Inverse Problems for end damped strings

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1. Introduction

The displacement $u$ of a string of unit length and density $\rho^2$, free at the left end, and in the presence of viscous damping at the right end, satisfies

$$\rho^2(x)u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t,$$

$$u_x(0,t) = u_x(1,t) + u_t(1,t) = 0, \quad 0 < t,$$

upon being set in motion by the initial disturbance

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x),$$

assumed an element of the energy space $X = H^1(0,1) \times L^2(0,1)$ with inner product

$$\langle [u,v], [w,z] \rangle = \int_0^1 u'w' + uw + vz \ dx.$$

We assume throughout that $\rho$ is measurable and that

$$0 < \alpha \leq \rho(x) \leq \beta < \infty \quad \text{a.e. in} \quad (0,1).$$

Let us observe, however, that the quantity

$$f(t) = u(1,t) + \int_0^1 \rho^2 u_t \ dx$$

remains constant along the trajectories. This is due to the lack of coercivity of the energy $E$ in the space $X$. Given any initial data $[u_0, u_1]$ we may decompose it as $[u_0, u_1] = [\tilde{u}_0, u_1] + [f(0), 0]$ where $\tilde{u}_0 = u_0 - f(0)$. Then the solution $[u, u_t]$ of (1.1) can be written

$$[u, u_t] = [\tilde{u}, u_t] + [f(0), 0]$$

where $[\tilde{u}, u_t]$ is the solution of (1.1) with initial data $[\tilde{u}_0, u_1]$ for which the corresponding quantity $f$ vanishes, i.e., $f(t) = 0$ for all $t \geq 0$.

With this decomposition in mind, the large time behavior of all solutions in $X$ is completely determined by the corresponding behavior of solutions that take their initial data from

$$V = \{ [u,v] \in X : u(1) + \int_0^1 \rho^2 v \ dx = 0 \}.$$
Note that $V$ is invariant under the flow given by (1.1) and is a closed subspace of $X$ in which the norm induced by the energy,

$$
\| [u, v] \|_V^2 \equiv \int_0^1 |u'|^2 + \rho^2 |v|^2 \, dx
$$
is equivalent to the one induced by $X$.

As in our study of the internally damped string, [2], we strive to identify $\omega(\rho)$ with the spectral abscissa of the matrix differential operator obtained on expressing (1.1) as the first order system $U_t = AU$. Here $U = [u, u_t]$, and $A : D(A) \to V$, is given by

$$
A = \begin{pmatrix} 0 & I \\ \frac{1}{\rho^2} \frac{d^2}{dx^2} & 0 \end{pmatrix}, \quad D(A) = \{ [u, v] \in (H^2(0, 1) \times H^1(0, 1)) \cap V : u'(0) = u'(1) + v(1) = 0 \}.
$$

We shall see, assuming no more than (1.2), that $A$ possesses a compact dissipative inverse on $V$. As a result $\sigma(A)$, the spectrum of $A$, is composed of at most a countable number of eigenvalues, each an element of the left half plane.

If $U = [y, z] \in D(A)$ is an eigenvector of $A$ with eigenvalue $\lambda$ then $z = \lambda y$ and $y'' = \lambda \rho^2 z$, or

$$
y'' = \lambda^2 \rho^2 y, \quad y'(0) = y'(1) + \lambda y(1) = 0.
$$

When $\rho$ is constant it follows that $y(x) = \cosh(\lambda \rho x)$ where $\lambda$ is determined by the right end condition, i.e., by $\lambda \rho \sinh(\lambda \rho) + \lambda \cosh(\lambda \rho) = 0$, or

$$
\lambda \rho \tanh(\lambda \rho) = -\lambda.
$$

That $\lambda = 0$ is not permitted follows from the fact that the associated eigenvector $U = [1, 0] \not\in V$. Hence, the eigenvalues are

$$
\lambda_n = -\frac{1}{\rho} \tanh^{-1} \frac{1}{\rho} = -\frac{1}{2\rho} \log \left| \frac{\rho + 1}{\rho - 1} \right| + i \frac{\pi}{\rho} \begin{cases} 
\frac{n}{\rho} & \text{if } \rho > 1 \\
\frac{n + 1}{2} & \text{if } \rho < 1
\end{cases} \quad n \in \mathbb{Z},
$$

while the corresponding eigenvectors are

$$
U_n(x) = \cosh(\lambda_n \rho x)[1, \lambda_n].
$$

**ex 2. The Existence of Eigenvalues**

The eigenvalues of $A$ are the poles of the resolvent $\lambda \mapsto (A - \lambda)^{-1}$. To solve $(A - \lambda)[v_1, v_2] = [f_1, f_2]$ in $V$ is to set $v_2 = \lambda v_1 + f_1$ and solve

$$
v_1'' - \lambda^2 \rho^2 v_1 = \rho^2 (\lambda f_1 + f_2)
$$

$$
v'_1(0) = 0, \quad v'_1(1) + \lambda v_1(1) + f_1(1) = 0,
$$

(2.1)_{\text{nhibp}}
subject to

\[ v_1(1) = -\lambda \int_0^1 \rho^2 v_1 \, dx - \int_0^1 \rho^2 f_1 \, dx. \tag{2.2}_{v\text{con}} \]

We note that \( \frac{1}{2} f_1(1)(1-x^2) \) satisfies the boundary conditions in (2.1) and so write \( v_1(x) = \frac{1}{2} f_1(1)(1-x^2) + w(x) \) where \( w \) must now solve

\[
\begin{align*}
  w'' - \lambda^2 \rho^2 w &= \rho^2 (\lambda f_1 + f_2 + \frac{1}{2} \lambda^2 f_1(1)(1-x^2)) + f_1(1), \\
  w'(0) &= w'(1) + \lambda w(1) = 0. 
\end{align*} \tag{2.3}_{\text{red}}
\]

We first show that this problem has a one parameter family of solutions when \( \lambda = 0 \) and that (2.2) selects a particular one. When \( \lambda = 0 \) we find

\[
\begin{align*}
  w'' &= \rho^2 f_2 + f_1(1), \\
  w'(0) &= w'(1) = 0. \tag{2.4}_{\text{baby}}
\end{align*}
\]

The solvability condition, \( \int_0^1 \rho^2 f_2 \, dx + f_1(1) = 0 \), that stems from the Fredholm Alternative exactly coincides with one of the requirements for \( [f_1, f_2] \in V \). As a result,

\[
w(x) = a + \int_0^x (x-s) (\rho^2(s) f_2(s) + f_1(1)) \, ds, \quad a \in \mathbb{R}. \]

