for which

$$|\Re\lambda_n| \le C_1 + C_2 |\Im\lambda_n|.$$

Majda [16], in the context of (***), has bettered this with

$$|\Re\lambda_n| \le C_1 + C_2|\Im\lambda_n|^{3/4}.$$

We shall soon see in fact that $|\Re\lambda_n| \le C_1$ when ρ is Lipschitz.

com 3. The Completeness of the Root Vectors

We follow Krein and Nudelman [11] in their application of the following result of Livšic. We assume throughout that ρ is not identically one.

Theorem 3.1. *([10, §2.5]) If H is Hilbert and $T : H \to H$ is linear and compact and $T_\Re \equiv \frac{1}{2}(T + T^*)$ is nonpositive and of finite trace then*

$$\sum_{\nu_n \in \sigma(T)} |\Re\nu_n| \le -\mathrm{tr}\,(T_\Re), \tag{3.1}$$ liv

where the ν_n are repeated according to their algebraic multiplicity. Equality holds in (3.1) *if and only if the root vectors of T are complete in H.*

We note that $(T^*)_\Re = T_\Re$ and that if T is real then $\sigma(T^*) = \sigma(T)$ including algebraic multiplicities. Hence, if T is real and equality holds in (3.1) then the root vectors of T^* are complete in H as well.

We show that $(A^{-1})_\Re$ is nonpositive and of rank one. In particular,

$$(A^{-1})^*[f_1, f_2] = \Bigg[\int_0^1 (1-s)\rho^2 f_2\,ds - \int_0^x (x-s)\rho^2 f_2\,ds + \int_0^1 \rho^2 f_1\,ds + 2\int_0^1 \rho^2\,ds \int_0^1 \rho^2 f_2\,ds,\ -f_1 - 2\int_0^1 \rho^2 f_2\,ds\Bigg],$$

and so

$$(A^{-1})_\Re[f_1, f_2] = \int_0^1 \rho^2 f_2\,dx \left[\int_0^1 \rho^2\,dx,\ -1\right]$$

is indeed rank–one while

$$\langle (A^{-1})_\Re[f_1, f_2], [f_1, f_2]\rangle = -\left|\int_0^1 \rho^2 f_2\,ds\right|^2$$

and

$$\mathrm{tr}\,(A^{-1})_\Re = -\int_0^1 \rho^2\,dx.$$

From Theorem 3.1 we may now draw

Corollary 3.2. *If ρ satisfies* (1.2) *then* $\displaystyle\sum_{\lambda_n\in\sigma(A)} \frac{|\Re\,\lambda_n|}{|\lambda_n|^2} \le \int_0^1 \rho^2\,dx.$

We now produce conditions under which equality holds in Corollary 3.2. In the constant case we offer the following elementary argument.

Theorem 3.3. *If ρ is a positive constant distinct from one then the $\lambda_n \in \sigma(A)$, see* (1.4)*, are each of algebraic multiplicity one and their associated eigenvectors are complete in V.*

Proof: We assume $\rho > 1$. The other case being similar. We sum the $|\Re\,\lambda_n|/|\lambda_n|^2$ without repetition,

$$\sum_n \frac{|\Re\,\lambda_n|}{|\lambda_n|^2} = \frac{\rho}{\nu} + 2\nu\rho\sum_{n=1}^{\infty}\frac{1}{\nu^2+n^2\pi^2}, \qquad \nu \equiv \tfrac{1}{2}\log\left(\frac{\rho+1}{\rho-1}\right). \tag*{(3.2)lsum}$$

As $\nu^2 + n^2\pi^2$ is the nth eigenvalue of $Lu \equiv -u'' + \nu^2 u$, $u \in H_0^1(0,1)$, it follows from the standard trace formula that

$$\sum_{n=1}^{\infty}\frac{1}{\nu^2+n^2\pi^2} = \int_0^1 g(x,x)\,dx$$

where

$$g(x,y) = \frac{\sinh\nu(x\vee y)\sinh\nu(1-(x\wedge y))}{\nu\sinh\nu}$$

is the Green's function for L. Hence,

$$\sum_{n=1}^{\infty}\frac{1}{\nu^2+n^2\pi^2} = \frac{\nu\cosh\nu - \sinh\nu}{2\nu^2\sinh\nu} = \frac{\rho}{2\nu} - \frac{1}{2\nu^2}.$$

inserting this sum in (3.2) we find that equality holds in Corollary 3.2. ∎

The fact that a (bi)normalized copy of $\{U_n\}_n$ for constant ρ is in fact a Riesz basis for V now follows easily. As in [2] our principal tool is the following result of Bari.

