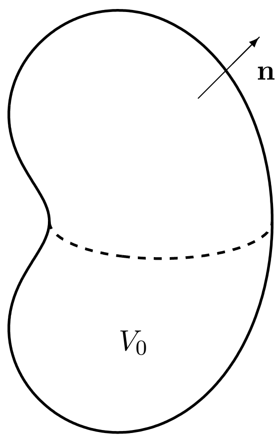
Drag Lecture 2: The Incompressible Navier–Stokes Equation

We continue to work from LANDAU & LIFSCHITZ, *Fluid Mechanics*, 2nd ed. In Lecture 1 we derived the Euler equations, which we will briefly summarize.



A reference volume V_0 in three dimensions with unit outward-pointing normal vector $\mathbf{n}$

▶ Review of Lecture 1

Consider the point $\mathbf{x}$ and the velocity $\mathbf{v}(\mathbf{x}, t)$, density $\rho(\mathbf{x}, t)$, and pressure $p(\mathbf{x}, t)$ at that point, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{v}(\mathbf{x}, t) = \begin{bmatrix} v_1(\mathbf{x}, t) \\ v_2(\mathbf{x}, t) \\ v_3(\mathbf{x}, t) \end{bmatrix}.$$

Last time, we derived:

$$\text{mass balance} \quad \Longrightarrow \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

$$\text{force balance} \quad \Longrightarrow \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \operatorname{grad})\mathbf{v} = -\frac{\operatorname{grad} p}{\rho} \quad (\textit{Euler equation}).$$

Euler's equation leads to Navier–Stokes 'via' momentum flux, first recast. (We assume summation over repeated indices.)

$$\begin{aligned}\text{mass balance} &\implies \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_k)}{\partial x_k}\\ \text{force balance} &\implies \frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}.\end{aligned}$$

So

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i &= -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial(\rho v_k)}{\partial x_k}\\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k)\\ &=: -\frac{\partial \Pi_{ik}}{\partial x_k}\end{aligned}$$

where $\mathbf{\Pi}$ is a 3×3 matrix whose (i, k) entry Π_{ik} is given by

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k,$$

where δ_{ik} is the *Kronecker delta*,

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Equivalently, we can write $\mathbf{\Pi}$ directly as

$$\mathbf{\Pi} = p\mathbf{I} + \rho \mathbf{v}\mathbf{v}^{\mathrm{T}}.$$

What does this mean physically?

Consider a reference volume V_0 with unit outward normal $\mathbf{n}$ (see earlier figure). Consider the rate of change of momentum over V_0:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{V_0} \rho v_i \, \mathrm{d}V &= -\int_{V_0} \frac{\partial \Pi_{ik}}{\partial x_k} \, \mathrm{d}V\\ &= -\int_{\partial V_0} \Pi_{ik} n_k \, \mathrm{d}S \qquad \text{(Divergence Theorem)}\\ &= -\int_{\partial V_0} (\mathbf{\Pi}\mathbf{n})_i \, \mathrm{d}S\\ &= -\int_{\partial V_0} (p\mathbf{n} + \rho\mathbf{v}(\mathbf{v}^{\mathrm{T}}\mathbf{n}))_i \, \mathrm{d}s.\end{aligned}$$

▶ Derivation of the Navier–Stokes Equations

We are now ready to derive the Navier–Stokes equations. Notice that the Euler Equation can be written as

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}.$$

Now we have wish to introduce a new term into Π to account for stress. (Euler neglected this term in his force balance; it was first accounted for by Stokes.) In particular, we now have

$$\begin{aligned}\Pi_{ik} &= p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} \\ &=: \sigma_{ik} + \rho v_i v_k,\end{aligned}$$

where

$$\sigma_{ik} \equiv -p\delta_{ik} + \sigma'_{ik}$$

is the *stress tensor* and σ'_{ik} is the *viscous stress tensor*. We consider this viscous stress tensor takes the form

$$\sigma'_{ik} = A_{ik\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta},$$

so $A_{ik\alpha\beta}$, with $i, k, \alpha, \beta = 1, 2, 3$ allows for 81 degrees of freedom in the specification of σ'_{ik}. We will impose some conventions to narrow this down considerably. In particular, symmetry and isotropy (no distinction in material properties in different directions) allows us to simplify this to *two* degrees of freedom:

$$\sigma'_{ik} = \eta\Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\ell}{\partial x_\ell}\Big) + \zeta\,\delta_{ik}\frac{\partial v_\ell}{\partial x_\ell},$$

where ζ is the *second viscosity*, which we will address momentarily. Hence the new analog of the Euler Equation takes the form

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathrm{grad})\mathbf{v}\Big) = -\mathrm{grad}\,p + \eta\Delta\mathbf{v} + (\zeta + \tfrac{1}{2}\eta)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v},$$

where η is the called the *dynamic viscosity*, and Δ denotes the Laplacian,

$$\Delta v_i = \frac{\partial^2 v_i}{\partial x_1^2} + \frac{\partial^2 v_i}{\partial x_2^2} + \frac{\partial^2 v_i}{\partial x_3^2} = \mathrm{div\ grad}\ v_i.$$

Now incompressibility imposes

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0,$$

which implies

$$\mathrm{div}\,\mathbf{v} = 0,$$

thus removing the term involving the second viscosity. Under this assumption, we arrive at the *incompressible Navier–Stokes equations*:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathrm{grad})\mathbf{v}\Big) = -\mathrm{grad}\,p + \eta\Delta\mathbf{v}.$$

▶ Simple reductions

We can remove the grad p from the Navier–Stokes equations by taking a curl:

$$\frac{\partial}{\partial t}(\mathrm{curl}\ \mathbf{v}) = \mathrm{curl}(\mathbf{v}\times\mathrm{curl}\ \mathbf{v}) + \nu\Delta\mathrm{curl}\ \mathbf{v},$$

where the *kinematic viscosity* is defined as

$$\nu \equiv \frac{\eta}{\rho}.$$

We can get p back by taking the divergence of the Navier–Stokes equation:

$$\Delta p = -\rho\frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_i}.$$

Consider div $\mathbf{v} = 0$. In two dimensions, this implies

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0,$$

which suggests

$$v_1 = \frac{\partial \psi}{\partial x_2}, \qquad v_2 = -\frac{\partial \psi}{\partial x_1}$$

for some unknown function ψ. So when

$$\mathbf{v} = \begin{bmatrix} \partial\psi/\partial x_2 \\ -\partial\psi/\partial x_1 \end{bmatrix}.$$

Plug this into the (curled) Navier–Stokes equations to obtain

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial \psi}{\partial x_1}\frac{\partial \Delta\psi}{\partial x_2} + \frac{\partial \psi}{\partial x_2}\frac{\partial \Delta\psi}{\partial x_1} - \nu\Delta\Delta\psi = 0.$$

In the next lecture, we will consider what happens when ν is large, leading to the Stokes equation $\Delta\Delta\psi = 0$.

[Steve Cox, 20 January 2009]