We choose \( a \) to satisfy (2.2), i.e., \( w(1) = -\int_0^1 \rho^2 f_1 \, dx \), and so arrive at

\[
v_1(x) = \int_0^x (x-s) \rho^2 f_2 \, ds - \int_0^1 (1-s) \rho^2 f_2 \, ds - \int_0^1 \rho^2 f_1 \, ds.
\]

Noting that \( v_2 = f_1 \) when \( \lambda = 0 \), we summarize the above in

\[
A^{-1}[f_1, f_2] = \left[ \int_0^x (x-s) \rho^2 f_2 \, ds - \int_0^1 (1-s) \rho^2 f_2 \, ds - \int_0^1 \rho^2 f_1 \, ds, f_1 \right].
\]

From the boundedness of \( \rho \) and the compactness of the imbedding of \( H^1(0,1) \) in to \( L^2(0,1) \) follows the compactness of \( A^{-1} \) on \( V \). As a result, the spectrum of \( A \) is composed of at most a countable number of eigenvalues, \( \{\lambda_k\}_k \).

We now return to (2.3), assume \( \lambda \neq 0 \), and characterize the \( \lambda_k \) as the zeros of a shooting function. In particular, we introduce \( \phi(x, \lambda) \) and \( \psi(x, \lambda) \), solutions of the respective initial and terminal value problems,

\[
\begin{align*}
  \phi'' - \lambda^2 \rho^2 \phi &= 0, \quad &\phi(0, \lambda) = 1, \quad \phi'(0, \lambda) = 0, \tag{2.5}_{\text{init}} \\
  \psi'' - \lambda^2 \rho^2 \psi &= 0, \quad &\psi(1, \lambda) = 1/\lambda, \quad \psi'(1, \lambda) = -1. \tag{2.6}_{\text{term}}
\end{align*}
\]

We note that \( \phi \) likewise satisfies the integral equation

\[
\phi(x, \lambda) = 1 + \lambda^2 \int_0^x (x-s) \rho^2(s) \phi(s, \lambda) \, ds, \tag{2.7}_{\text{pint}}
\]
and denote the corresponding shooting function by

\[ Q(\lambda) \equiv \phi'(1, \lambda)/\lambda + \phi(1, \lambda). \] (2.8)shoot

Clearly \( \sigma(A) \) coincides with the set of zeros of \( Q \). In the introduction we found that if \( \rho \equiv 1 \) then \( Q \) never vanishes and \( \sigma(A) = \emptyset \). We now show that to be the only pathological case.

**Theorem 2.1.** If \( \rho \) satisfies (1.2) and \( \rho \) is not identically one then \( \sigma(A) \neq \emptyset \).

Proof: Dym and McKean [3, §6.3(6)] demonstrate that \( Q \) is of exponential type

\[ \int_0^1 \rho \, dx = \limsup_{R \to \infty} R^{-1} \max_{\theta \in [0,2\pi]} \log |Q(Re^{i\theta})|. \]

It then follows from Hadamard’s Factorization Theorem that

\[ Q(\lambda) = e^{a\lambda} \prod_n (1 - \lambda/\lambda_n)e^{\lambda/\lambda_n}, \] (2.9)hada

where \( \{\lambda_n\}_n \) is the zero set of \( Q \) and \( a \) is a complex constant. If this zero set is empty then the product defaults to one and \( Q(\lambda) = e^{a\lambda} \). As \( Q(\lambda) \) is real for real \( \lambda \) it follows that \( a \) is real and in fact \( a = \int_0^1 \rho \, dx \). We now deduce the restraints this places on \( \phi \). We follow Kac and Krein [7, §2] and develop \( \phi \), as the solution of (2.7), in powers of \( \lambda^2 \),

\[ \phi(x, \lambda) = \sum_{n=0}^{\infty} \phi_n(x)\lambda^{2n}, \quad \phi_0(x) \equiv 1, \quad \phi_{j+1}(x) = \int_0^x (x-s)\rho^2 \phi_j \, ds. \] (2.10)ps

It follows that \( \lambda \mapsto \phi(1, \lambda) \) and \( \lambda \mapsto \phi'(1, \lambda)/\lambda \) are power series in \( \lambda^2 \) and \( \lambda \) respectively. As \( Q(\lambda) = e^{a\lambda} = \cosh(a\lambda) + \sinh(a\lambda) \) we find the explicit representations

\[ \phi(1, \lambda) = \cosh(a\lambda) \quad \text{and} \quad \phi'(1, \lambda) = \lambda \sinh(a\lambda). \] (2.11)dsh

We recognize the former as the shooting function for the Neumann–Dirichlet problem

\[ \eta'' = \nu^2 \rho^2 \eta, \quad \eta'(0) = \eta(1) = 0, \]

and the latter as the shooting function for the Neumann–Neumann problem

\[ \zeta'' = \chi^2 \rho^2 \zeta, \quad \zeta'(0) = \zeta'(1) = 0. \]

In particular, from (2.11) it follows that

\[ \nu_n = \frac{i\pi}{2a}(2n-1) \quad \text{and} \quad \chi_n = \frac{i\pi}{a}(n-1), \quad n = 1, 2, \ldots \]
On recalling the well known fact, see, e.g., [3, §6.6], that two such spectra uniquely determine a bounded \( \rho \) we immediately conclude that \( \rho \) is identically \( a \). Hence, it suffices to restrict ourselves to constant \( \rho \).

From the introduction we now recall that \( Q(\lambda) = e^{\rho \lambda} \) if and only if \( \rho = 1 \).

If \( \lambda_n \in \sigma(A) \) we note that as a consequence of the uniqueness of \( \phi(1, \lambda_n) \) that each eigenvector of \( A \) corresponding to \( \lambda_n \) must be a scalar multiple of \( U_n(x) \equiv \phi(x, \lambda_n)[1, \lambda_n] \). In other words, the geometric multiplicity of each eigenvalue is one. As a result, the algebraic multiplicity of an eigenvalue is its order as pole of \( (A - \lambda)^{-1} \). We now associate this order with \( Q \).