Theorem 3.4. *[4, Theorem 2.1, Chapter VI] $\{f_n\}_n$ is a Riesz basis for the Hilbert space H if and only if $\{f_n\}_n$ is complete in H and there corresponds to it a complete biorthogonal sequence $\{g_n\}_n$, and for any $f \in H$ both $\{\langle f_n, f\rangle\}_n$ and $\{\langle g_n, f\rangle\}_n$ are square summable.*

We shall also make use of the equivalent statement that $\{f_n\}_n$ is a Riesz basis for H iff $\{f_n\}_n$ is complete and there exist two constants c_0 and c_1 such

$$c_0\sum_{n=1}^{N}|a_n|^2 \le \left\|\sum_{n=1}^{N}a_n f_n\right\|_H^2 \le c_1\sum_{n=1}^{N}|a_n|^2 \tag*{(3.3)b2}$$

for each N and each $\{a_n\}_n \in \mathbf{C}^N$.

The sequence biorthogonal to $\{U_n\}_n$ is built from the eigenvectors of the adjoint of A,

$$A^* = -\begin{pmatrix} 0 & I \\ \frac{1}{\rho^2}\frac{d^2}{dx^2} & 0 \end{pmatrix} \qquad \begin{aligned} D(A^*) = \{[u,v] \in (H^2(0,1)\times H^1(0,1)) \cap V : \\ u'(0) = u'(1) - v(1) = 0\}. \end{aligned}$$

We note that $\sigma(A^*) = \sigma(A)$ and that if W_k is an eigenvector of A^* corresponding to $\overline{\lambda}_k$ then, for constant ρ, $W_k(x) = \cosh(\overline{\lambda}_k \rho x)[-1, \overline{\lambda}_k]$ and $\langle U_j, W_k\rangle = \lambda_k^2\rho^2\delta_{jk}$. As a result

$$\begin{aligned} \tilde{U}_k(x) &= \frac{1}{\lambda_k\rho}\cosh(\lambda_k\rho x)[1,\lambda_k], \\ \tilde{W}_k(x) &= \frac{1}{\overline{\lambda}_k\rho}\cosh(\overline{\lambda}_k\rho x)[-1,\overline{\lambda}_k] \end{aligned} \tag*{(3.4)rncon}$$

satisfy $\langle \tilde{U}_j, \tilde{W}_k\rangle = \delta_{jk}$, i.e., they constitute a biorthogonal set in V. That $\{\tilde{W}_n\}_n$ is complete in V follows from the remark following Theorem 3.1. It remains to select $[f_1, f_2] \in V$ and check $\{\langle \tilde{U}_n, [f_1, f_2]\rangle\} \in \ell^2(\mathbf{C})$. We suppose $\rho > 1$, the other case being similar, and recall the definition of ν in (3.2). That

$$\begin{aligned} \langle \tilde{U}_n, [f_1,f_2]\rangle &= \int_0^1 \sinh(\lambda_n\rho x)\overline{f}_1' + \rho\cosh(\lambda_n\rho x)\overline{f}_2\,dx \\ &= \int_0^1 (\sinh(\nu)\overline{f}_1' + \rho\cosh(\nu)\overline{f}_2)\cos(n\pi x)\,dx + \\ &\qquad i\int_0^1 (\cosh(\nu)\overline{f}_1' + \rho\sinh(\nu)\overline{f}_2)\sin(n\pi x)\,dx \end{aligned}$$

is square summable now follows directly from $[f_1, f_2] \in V$. The verification that $\{\langle \tilde{W}_n, [f_1, f_2]\rangle\}_n \in \ell^2(\mathbf{C})$ is just as simple. From Theorem 3.4. we may now deduce

Theorem 3.5. *If ρ is a positive constant distinct from one then the eigenvectors $\{\tilde{U}_n\}_n$, see* (3.4)*, constitute a Riesz basis for V. If we append to this sequence the vector* $[1,0]$*, corresponding to the zero eigenvalue of A in X, we obtain a Riesz basis for X.*