If \( \lambda \not\in \sigma(A) \) then (2.3) has a unique solution that is consistent with (2.2). In particular,

\[
w(x) = G(\lambda)\{\rho^2(\lambda f_1 + f_2 + \frac{1}{2}\lambda^2 f_1(1-x^2)) + f_1(1)\},
\]

where \( G(\lambda) \) is the Green’s operator

\[
G(\lambda)\eta(x) = \int_0^1 g(x, \xi, \lambda)\eta(\xi) \, d\xi, \quad \text{where}
\]

\[
g(x, \xi, \lambda) = \begin{cases} \frac{\psi(x, \lambda)\phi(\xi, \lambda)}{Q(\lambda)} & \text{if } 0 \leq \xi \leq x \leq 1 \\ \frac{\psi(\xi, \lambda)\phi(x, \lambda)}{Q(\lambda)} & \text{if } 0 \leq x \leq \xi \leq 1. \end{cases}
\]

Hence the algebraic multiplicity of \( \lambda_n \in \sigma(A) \) is its order as a zero of \( Q \).

Some preliminary information is obtained upon taking the \( L^2(0,1) \) inner product of (2.5) at \( \lambda = \lambda_n \) with \( \phi_n(x) \equiv \phi(x, \lambda_n) \). One finds

\[
\lambda_n^2 \int_0^1 \rho^2|\phi_n|^2 \, dx + \lambda_n|\phi_n(1)|^2 + \int_0^1 |\phi_n'|^2 \, dx = 0,
\]

and therefore

\[
\lambda_n = \frac{-|\phi_n(1)|^2 \pm \sqrt{|\phi_n(1)|^4 - 4 \int_0^1 \rho^2|\phi_n|^2 \, dx \int_0^1 |\phi_n'|^2 \, dx}}{2 \int_0^1 \rho^2|\phi_n|^2 \, dx}.
\]

It follows that \( \Re \lambda_n \leq 0 \), in fact \( \Re \lambda_n < 0 \), for equality would force \( \phi_n(1) = \phi_n'(1) = 0 \) and hence \( \phi_n \equiv 0 \). Dym and McKean [3, §6.3] also show that

\[
\int_{-\infty}^{\infty} \frac{\log_+ |Q(-ix)|}{1 + x^2} \, dx < \infty,
\]

from which it follows, see, e.g., Levin [14, §V.4, Theorem 11], that there exist \( C_1 \) and \( C_2 \) for which

\[
|\Re \lambda_n| \leq C_1 + C_2|3\lambda_n|.
\]
Majda [16], in the context of (**), has bettered this with

\[ |\Re \lambda_n| \leq C_1 + C_2 |\Im \lambda_n|^{3/4}. \]

We shall soon see in fact that \( |\Re \lambda_n| \leq C_1 \) when \( \rho \) is Lipschitz.

**3. The Completeness of the Root Vectors**

We follow Krein and Nudelman [11] in their application of the following result of Livšic. We assume throughout that \( \rho \) is not identically one.

**Theorem 3.1.** ([10, §2.5]) If \( H \) is Hilbert and \( T : H \to H \) is linear and compact and \( T_{\Re} \equiv \frac{1}{2} (T + T^*) \) is nonpositive and of finite trace then

\[ \sum_{\nu_n \in \sigma(T)} |\Re \nu_n| \leq -\text{tr} (T_{\Re}), \quad (3.1)_{\text{liv}} \]

where the \( \nu_n \) are repeated according to their algebraic multiplicity. Equality holds in (3.1) if and only if the root vectors of \( T \) are complete in \( H \).

We note that \( (T^*)_{\Re} = T_{\Re} \) and that if \( T \) is real then \( \sigma(T^*) = \sigma(T) \) including algebraic multiplicities. Hence, if \( T \) is real and equality holds in (3.1) then the root vectors of \( T^* \) are complete in \( H \) as well.

We show that \( (A^{-1})_{\Re} \) is nonpositive and of rank one. In particular,

\[
(A^{-1})^*[f_1, f_2] = \left[ \int_0^1 (1 - s) \rho^2 f_2 \, ds - \int_0^x (x - s) \rho^2 f_2 \, ds + \int_0^1 \rho^2 f_1 \, ds + 2 \int_0^1 \rho^2 \, ds \int_0^1 \rho^2 f_2 \, ds, -f_1 - 2 \int_0^1 \rho^2 f_2 \, ds \right],
\]

and so

\[
(A^{-1})_{\Re}[f_1, f_2] = \int_0^1 \rho^2 f_2 \, dx \left[ \int_0^1 \rho^2 \, dx, -1 \right]
\]

is indeed rank-one while

\[
\langle (A^{-1})_{\Re}[f_1, f_2], [f_1, f_2] \rangle = -\left| \int_0^1 \rho^2 f_2 \, ds \right|^2
\]

and

\[
\text{tr} (A^{-1})_{\Re} = -\int_0^1 \rho^2 \, dx.
\]

From Theorem 3.1 we may now draw
Corollary 3.2. If $\rho$ satisfies (1.2) then 
$$ \sum_{\lambda_n \in \sigma(A)} \frac{\Re \lambda_n}{|\lambda_n|^2} \leq \int_0^1 \rho^2 \, dx. $$

We now produce conditions under which equality holds in Corollary 3.2. In the constant case we offer the following elementary argument.

Theorem 3.3. If $\rho$ is a positive constant distinct from one then the $\lambda_n \in \sigma(A)$, see (1.4), are each of algebraic multiplicity one and their associated eigenvectors are complete in $V$.

Proof: We assume $\rho > 1$. The other case being similar. We sum the $|\Re \lambda_n|/|\lambda_n|^2$ without repetition,
$$ \sum_n \frac{|\Re \lambda_n|}{|\lambda_n|^2} = \frac{\rho}{\nu} + 2\nu \rho \sum_{n=1}^{\infty} \frac{1}{\nu^2 + n^2 \pi^2}, \quad \nu \equiv \frac{1}{2} \log \left( \frac{\rho + 1}{\rho - 1} \right). \quad (3.2)_{\text{sum}} $$

As $\nu^2 + n^2 \pi^2$ is the $n$th eigenvalue of $Lu \equiv -u'' + \nu^2 u$, $u \in H^1_0(0,1)$, it follows from the standard trace formula that
$$ \sum_{n=1}^{\infty} \frac{1}{\nu^2 + n^2 \pi^2} = \int_0^1 g(x,x) \, dx $$

where
$$ g(x,y) = \frac{\sinh \nu (x \vee y) \sinh \nu (1 - (x \wedge y))}{\nu \sinh \nu} $$
is the Green’s function for $L$. Hence,
$$ \sum_{n=1}^{\infty} \frac{1}{\nu^2 + n^2 \pi^2} = \frac{\nu \cosh \nu - \sinh \nu}{2\nu^2 \sinh \nu} = \frac{\rho}{2\nu} - \frac{1}{2\nu^2}. $$

inserting this sum in (3.2) we find that equality holds in Corollary 3.2. \qed

The fact that a (bi)normalized copy of $\{U_n\}_n$ for constant $\rho$ is in fact a Riesz basis for $V$ now follows easily. As in [2] our principal tool is the following result of Bari.

Theorem 3.4. [4, Theorem 2.1, Chapter VI] $\{f_n\}_n$ is a Riesz basis for the Hilbert space $H$ if and only if $\{f_n\}_n$ is complete in $H$ and there corresponds to it a complete biorthogonal sequence $\{g_n\}_n$, and for any $f \in H$ both $\{\langle f_n, f \rangle\}_n$ and $\{\langle g_n, f \rangle\}_n$ are square summable.

We shall also make use of the equivalent statement that $\{f_n\}_n$ is a Riesz basis for $H$ iff $\{f_n\}_n$ is complete and there exist two constants $c_0$ and $c_1$ such
$$ c_0 \sum_{n=1}^{N} |a_n|^2 \leq \left\| \sum_{n=1}^{N} a_n f_n \right\|_H^2 \leq c_1 \sum_{n=1}^{N} |a_n|^2 \quad (3.3)_{b2} $$

for each $N$ and each $\{a_n\}_n \in \mathbb{C}^N$. 

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The sequence biorthogonal to \( \{U_n\}_n \) is built from the eigenvectors of the adjoint of \( A \),
\[
A^* = -\begin{pmatrix} 0 & d^2 \rho x \frac{d^2}{dx^2} \\ \rho^2 x \frac{d^2}{dx^2} & 0 \end{pmatrix} \quad D(A^*) = \{ [u, v] \in (H^2(0, 1) \times H^1(0, 1)) \cap V : u'(0) = u'(1) - v(1) = 0 \}.
\]

We note that \( \sigma(A^*) = \sigma(A) \) and that if \( W_k \) is an eigenvector of \( A^* \) corresponding to \( \overline{\lambda}_k \) then, for constant \( \rho \), \( W_k(x) = \cosh(\overline{\lambda}_k \rho x)[-1, \overline{\lambda}_k] \) and \( \langle U_j, W_k \rangle = \lambda^2_k \rho^2 \delta_{jk} \). As a result
\[
\begin{align*}
\tilde{U}_k(x) &= \frac{1}{\lambda_k \rho} \cosh(\lambda_k \rho x)[1, \lambda_k], \\
\tilde{W}_k(x) &= \frac{1}{\lambda_k \rho} \cosh(\overline{\lambda}_k \rho x)[-1, \overline{\lambda}_k] \tag{3.4}_{\text{rncon}}
\end{align*}
\]
satisfy \( \langle \tilde{U}_j, \tilde{W}_k \rangle = \delta_{jk} \), i.e., they constitute a biorthogonal set in \( V \). That \( \{\tilde{W}_n\}_n \) is complete in \( V \) follows from the remark following Theorem 3.1. It remains to select \([f_1, f_2] \in V \) and check \( \{\langle \tilde{U}_n, [f_1, f_2] \rangle \} \in \ell^2(C) \). We suppose \( \rho > 1 \), the other case being similar, and recall the definition of \( \nu \) in (3.2). That
\[
\langle \tilde{U}_n, [f_1, f_2] \rangle = \int_0^1 \sinh(\lambda_n \rho x) \overline{f}_1 + \rho \cosh(\lambda_n \rho x) \overline{f}_2 \ dx
\]
\[
= \int_0^1 (\sinh(\nu) \overline{f}_1 + \rho \cosh(\nu) \overline{f}_2) \cos(n \pi x) \ dx + i \int_0^1 (\cosh(\nu) \overline{f}_1 + \rho \sinh(\nu) \overline{f}_2) \sin(n \pi x) \ dx
\]
is square summable now follows directly from \([f_1, f_2] \in V \). The verification that \( \{\langle \tilde{W}_n, [f_1, f_2] \rangle \}_n \in \ell^2(C) \) is just as simple. From Theorem 3.4. we may now deduce

**Theorem 3.5.** If \( \rho \) is a positive constant distinct from one then the eigenvectors \( \{\tilde{U}_n\}_n \), see (3.4), constitute a Riesz basis for \( V \). If we append to this sequence the vector \([1, 0] \), corresponding to the zero eigenvalue of \( A \) in \( X \), we obtain a Riesz basis for \( X \).

We now return to the variable coefficient case and address the extent to which the root vectors of \( A \) are complete in \( V \). In particular, we return to [11] and equate the power series representation of \( Q \) stemming from (2.10) with a refinement of (2.9). First, it follows directly from (2.8) and (2.10) that
\[
Q(\lambda) = 1 + \lambda \int_0^1 \rho^2 dx + O(\lambda^2). \tag{3.5}_{\text{rep1}}
\]
Next, from the summability of \(|\Re\lambda_n|/|\lambda_n|^2 \) comes the fact that one may remove the exponential factors in (2.9), i.e.,
\[
Q(\lambda) = e^{a\lambda} \prod_{\lambda_n > 0} \left( 1 - \frac{\lambda}{\lambda_n} \right) \left( 1 - \frac{\lambda}{\lambda_n} \right) \prod_{\lambda_n = 0} \left( 1 - \frac{\lambda}{\lambda_n} \right) \tag{3.6}_{\text{rep2}}
\]
See [3, §6.2] for an independent proof. From the easily verified fact that if \( \rho(x) = 1 \) for \( x \in (\ell, 1) \) then \( \sigma(A) \) coincides with the spectrum associated with (1.1) for \( x \in (0, \ell) \) subject to \( u_x(0, t) = u_x(\ell, t) + u_t(\ell, t) = 0 \) Krein and Nudelman next deduce that \( a \) is in fact the largest number \( b \) for which \( \rho \) is identically one on the interval \( (1 - b, 1) \). Identifying the coefficients of \( \lambda \) in (3.5) and (3.6) we find

\[
\int_0^1 \rho^2 \, dx = a - \sum_{\lambda_n \in \sigma(A)} \frac{\Re \lambda_n}{|\lambda_n|^2}.
\]

As a result, analogous to [11, Theorem 2], we have

**Theorem 3.6.** If \( \rho \) satisfies (1.2) then the root vectors of \( A \) are complete in \( V \) if and only if \( \rho \) is not identically one on any interval of the form \( (\ell, 1) \).