We now return to the variable coefficient case and address the extent to which the root vectors of A are complete in V. In particular, we return to [11] and equate the power series representation of Q stemming from (2.10) with a refinement of (2.9). First, it follows directly from (2.8) and (2.10) that

$$Q(\lambda) = 1 + \lambda\int_0^1 \rho^2\,dx + O(\lambda^2). \tag*{(3.5)rep1}$$

Next, from the summability of $|\Re\lambda_n|/|\lambda_n|^2$ comes the fact that one may remove the exponential factors in (2.9), i.e.,

$$Q(\lambda) = e^{a\lambda}\prod_{\Im\lambda_n>0}\left(1 - \frac{\lambda}{\lambda_n}\right)\left(1 - \frac{\lambda}{\overline{\lambda}_n}\right)\prod_{\Im\lambda_n=0}\left(1 - \frac{\lambda}{\lambda_n}\right). \tag*{(3.6)rep2}$$

See [3, §6.2] for an independent proof. From the easily verified fact that if $\rho(x) = 1$ for $x \in (\ell, 1]$ then $\sigma(A)$ coincides with the spectrum associated with (1.1) for $x \in (0, \ell)$ subject to $u_x(0,t) = u_x(\ell,t) + u_t(\ell,t) = 0$ Krein and Nudelman next deduce that a is in fact the largest number b for which ρ is identically one on the interval $(1-b, 1)$. Identifying the coefficients of λ in (3.5) and (3.6) we find

$$\int_0^1 \rho^2\,dx = a - \sum_{\lambda_n \in \sigma(A)} \frac{\Re\lambda_n}{|\lambda_n|^2}.$$

As a result, analogous to [11, Theorem 2], we have

Theorem 3.6. *If ρ satisfies* (1.2) *then the root vectors of A are complete in V if and only if ρ is not identically one on any interval of the form* $(\ell, 1)$.

The proof that these root vectors indeed make up a basis for V will require a considerably more detailed study of the λ_n. In the next three sections we respectively analyze the real eigenvalues, establish a crude lower bound on $\Re\lambda_n$, and develop asymptotic estimates for λ_n.

high 4. High Frequencies

We now develop asymptotic formulas for λ_n and U_n as $|\lambda_n| \to \infty$. Our development can be seen both as an elaboration of [6], where few details are provided and more function theory is invoked, and as a special case of [22], where, in her desire to permit ρ to either vanish or become infinite at the damped end, Shubov requires a twenty page immersion in special functions. At the heart of both approaches, as with our previous work [2], is a fake potential that has the advantage that its introduction into (2.6) permits one to find an explicit solution. One then argues that the fake potential has a negligible effect on the high frequencies. The fake potential in this case

$$q(x) \equiv \rho^{1/2}(x)\left(\rho^{-1/2}(x)\right)'' = \frac{3}{4}\left(\frac{\rho'(x)}{\rho(x)}\right)^2 - \frac{\rho''(x)}{2\rho(x)} \qquad (4.1)_{\text{q}}$$

lies in $L^2(0,1)$ so long as, in addition to (1.2), $\rho \in H^2(0,1)$. Its addition to (2.6) brings us to

$$z'' - qz = \lambda^2\rho^2 z, \quad z(1) = \lambda^{-1},\ z'(1) = -1, \qquad (4.2)_{\text{fake}}$$

the solution to which is

$$\begin{aligned} z(x,\lambda) &= \frac{\rho_1 \cosh(\lambda \int_x^1 \rho\,ds) + \sinh(\lambda \int_x^1 \rho\,ds)}{\lambda\sqrt{\rho_1\rho(x)}} - \frac{(\log\rho)'(1)\sinh(\lambda\int_x^1 \rho\,ds)}{2\lambda^2\sqrt{\rho_1\rho(x)}} \\ &\equiv w_1(x,\lambda) + O(|\lambda|^{-2}). \end{aligned}$$

Recall that in (***) we defined $\rho_1 \equiv \rho(1)$. As $z'(0,\lambda)$ will serve as our fake shooting function, we require

$$\begin{aligned} z'(x,\lambda) = &-\frac{\sqrt{\rho(x)}}{\sqrt{\rho_1}}\left\{\rho_1 \sinh(\lambda \textstyle\int_x^1 \rho\,ds) + \cosh(\lambda \textstyle\int_x^1 \rho\,ds)\right\} - \\ &\frac{\rho'(x)}{2\lambda\rho^{3/2}(x)\sqrt{\rho_1}}\left\{\rho_1 \cosh(\lambda \textstyle\int_x^1 \rho\,ds) + \sinh(\lambda \textstyle\int_x^1 \rho\,ds)\right\} + \\ &\frac{\sqrt{\rho(x)}(\log\rho)'(1)}{2\lambda\sqrt{\rho_1}}\cosh(\lambda \textstyle\int_x^1 \rho\,ds) + \frac{\rho'(x)(\log\rho)'(1)}{4\lambda^2\rho^{3/2}(x)\sqrt{\rho_1}}\sinh(\lambda \textstyle\int_x^1 \rho\,ds) \\ &\equiv w_2(x,\lambda) + O(|\lambda|^{-1}). \end{aligned}$$