The proof that these root vectors indeed make up a basis for \( V \) will require a considerably more detailed study of the \( \lambda_n \). In the next three sections we respectively analyze the real eigenvalues, establish a crude lower bound on \( \Re \lambda_n \), and develop asymptotic estimates for \( \lambda_n \).

**high 4. High Frequencies**

We now develop asymptotic formulas for \( \lambda_n \) and \( U_n \) as \( |\lambda_n| \to \infty \). Our development can be seen both as an elaboration of [6], where few details are provided and more function theory is invoked, and as a special case of [22], where, in her desire to permit \( \rho \) to either vanish or become infinite at the damped end, Shubov requires a twenty page immersion in special functions. At the heart of both approaches, as with our previous work [2], is a fake potential that has the advantage that its introduction into (2.6) permits one to find an explicit solution. One then argues that the fake potential has a negligible effect on the high frequencies. The fake potential in this case

\[
q(x) \equiv \rho^{1/2}(x) \left( \rho^{-1/2}(x) \right)' = \frac{3}{4} \left( \frac{\rho'(x)}{\rho(x)} \right)^2 - \frac{\rho''(x)}{2\rho(x)},
\]

lies in \( L^2(0, 1) \) so long as, in addition to (1.2), \( \rho \in H^2(0, 1) \). Its addition to (2.6) brings us to

\[
z'' - qz = \lambda^2 \rho^2 z, \quad z(1) = \lambda^{-1}, \quad z'(1) = -1,
\]

the solution to which is

\[
z(x, \lambda) = \frac{\rho_1 \cosh(\lambda \int_x^1 \rho \, ds) + \sinh(\lambda \int_x^1 \rho \, ds)}{\lambda \sqrt{\rho_1 \rho(x)}} - \frac{(\log \rho)'(1) \sinh(\lambda \int_x^1 \rho \, ds)}{2\lambda^2 \sqrt{\rho_1 \rho(x)}}
\]

\[
\equiv w_1(x, \lambda) + O(|\lambda|^{-2}).
\]
Recall that in (***) we defined \( \rho_1 \equiv \rho(1) \). As \( z'(0, \lambda) \) will serve as our fake shooting function, we require
\[
 z'(x, \lambda) = -\frac{\sqrt{\rho(x)}}{\sqrt{\rho_1}} \left\{ \rho_1 \sinh(\lambda \int_x^1 \rho ds) + \cosh(\lambda \int_x^1 \rho ds) \right\} - \\
\frac{\rho'(x)}{2\lambda \rho^{3/2}(x) \sqrt{\rho_1}} \left\{ \rho_1 \cosh(\lambda \int_x^1 \rho ds) + \sinh(\lambda \int_x^1 \rho ds) \right\} + \\
\frac{\sqrt{\rho(x)(\log \rho)'(1)}}{2\lambda \sqrt{\rho_1}} \cosh(\lambda \int_x^1 \rho ds) + \frac{\rho'(x)(\log \rho)'(1)}{4\lambda^2 \rho^{3/2}(x) \sqrt{\rho_1}} \sinh(\lambda \int_x^1 \rho ds)
\]
\[
\equiv w_2(x, \lambda) + O(|\lambda|^{-1}).
\]

We see on inspection that \( z(x, \cdot) \) and \( z'(x, \cdot) \) are asymptotically close to \( w_1(x, \cdot) \) and \( w_2(x, \cdot) \). We now show in fact that \( \psi(x, \lambda) \) and \( \psi'(x, \lambda) \) are asymptotically close to \( w_1(x, \cdot) \) and \( w_2(x, \cdot) \), where \( \psi(x, \lambda) \) is the actual solution to (2.6). By Theorem ???.1 it suffices to work in the band \(-\kappa \leq \Re \lambda \leq 0\).

**Theorem 4.1.** Assume that \( \rho \in H^2(0, 1) \), \( \rho_1 \neq 1 \), and \( \rho \) satisfies (1.2). Then there exist constants \( C_0 \) and \( C_1 \) such that
\[
|\psi(x, \lambda) - w_1(x, \lambda)| \leq C_0|\lambda|^{-2}, \quad (4.3)_{\text{pest}}
\]
\[
|\psi'(x, \lambda) - w_2(x, \lambda)| \leq C_1|\lambda|^{-1}, \quad (4.4)_{\text{pest}}
\]
uniformly for \( 0 < x < 1 \) and
\[
-\kappa \leq \Re \lambda \leq 0 \quad |\lambda| \geq \max\{1, \frac{\|\rho\|_\infty}{2\alpha}\}. \quad (4.5)_{\text{labnd}}
\]

**Proof:** We note that \( \psi \) satisfies
\[
\psi'' - q\psi - \lambda^2 \rho^2 \psi = -q\psi, \quad \psi(1, \lambda) = 1/\lambda, \quad \psi'(1, \lambda) = -1,
\]
and therefore the integral equation
\[
\psi(x, \lambda) = z(x, \lambda) - \int_x^1 K(x, t, \lambda) q(t) \psi(t, \lambda) \, dt \quad (4.6)_{\text{intp}}
\]
with \( K(x, t, \lambda) = z(x, \lambda) \tilde{z}(t, \lambda) - z(t, \lambda) \tilde{z}(x, \lambda) \) where
\[
\tilde{z}(x, \lambda) = \frac{-1}{\sqrt{\rho_1 \rho(x)}} \sinh(\lambda \int_x^1 \rho ds),
\]
satisfies the same differential equation as \( z \) but with terminal data \( \tilde{z}(1, \lambda) = 0, \tilde{z}'(1, \lambda) = \lambda \).

We solve the integral equation (4.6) in series form
\[
\psi(x, \lambda) = \sum_{n=0}^\infty S_n(x, \lambda), \quad (4.7)_{\text{ser}}
\]
where $S_0 = z$ and

$$S_n(x, \lambda) = - \int_x^1 K(x, t, \lambda) q(t) S_{n-1}(t, \lambda) \, dt$$

$$= (-1)^n \int_1^{t_1 \geq \cdots \geq t_{n+1} = x} z(t_1, \lambda) \prod_{i=1}^n [K(t_{i+1}, t_i, \lambda) q(t_i)] \, dt_1 \cdots dt_n.$$ 

In order to establish the convergence of (4.7) we proceed to derive uniform estimates for the $S_n$. To begin, we recall $m = \int_0^1 \rho \, dx$ and observe that

$$|z(x, \lambda)| \leq \frac{(\beta + 1)e^{km}}{\alpha |\lambda|} + \frac{\|\rho'\|_\infty e^{km}}{2\alpha^2|\lambda|} \leq \frac{(\beta + 2)e^{km}}{\alpha |\lambda|}$$

when $\lambda$ obeys (4.5). Likewise, as

$$|\tilde{z}(x, \lambda)| \leq \frac{e^{km}}{\alpha}$$

it follows that

$$|K(x, t, \lambda)| \leq \frac{2(\beta + 2)e^{2km}}{\alpha^2|\lambda|}$$

and hence

$$|S_n(x, \lambda)| \leq \frac{(\beta + 2)e^{km}}{\alpha |\lambda|} \left( \frac{2(\beta + 2)e^{2km}}{\alpha^2|\lambda|} \right)^n \frac{\|q\|_1^n}{n!}$$

$$\leq \frac{(\beta + 2)e^{km}}{\alpha |\lambda|^2} \frac{(2\alpha^{-2}(\beta + 2)\|q\|_2^2 e^{2km})^n}{n!}.$$ 

From the Weierstrass comparison test it now follows that (4.7) converges uniformly for $0 \leq x \leq 1$ and $\lambda$ satisfying (4.5). Moreover,

$$|\psi(x, \lambda) - w_1(x, \lambda)| \leq |\psi(x, \lambda) - z(x, \lambda)| + |z(x, \lambda) - w_1(x, \lambda)|$$

$$\leq \sum_{n=1}^\infty |S_n(x, \lambda)| + \frac{e^{km}\|\rho'\|_\infty}{2\alpha|\lambda|^2}$$

$$\leq \frac{e^{km}}{\alpha |\lambda|^2} \left( \frac{1}{2}\|\rho'\|_\infty + (\beta + 2)e^{2\alpha^{-2}(\beta + 2)\|q\|_2 e^{2km}} \right)$$

$$\equiv C_0|\lambda|^{-2}$$

now establishes (4.3).

Regarding the estimate for $\psi'$ we differentiate (4.6) and find

$$\psi'(x, \lambda) - z'(x, \lambda) = - \int_x^1 K_x(x, t, \lambda) q(t) \psi(t, \lambda) \, dt,$$  

(4.8)pre
and so it remains to simply bound $\psi$ and $K_x$. With respect to the former

$$|\psi(x, \lambda)| \leq |\psi(x, \lambda) - w_1(x, \lambda)| + |w_1(x, \lambda)| \leq \frac{C_0 + (1 + \beta)\alpha^{-1}e^{\kappa m}}{\lambda},$$

while the latter requires both

$$|z'(x, \lambda)| \leq e^{\kappa m} \left(1 + \frac{1}{\alpha} + \sqrt{\frac{\beta}{\alpha}}\right)(1 + \beta) \quad \text{and} \quad |\tilde{z}'(x, \lambda)| \leq \frac{|\lambda|e^{\kappa m}}{\alpha}(1 + \beta).$$

Assembling these bounds we find

$$|K_x(x, t, \lambda)| \leq |z'(x, \lambda)||\tilde{z}(t, \lambda)| + |z(t, \lambda)||\tilde{z}'(x, \lambda)|$$

$$\leq \frac{e^{2\kappa m}}{\alpha} \left(1 + \frac{1}{\alpha} + \sqrt{\frac{\beta}{\alpha}}\right)(1 + \beta) + \frac{e^{2\kappa m}}{\alpha^2}(2 + \beta)(1 + \beta)$$

$$\leq \frac{4e^{2\kappa m}}{\alpha^2}(1 + \beta)^2.$$

Now (4.8) yields (4.4),

$$|\psi'(x, \lambda) - w_2(x, \lambda)| \leq |\psi'(x, \lambda) - z'(x, \lambda)| + |z'(x, \lambda) - w_2(x, \lambda)|$$

$$\leq \int_0^1 |K_x(x, t, \lambda)||\psi(t, \lambda)||q(t)||dt + \frac{e^{\kappa m}\|\rho'\|_\infty}{2|\lambda|\alpha^2}(1 + \beta)^2$$

$$\leq \frac{4(1 + \beta)^3 e^{3\kappa m}}{\alpha^2|\lambda|}((C_0 + \alpha^{-1})\|q\|_2 + \|\rho'\|_\infty)$$

$$\equiv C_1|\lambda|^{-1}. \blacksquare$$

On close inspection of the estimate for $S_n$ it follows that $dq$ need only be a finite measure, i.e., it suffices to require that $\rho'$ simply have finite total variation.

We now show that the zeros of $\psi'(0, \lambda)$ are close to the zeros of $w_2'(0, \lambda)$, these being

$$\mu_n = -\frac{1}{2m} \log \left|\frac{1 + \rho_1}{\rho_1 - 1}\right| + \frac{i\pi}{m} \left\{ \begin{array}{ll} n & \text{if } \rho_1 > 1 \\ n + \frac{1}{2} & \text{if } \rho_1 < 1 \end{array} \right. \quad n \in \mathbb{Z}. \quad (4.9)_{\text{mus}}$$

This is done by choosing $r_n$ in

$$\Gamma_n = \{\lambda \in \mathbb{C} : |\lambda - \mu_n| = r_n\}$$

in such a way that the $\Gamma_n$ do not intersect and

$$|\psi'(0, \lambda) - w_2(0, \lambda)| < |w_2(0, \lambda)|, \quad \lambda \in \Gamma_n.$$
By the previous Theorem it suffices to show that

\[ |w_2(0, \lambda)| > \frac{C_1}{|\lambda|}, \quad \lambda \in \Gamma_n. \]