We see on inspection that $z(x,\cdot)$ and $z'(x,\cdot)$ are asymptotically close to $w_1(x,\cdot)$ and $w_2(x,\cdot)$. We now show in fact that $\psi(x,\cdot)$ and $\psi'(x,\cdot)$ are asymptotically close to $w_1(x,\cdot)$ and $w_2(x,\cdot)$, where $\psi(x,\lambda)$ is the actual solution to (2.6). By Theorem ??.1 it suffices to work in the band $-\kappa \le \Re\lambda \le 0$.

Theorem 4.1. *Assume that* $\rho \in H^2(0,1)$, $\rho_1 \neq 1$, *and* ρ *satisfies* (1.2). *Then there exist constants* C_0 *and* C_1 *such that*

$$|\psi(x,\lambda) - w_1(x,\lambda)| \le C_0|\lambda|^{-2} \qquad (4.3)_{\text{pest}}$$

$$|\psi'(x,\lambda) - w_2(x,\lambda)| \le C_1|\lambda|^{-1}, \qquad (4.4)_{\text{ppest}}$$

uniformly for $0 < x < 1$ *and*

$$-\kappa \le \Re\lambda \le 0 \quad |\lambda| \ge \max\{1, \tfrac{\|\rho'\|_\infty}{2\alpha}\}. \qquad (4.5)_{\text{labnd}}$$

Proof: We note that ψ satisfies

$$\psi'' - q\psi - \lambda^2\rho^2\psi = -q\psi, \quad \psi(1,\lambda) = 1/\lambda,\ \psi'(1,\lambda) = -1,$$

and therefore the integral equation

$$\psi(x,\lambda) = z(x,\lambda) - \int_x^1 K(x,t,\lambda)q(t)\psi(t,\lambda)\,dt \qquad (4.6)_{\text{intp}}$$

with $K(x,t,\lambda) = z(x,\lambda)\tilde{z}(t,\lambda) - z(t,\lambda)\tilde{z}(x,\lambda)$ where

$$\tilde{z}(x,\lambda) = \frac{-1}{\sqrt{\rho_1\rho(x)}}\sinh(\lambda \textstyle\int_x^1 \rho\,ds),$$

satisfies the same differential equation as z but with terminal data $\tilde{z}(1,\lambda) = 0$, $\tilde{z}'(1,\lambda) = \lambda$. We solve the integral equation (4.6) in series form

$$\psi(x,\lambda) = \sum_{n=0}^{\infty} S_n(x,\lambda), \qquad (4.7)_{\text{ser}}$$

where $S_0 = z$ and

$$\begin{aligned} S_n(x,\lambda) &= -\int_x^1 K(x,t,\lambda)q(t)S_{n-1}(t,\lambda)\,dt \\ &= (-1)^n \int^{1\ge t_1 \ge \cdots \ge t_{n+1}=x} z(t_1,\lambda)\prod_{i=1}^n [K(t_{i+1},t_i,\lambda)q(t_i)]\,dt_1\cdots dt_n. \end{aligned}$$

In order to establish the convergence of (4.7) we proceed to derive uniform estimates for the S_n. To begin, we recall $m = \int_0^1 \rho\,dx$ and observe that

$$|z(x,\lambda)| \le \frac{(\beta+1)e^{\kappa m}}{\alpha|\lambda|} + \frac{\|\rho'\|_\infty e^{\kappa m}}{2\alpha^2|\lambda|^2} \le \frac{(\beta+2)e^{\kappa m}}{\alpha|\lambda|}$$

when λ obeys (4.5). Likewise, as

$$|\tilde z(x,\lambda)| \le \frac{e^{\kappa m}}{\alpha}$$

it follows that

$$|K(x,t,\lambda)| \le \frac{2(\beta+2)e^{2\kappa m}}{\alpha^2|\lambda|}$$

and hence

$$\begin{aligned} |S_n(x,\lambda)| &\le \frac{(\beta+2)e^{\kappa m}}{\alpha|\lambda|}\left(\frac{2(\beta+2)e^{2\kappa m}}{\alpha^2|\lambda|}\right)^n \frac{\|q\|_1^n}{n!} \\ &\le \frac{(\beta+2)e^{\kappa m}}{\alpha|\lambda|^2}\frac{(2\alpha^{-2}(\beta+2)\|q\|_2 e^{2\kappa m})^n}{n!}. \end{aligned}$$

From the Weierstrass comparison test it now follows that (4.7) converges uniformly for $0 \le x \le 1$ and λ satisfying (4.5). Moreover,

$$\begin{aligned} |\psi(x,\lambda) - w_1(x,\lambda)| &\le |\psi(x,\lambda) - z(x,\lambda)| + |z(x,\lambda) - w_1(x,\lambda)| \\ &\le \left|\sum_{n=1}^\infty S_n(x,\lambda)\right| + \frac{e^{\kappa m}\|\rho'\|_\infty}{2\alpha|\lambda|^2} \\ &\le \frac{e^{\kappa m}}{\alpha|\lambda|^2}\left(\tfrac{1}{2}\|\rho'\|_\infty + (\beta+2)e^{2\alpha^{-2}(\beta+2)\|q\|_2 e^{2\kappa m}}\right) \\ &\equiv C_0|\lambda|^{-2} \end{aligned}$$

now establishes (4.3).

Regarding the estimate for ψ' we differentiate (4.6) and find

$$\psi'(x,\lambda) - z'(x,\lambda) = -\int_x^1 K_x(x,t,\lambda)q(t)\psi(t,\lambda)\,dt, \qquad (4.8)_{\text{ppre}}$$

and so it remains to simply bound ψ and K_x. With respect to the former

$$|\psi(x,\lambda)| \leq |\psi(x,\lambda) - w_1(x,\lambda)| + |w_1(x,\lambda)| \leq \frac{C_0 + (1+\beta)\alpha^{-1}e^{\kappa m}}{|\lambda|},$$

while the latter requires both

$$|z'(x,\lambda)| \leq e^{\kappa m}\left(1 + \tfrac{1}{\alpha} + \sqrt{\tfrac{\beta}{\alpha}}\right)(1+\beta) \quad \text{and} \quad |\tilde{z}'(x,\lambda)| \leq \frac{|\lambda|e^{\kappa m}}{\alpha}(1+\beta).$$

Assembling these bounds we find

$$\begin{aligned}|K_x(x,t,\lambda)| &\leq |z'(x,\lambda)||\tilde{z}(t,\lambda)| + |z(t,\lambda)||\tilde{z}'(x,\lambda)| \\ &\leq \frac{e^{2\kappa m}}{\alpha}\left(1 + \tfrac{1}{\alpha} + \sqrt{\tfrac{\beta}{\alpha}}\right)(1+\beta) + \frac{e^{2\kappa m}}{\alpha^2}(2+\beta)(1+\beta) \\ &\leq \frac{4e^{2\kappa m}}{\alpha^2}(1+\beta)^2.\end{aligned}$$

Now (4.8) yields (4.4),

$$\begin{aligned}|\psi'(x,\lambda) - w_2(x,\lambda)| &\leq |\psi'(x,\lambda) - z'(x,\lambda)| + |z'(x,\lambda) - w_2(x,\lambda)| \\ &\leq \int_0^1 |K_x(x,t,\lambda)||\psi(t,\lambda)||q(t)|\,dt + \frac{e^{\kappa m}\|\rho'\|_\infty}{2|\lambda|\alpha^2}(1+\beta)^2 \\ &\leq \frac{4(1+\beta)^3 e^{3\kappa m}}{\alpha^2|\lambda|}((C_0+\alpha^{-1})\|q\|_2 + \|\rho'\|_\infty) \\ &\equiv C_1|\lambda|^{-1}. \blacksquare\end{aligned}$$

On close inspection of the estimate for S_n it follows that dq need only be a finite measure, i.e, it suffices to require that ρ' simply have finite total variation.

We now show that the zeros of $\psi'(0,\lambda)$ are close to the zeros of $w_2(0,\lambda)$, these being

$$\mu_n = -\frac{1}{2m}\log\left|\frac{1+\rho_1}{\rho_1 - 1}\right| + \frac{i\pi}{m}\begin{cases} n & \text{if } \rho_1 > 1 \\ n + \frac{1}{2} & \text{if } \rho_1 < 1 \end{cases} \qquad n \in \mathbf{Z}. \qquad (4.9)\text{mus}$$

This is done by choosing r_n in

$$\Gamma_n = \{\lambda \in \mathbf{C} : |\lambda - \mu_n| = r_n\}$$

in such a way that the Γ_n do not intersect and

$$|\psi'(0,\lambda) - w_2(0,\lambda)| < |w_2(0,\lambda)|, \quad \lambda \in \Gamma_n.$$