We proceed under the assumption that \( \rho_1 > 1 \), the other case following similarly. If \( \lambda \in \Gamma_n \) then \( \lambda = \mu_n + r_ne^{i\theta} \) where \( \theta \in [0, 2\pi) \) and

\[
\begin{align*}
\lambda w_2(0, \lambda) &= -\lambda \sqrt{\rho_0/\rho_1} \{ \rho_1 \sinh(m(\mu_n + r_ne^{i\theta})) + \cosh(m(\mu_n + r_ne^{i\theta})) \} \\
&= -\lambda \sqrt{\rho_0/\rho_1} (\rho_1^2 - 1) \sinh(m\mu_n) \sinh(mr_ne^{i\theta}) \\
&= (-1)^n \sqrt{\rho_0/\rho_1} (\rho_1^2 - 1) \sinh(m\mu_0)(\mu_n + r_ne^{i\theta}) \sinh(mr_ne^{i\theta})
\end{align*}
\]

Hence, if \( C' \equiv \sqrt{\rho_0/\rho_1}(1 - \rho_1^2) \sinh(m\mu_0) \) then

\[
|\lambda||w_2(0, \lambda)| > C'(|n|\pi/m - r_n)|\sinh(mr_ne^{i\theta})| \\
\geq C'(|n|\pi/m - r_n)(mr_n - \frac{1}{2}m^2r_n^2) \\
\geq C'r_n(|n|\pi - mr_n(1 + \frac{1}{2}|n|\pi)).
\]

One makes the obvious guess \( r_n = \frac{2C_1}{C'|n|\pi} \) and finds that

\[
|\lambda||w_2(0, \lambda)| \geq C'r_n(|n|\pi - mr_n(1 + \frac{1}{2}|n|\pi)) \\
= C_1 \left( 2 - \frac{4C_1m}{C'|n|\pi} \left( \frac{1}{2} + \frac{1}{|n|\pi} \right) \right) \\
> C_1, \quad \text{when } |n| > N_1 = \left\lceil \frac{4C_1m}{C'|\pi} \right\rceil,
\]

where \( \lceil x \rceil \) denotes the least integer greater than \( x \). Furthermore, this choice of \( N \) renders \( r_n < 1/(2m) \). As the distance between centers of the \( \Gamma_n \) is \( \pi/m \) it follows that the contours are nonintersecting. To capture the remaining eigenvalues we consider

\[ Z_n = \left\{ \lambda \in \mathbb{C} : |\lambda - \mu_0| = \frac{\pi}{m}(n + \frac{1}{2}) \right\}, \]

and denote by \( N_2 \) the smallest integer \( n > 0 \) for which \( Z_n \) encircles the disk of radius \( C_1/C' \) centered at the origin. For each \( n \) we note that if \( \lambda \in Z_n \) then \( |w_2(0, \lambda)| \geq C' \) while when \( n > N_2 \) we find \( |\lambda| > C_1/C' \) as well. As a result,

\[
|\lambda||w_2(0, \lambda)| > C_1, \quad \lambda \in Z_n, \quad n \geq N_2.
\]

With \( N \equiv \max\{N_1, N_2\} \), from the Theorem of Rouché follows
**Theorem 4.2.** If \( \rho \in H^2(0,1), \rho_1 \neq 1, \) and \( \rho \) satisfies (1.2) then \( A(\rho) \) has exactly \( 2N + 1 \) eigenvalues, including multiplicity, in \( \mathbb{Z}_N \) and one simple eigenvalue in \( \Gamma_n \) for \( |n| > N \). This exhausts the spectrum of \( A \).

From the estimate \( \lambda_n = \mu_n + O(|n|^{-1}) \) one easily improves those of Theorem 4.1 at \( \lambda = \lambda_n \).

**Corollary 4.3.** If \( \rho \in H^2(0,1), \rho_1 \neq 1, \) and \( \rho \) satisfies (1.2) then, uniformly for \( 0 < x < 1 \),

\[
\psi(x, \lambda_n) = w_1(x, \mu_n) + O(|n|^{-2}) \\
\psi'(x, \lambda_n) = w_2(x, \mu_n) + O(|n|^{-1}).
\]

### Root 5. The Root Vectors Comprise a Riesz Basis

We denote the algebraic multiplicity of \( \lambda_n \) by \( \nu_n \) and to \( \lambda_n \) associate the Jordan Chain of root vectors, \( \{U_{n,j}\}_{j=0}^{\nu_n-1} \),

\[
U_{n,0}(x) = \psi(x, \lambda_n)[1, \lambda_n], \\
AU_{n,j} = \lambda_n U_{n,j} + U_{n,j-1}, \quad \langle U_{n,j}, U_{n,0} \rangle = 0, \quad j = 1, \ldots, \nu_n - 1.
\]

Clearly, \( U_{n,0} \) is an eigenvector and the chain is a basis for the root subspace

\[
\mathcal{L}_n \equiv \{ U : (A - \lambda_n)^{\nu_n} U = 0 \}.
\]

We construct a biorthogonal sequence to \( \{\tilde{U}_{n,j}\}_{n,j} \) from the eigenvectors of the adjoint, \( A^* \). We recall that \( \sigma(A) = \sigma(A^*) \), including multiplicities, and to \( \overline{\lambda}_n \) we associate the Jordan Chain of root vectors, \( \{W_{n,j}\}_{j=0}^{\nu_n-1} \), where

\[
W_{n,0}(x) = \psi(x, \overline{\lambda}_n)[1, -\overline{\lambda}_n], \\
A^* W_{n,j} = \overline{\lambda}_n W_{n,j} + W_{n,j-1}, \quad \langle W_{n,j}, V_{n,\nu_n-1} \rangle = 0, \quad j = 1, \ldots, \nu_n - 1.
\]

Observe that \( W_{n,0} \) is an eigenvector for \( A^* \) and that the subsequent \( W_{n,j} \) are uniquely determined so long as \( \langle W_{n,0}, V_{n,\nu_n-1} \rangle \neq 0 \). In addition, the chain \( \{W_{n,j}\}_{j=0}^{\nu_n-1} \) is a basis for the root subspace

\[
\mathcal{L}_n^* \equiv \{ W : (A^* - \overline{\lambda}_n)^{\nu_n} W = 0 \}.
\]

**Lemma 5.1.** If \( \rho \in H^2(0,1), \rho_1 \neq 1, \) and \( \rho \) satisfies (1.2) then there exists a \( c > 0 \) such that

\[
|\langle U_{n,p}, W_{j,k} \rangle| = |\langle U_{n,p}, W_{n,\nu_n-1-p} \rangle| \delta_{n,j} \delta_{\nu_n-1-p,k} \geq c \delta_{n,j} \delta_{\nu_n-1-p,k}.
\]