By the previous Theorem it suffices to show that

$$|w_2(0,\lambda)| > \frac{C_1}{|\lambda|}, \quad \lambda \in \Gamma_n.$$

We proceed under the assumption that $\rho_1 > 1$, the other case following similarly. If $\lambda \in \Gamma_n$ then $\lambda = \mu_n + r_n e^{i\theta}$ where $\theta \in [0, 2\pi)$ and

$$\begin{aligned} \lambda w_2(0,\lambda) &= -\lambda\sqrt{\rho_0/\rho_1}\{\rho_1 \sinh(m(\mu_n + r_n e^{i\theta})) + \cosh(m(\mu_n + r_n e^{i\theta}))\} \\ &= -\lambda\sqrt{\rho_0/\rho_1}(\rho_1^2 - 1)\sinh(m\mu_n)\sinh(m r_n e^{i\theta}) \\ &= (-1)^n\sqrt{\rho_0/\rho_1}(\rho_1^2 - 1)\sinh(m\mu_0)(\mu_n + r_n e^{i\theta})\sinh(m r_n e^{i\theta}) \end{aligned}$$

Hence, if $C' \equiv \sqrt{\rho_0/\rho_1}(1-\rho_1^2)\sinh(m\mu_0)$ then

$$\begin{aligned} |\lambda||w_2(0,\lambda)| &> C'(|n|\pi/m - r_n)|\sinh(m r_n e^{i\theta})| \\ &\geq C'(|n|\pi/m - r_n)(m r_n - \tfrac{1}{2}m^2 r_n^2) \\ &\geq C' r_n(|n|\pi - m r_n(1 + \tfrac{1}{2}|n|\pi)). \end{aligned}$$

One makes the obvious guess $r_n = \dfrac{2C_1}{C'|n|\pi}$ and finds that

$$\begin{aligned} |\lambda||w_2(0,\lambda)| &\geq C' r_n(|n|\pi - m r_n(1 + \tfrac{1}{2}|n|\pi)) \\ &= C_1\left(2 - \frac{4C_1 m}{C'|n|\pi}\left(\frac{1}{2} + \frac{1}{|n|\pi}\right)\right) \\ &> C_1, \quad \text{when } |n| > N_1 = \left\lceil \frac{4C_1 m}{C'\pi} \right\rceil, \end{aligned}$$

where $\lceil x \rceil$ denotes the least integer greater than x. Furthermore, this choice of N renders $r_n < 1/(2m)$. As the distance between centers of the Γ_n is π/m it follows that the contours are nonintersecting. To capture the remaining eigenvalues we consider

$$Z_n = \left\{\lambda \in \mathbf{C} : |\lambda - \mu_0| = \frac{\pi}{m}(n + \tfrac{1}{2})\right\},$$

and denote by N_2 the smallest integer $n > 0$ for which Z_n encircles the disk of radius C_1/C' centered at the origin. For each n we note that if $\lambda \in Z_n$ then $|w_2(0,\lambda)| \geq C'$ while when $n > N_2$ we find $|\lambda| > C_1/C'$ as well. As a result,

$$|\lambda||w_2(0,\lambda)| > C_1, \quad \lambda \in Z_n,\ n \geq N_2.$$

With $N \equiv \max\{N_1, N_2\}$, from the Theorem of Rouché follows

Theorem 4.2. *If* $\rho \in H^2(0,1)$, $\rho_1 \neq 1$, *and* ρ *satisfies* (1.2) *then* $A(\rho)$ *has exactly* $2N+1$ *eigenvalues, including multiplicity, in* Z_N *and one simple eigenvalue in* Γ_n *for* $|n| > N$. *This exhausts the spectrum of* A.

From the estimate $\lambda_n = \mu_n + O(|n|^{-1})$ one easily improves those of Theorem 4.1 at $\lambda = \lambda_n$.

Corollary 4.3. *If* $\rho \in H^2(0,1)$, $\rho_1 \neq 1$, *and* ρ *satisfies* (1.2) *then, uniformly for* $0 < x < 1$,

$$\begin{aligned} \psi(x,\lambda_n) &= w_1(x,\mu_n) + O(|n|^{-2}) \\ \psi'(x,\lambda_n) &= w_2(x,\mu_n) + O(|n|^{-1}). \end{aligned}$$

root 5. The Root Vectors Comprise a Riesz Basis

We denote the algebraic multiplicity of λ_n by ν_n and to λ_n associate the Jordan Chain of root vectors, $\{U_{n,j}\}_{j=0}^{\nu_n-1}$,

$$\begin{aligned} U_{n,0}(x) &= \psi(x,\lambda_n)[1,\lambda_n], \\ AU_{n,j} &= \lambda_n U_{n,j} + U_{n,j-1}, \quad \langle U_{n,j}, U_{n,0}\rangle = 0, \quad j = 1,\ldots,\nu_n - 1. \end{aligned}$$

Clearly, $U_{n,0}$ is an eigenvector and the chain is a basis for the root subspace

$$\mathcal{L}_n \equiv \{U : (A-\lambda_n)^{\nu_n} U = 0\}.$$

We construct a biorthogonal sequence to $\{\tilde{U}_{n,j}\}_{n,j}$ from the eigenvectors of the adjoint, A^*. We recall that $\sigma(A) = \sigma(A^*)$, including multiplicities, and to $\overline{\lambda}_n$ we associate the Jordan Chain of root vectors, $\{W_{n,j}\}_{j=0}^{\nu_n-1}$, where

$$\begin{aligned} W_{n,0}(x) &= \psi(x,\overline{\lambda}_n)[1,-\overline{\lambda}_n], \\ A^*W_{n,j} &= \overline{\lambda}_n W_{n,j} + W_{n,j-1}, \quad \langle W_{n,j}, V_{n,\nu_n-1}\rangle = 0, \quad j = 1,\ldots,\nu_n - 1. \end{aligned}$$

Observe that $W_{n,0}$ is an eigenvector for A^* and that the subsequent $W_{n,j}$ are uniquely determined so long as $\langle W_{n,0}, V_{n,\nu_n-1}\rangle \neq 0$. In addition, the chain $\{W_{n,j}\}_{j=0}^{\nu_n-1}$ is a basis for the root subspace

$$\mathcal{L}_n^* \equiv \{W : (A^* - \overline{\lambda}_n)^{\nu_n} W = 0\}.$$

Lemma 5.1. *If* $\rho \in H^2(0,1)$, $\rho_1 \neq 1$, *and* ρ *satisfies* (1.2) *then there exists a* $c > 0$ *such that*

$$|\langle U_{n,p}, W_{j,k}\rangle| = |\langle U_{n,p}, W_{n,\nu_n-1-p}\rangle| \delta_{n,j}\delta_{\nu_n-1-p,k} \geq c\,\delta_{n,j}\delta_{\nu_n-1-p,k}.$$

Proof: The biorthogonality is an algebraic result that follows essentially by construction. For details see [2, Lemma 6.2]. The fact that $c > 0$, i.e., that the two sequences may be binormalized, is a consequence of the asymptotic formulas of Corollary 4.3. In particular, we recall from Theorem 4.2 that $\mu_n = 1$ for $n > |N|$ and proceed for such n to establish

$$\begin{aligned}\langle U_{n,0}, W_{n,0}\rangle &= \langle \psi(x,\lambda_n)[1,\lambda_n], \psi(x,\overline{\lambda}_n)[1,-\overline{\lambda}_n]\rangle \\ &= \int_0^1 (\psi'(x,\lambda_n))^2 - \lambda_n^2\rho^2\psi^2(x,\lambda_n)\,dx \\ &= \int_0^1 w_2^2(x,\mu_n) - \mu_n^2\rho^2 w_1^2(x,\mu_n)\,dx + O(|n|^{-1}) \\ &= \int_0^1 \frac{\rho}{\rho_1}\Bigg(\Big\{\rho_1 \sinh(\mu_n \textstyle\int_x^1 \rho\,ds) + \cosh(\mu_n \int_x^1 \rho\,ds)\Big\}^2 \\ &\qquad - \Big\{\rho_1 \cosh(\mu_n \textstyle\int_x^1 \rho\,ds) + \sinh(\mu_n \int_x^1 \rho\,ds)\Big\}^2\Bigg)\,dx + O(|n|^{-1}) \\ &= \frac{m}{\rho_1}(1-\rho_1^2) + O(|n|^{-1})\end{aligned}$$

As $\rho_1 \neq 1$ it follows that $|\langle U_{n,0}, W_{n,0}\rangle| \geq \dfrac{m}{2\rho_1}(1-\rho_1^2)$ for n of sufficient magnitude. ∎