Proof: The biorthogonality is an algebraic result that follows essentially by construction. For details see [2, Lemma 6.2]. The fact that \( c > 0 \), i.e., that the two sequences may be
binormalized, is a consequence of the asymptotic formulas of Corollary 4.3. In particular, we recall from Theorem 4.2 that \( \mu_n = 1 \) for \( n > |N| \) and proceed for such \( n \) to establish

\[
\langle U_{n,0}, W_{n,0} \rangle = \langle \psi(x, \lambda_n)[1, \lambda_n], \psi(x, \lambda_n)[1, -\lambda_n] \rangle \\
= \int_0^1 (\psi'(x, \lambda_n))^2 - \lambda_n^2 \rho^2 \psi^2(x, \lambda_n) \, dx \\
= \int_0^1 w_2^2(x, \mu_n) - \mu_n^2 \rho^2 w_1^2(x, \mu_n) \, dx + O(|n|^{-1}) \\
= \int_0^1 \frac{\rho}{\rho_1} \left( \left\{ \rho_1 \sinh(\mu_n \int_x^1 \rho \, ds) + \cosh(\mu_n \int_x^1 \rho \, ds) \right\}^2 \\
- \left\{ \rho_1 \cosh(\mu_n \int_x^1 \rho \, ds) + \sinh(\mu_n \int_x^1 \rho \, ds) \right\}^2 \right) \, dx + O(|n|^{-1}) \\
= \frac{m}{\rho_1} (1 - \rho_1^2) + O(|n|^{-1})
\]

As \( \rho_1 \neq 1 \) it follows that \( |\langle U_{n,0}, W_{n,0} \rangle| \geq \frac{m}{2 \rho_1} (1 - \rho_1^2) \) for \( n \) of sufficient magnitude.

We may therefore binormalize,

\[
\tilde{U}_{n,0}(x) = \langle U_{n,0}, W_{n,0} \rangle^{-1/2} U_{n,0}(x) = U_{n,0}(x) + O(1/|n|), \quad \text{and} \\
\tilde{W}_{n,0}(x) = \langle U_{n,0}, W_{n,0} \rangle^{-1/2} W_{n,0}(x) = W_{n,0}(x) + O(1/|n|)
\]

for \( |n| > N \). Having demonstrated completeness, in order to invoke Bari’s Theorem it suffices to check that \( \{\langle \tilde{U}_{n,0}, [f, g] \rangle\}_n \in \ell^2(C) \) for each \([f, g] \in V\). Drawing once again on Corollary 4.3 we find

\[
\langle \tilde{U}_{n,0}, [f, g] \rangle = \int_0^1 \psi(x, \lambda_n) f' + \rho^2 \lambda_n \psi(x, \lambda_n) g \, dx + O(|n|^{-1}) \\
= \int_0^1 w_2(x, \mu_n) f' + \rho^2 \mu_n w_1(x, \mu_n) g \, dx + O(|n|^{-1}) \\
= \int_0^1 \frac{\sqrt{\rho}}{\sqrt{\rho_1}} \left( \sinh(\mu_n \int_x^1 \rho \, ds)(\rho g - \rho_1 f') \\
+ \cosh(\mu_n \int_x^1 \rho \, ds)(\rho_1 \rho g - f') \right) \, dx + O(|n|^{-1}).
\]

Its square summability is evidently determined by that of terms of the form

\[
\int_0^1 h(x) \sin(n \pi \xi(x)) \, dx, \quad \text{where} \quad \xi(x) = \frac{\int_x^1 \rho \, ds}{\int_0^1 \rho \, ds}
\]
for \( h \in L^2(0, 1) \). On performing the change of variables \( t = \xi(x) \) we find

\[
\int_0^1 h(x) \sin(n\pi \xi(x)) \, dx = \int_0^1 \frac{h(\xi^{-1}(t))}{\rho(\xi^{-1}(t))} \sin(n\pi t) \, dt,
\]

without doubt an element of \( \ell^2(C) \). Hence,

**Theorem 5.2.** If \( \rho \in H^2(0, 1), \rho_1 \neq 1, \) and \( \rho \) satisfies (1.2) then the root vectors of \( A(\rho) \) comprise a Riesz basis for \( V \) and \( \omega(\rho) = \mu(\rho) \).

6. **Finer Asymptotics**

We set

\[
m \equiv \int_0^1 \rho(y) \, dy \quad \text{and} \quad \tilde{\rho}(x) \equiv \rho(x) - m
\]

and refine our asymptotic result via the simple identity

\[
\lambda_n(\rho) - \lambda_n(m) = \int_0^1 \frac{d}{dt} \lambda_n(m + t\tilde{\rho}) \, dt
\]

\[
= \int_0^1 \frac{\langle A'(m + t\tilde{\rho})U_n(m + t\tilde{\rho}), W_n(m + t\tilde{\rho}) \rangle}{\langle U_n(m + t\tilde{\rho}), W_n(m + t\tilde{\rho}) \rangle} \, dt
\]

It remains to assemble the integrand. From

\[
A(m + t\tilde{\rho}) = \begin{pmatrix}
0 & 1 \\
\frac{1}{(m + t\tilde{\rho})^2} \frac{d^2}{dx^2} & I
\end{pmatrix}
\]

we glean

\[
A'(m + t\tilde{\rho}) = \begin{pmatrix}
0 & 0 \\
\frac{-2\tilde{\rho}}{(m + t\tilde{\rho})^2} \frac{d^2}{dx^2} & 0
\end{pmatrix}
\]

Next, recalling §5

\[
U_n(m + t\tilde{\rho}) = \psi(x, \lambda_n)[1, \lambda_n] \quad \text{and} \quad W_n(m + t\tilde{\rho}) = \psi(x, \overline{\lambda_n})[1, -\lambda_n]
\]

Hence

\[
A'(m + t\tilde{\rho})U_n(m + t\tilde{\rho}) = [0, \frac{-2\tilde{\rho}}{(m + t\tilde{\rho})^2}\psi''(x, \lambda_n)] = \left[0, \frac{-2\tilde{\rho}\lambda_n^2(m + t\tilde{\rho})}{m + t\tilde{\rho}}\psi(x, \lambda_n(m + t\tilde{\rho}))\right],
\]

and

7. **REFERENCES**


[22] Shubov, M.A., Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustic waves by a spherically symmetric inhomogeneity of the density, Differential and Integral Eqns., to appear.