We may therefore binormalize,

$$\begin{aligned}\tilde{U}_{n,0}(x) &= \langle U_{n,0}, W_{n,0}\rangle^{-1/2} U_{n,0}(x) = U_{n,0}(x) + O(1/|n|), \quad \text{and} \\ \tilde{W}_{n,0}(x) &= \langle U_{n,0}, W_{n,0}\rangle^{-1/2} W_{n,0}(x) = W_{n,0}(x) + O(1/|n|)\end{aligned}$$

for $|n| > N$. Having demonstrated completeness, in order to invoke Bari's Theorem it suffices to check that $\{\langle \tilde{U}_{n,0}, [f,g]\rangle\}_n \in \ell^2(\mathbf{C})$ for each $[f,g] \in V$. Drawing once again on Corollary 4.3 we find

$$\begin{aligned}\langle \tilde{U}_{n,0}, [f,g]\rangle &= \int_0^1 \psi'(x,\lambda_n)\overline{f}' + \rho^2\lambda_n\psi(x,\lambda_n)\overline{g}\,dx + O(|n|^{-1}) \\ &= \int_0^1 w_2(x,\mu_n)\overline{f}' + \rho^2\mu_n w_1(x,\mu_n)\overline{g}\,dx + O(|n|^{-1}) \\ &= \int_0^1 \frac{\sqrt{\rho}}{\sqrt{\rho_1}}\Big(\sinh(\mu_n \textstyle\int_x^1 \rho\,ds)(\rho\overline{g} - \rho_1\overline{f}') \\ &\qquad\qquad + \cosh(\mu_n \textstyle\int_x^1 \rho\,ds)(\rho_1\rho\overline{g} - \overline{f}')\Big)\,dx + O(|n|^{-1}).\end{aligned}$$

Its square summability is evidently determined by that of terms of the form

$$\int_0^1 h(x)\sin(n\pi\xi(x))\,dx, \quad \text{where} \quad \xi(x) = \frac{\int_x^1 \rho\,ds}{\int_0^1 \rho\,ds}$$

for $h \in L^2(0,1)$. On performing the change of variables $t = \xi(x)$ we find

$$\int_0^1 h(x)\sin(n\pi\xi(x))\,dx = \int_0^1 \frac{h(\xi^{-1}(t))}{\rho(\xi^{-1}(t))}\sin(n\pi t)\,dt,$$

without doubt an element of $\ell^2(\mathbf{C})$. Hence,

Theorem 5.2. *If $\rho \in H^2(0,1)$, $\rho_1 \neq 1$, and ρ satisfies (1.2) then the root vectors of $A(\rho)$ comprise a Riesz basis for V and $\omega(\rho) = \mu(\rho)$.*

od 6. Finer Asymptotics

We set

$$m \equiv \int_0^1 \rho(y)\,dy \quad \text{and} \quad \tilde\rho(x) \equiv \rho(x) - m$$

and refine our asymptotic result via the simple identity

$$\begin{aligned}\lambda_n(\rho) - \lambda_n(m) &= \int_0^1 \frac{d}{dt}\lambda_n(m+t\tilde\rho)\,dt \\ &= \int_0^1 \frac{\langle A'(m+t\tilde\rho)U_n(m+t\tilde\rho), W_n(m+t\tilde\rho)\rangle}{\langle U_n(m+t\tilde\rho), W_n(m+t\tilde\rho)\rangle}\,dt\end{aligned}$$

It remains to assemble the integrand. From

$$A(m+t\tilde\rho) = \begin{pmatrix} 0 & I \\ \frac{1}{(m+t\tilde\rho)^2}\frac{d^2}{dx^2} & 0 \end{pmatrix}$$

we glean

$$A'(m+t\tilde\rho) = \begin{pmatrix} 0 & 0 \\ \frac{-2\tilde\rho}{(m+t\tilde\rho)^3}\frac{d^2}{dx^2} & 0 \end{pmatrix}$$

Next, recalling §5

$$U_n(m+t\tilde\rho) = \psi(x,\lambda_n)[1,\ \lambda_n] \quad \text{and} \quad W_n(m+t\tilde\rho) = \psi(x,\overline{\lambda}_n)[1,\ -\overline{\lambda}_n]$$

Hence

$$A'(m+t\tilde\rho)U_n(m+t\tilde\rho) = [0,\ \frac{-2\tilde\rho}{(m+t\tilde\rho)^3}\psi''(x,\lambda_n)] = \left[0,\ \frac{-2\tilde\rho\lambda_n^2(m+t\tilde\rho)}{m+t\tilde\rho}\psi(x,\lambda_n(m+t\tilde\rho))\right],$$

and

7. REFERENCES

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