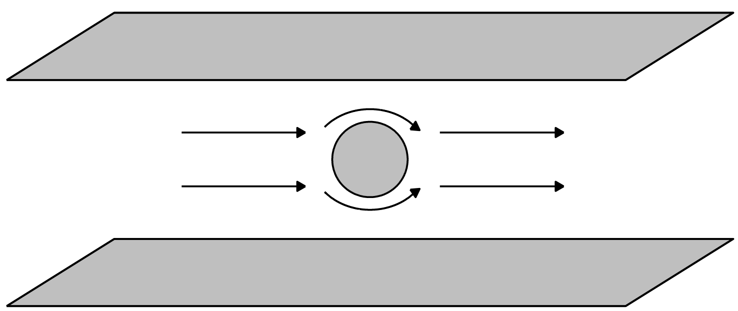
Drag Lecture 3: Energy Dissipation and the Reynolds Number

Last time we derived the incompressible Navier–Stokes equations:

$$\begin{aligned}\operatorname{div}\mathbf{v} &= 0\\ \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\operatorname{grad})\mathbf{v} &= -\frac{\operatorname{grad}p}{\rho} + \frac{\eta}{\rho}\Delta\mathbf{v}.\end{aligned}$$

We now address boundary conditions. We are thinking of a scenario such as flow between two infinite parallel plates with a sphere in the middle.



At a solid surface, such as a plate or the surface of the sphere, on ∂V_0 we require $\mathbf{v} = \mathbf{0}$. The associated momentum flux is

$$\begin{aligned}\Pi_{ik}n_k &= (\rho v_i v_k - \sigma_{ik})n_k\\ &= -\sigma_{ik}n_k\\ &= \underbrace{pn_i}_{\text{pressure}} - \underbrace{\sigma'_{ik}n_k}_{\text{viscous}}.\end{aligned}$$

▶ Landau & Lifshitz, §16: Energy dissipation

We now wish to investigate the kinetic energy of the solution to the Navier–Stokes equations. This kinetic energy is defined as

$$E(t) \equiv \frac{1}{2}\rho\int_{V_0}|\mathbf{v}|^2\,\mathrm{d}V,$$

where $|\mathbf{v}|$ denotes the usual Euclidean norm of the vector $\mathbf{v}$, the square-root of the sum of squares of the components:

$$|\mathbf{v}|^2 \equiv v_1(x,t)^2 + v_2(x,t)^2 + v_3(x,t)^2.$$

We wish to show that $E(t)$ is decreasing in time, i.e.,

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) \leq 0.$$

Now we compute

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \rho \int_{V_0} \Big(v_1 \frac{\partial v_1}{\partial t} + v_2 \frac{\partial v_2}{\partial t} + v_3 \frac{\partial v_3}{\partial t} \Big) \, \mathrm{d}V \\ &= \rho \int_{V_0} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \, \mathrm{d}V \\ &= \rho \int_{V_0} v_i \frac{\partial v_i}{\partial t} \, \mathrm{d}V, \end{aligned}$$

where this last formula uses the summation convention.

Substituting the Navier–Stokes equations, we have

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \rho \int_{V_0} v_i \Big(- v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \sigma'_{ik}}{\partial x_k} \Big) \, \mathrm{d}V \qquad (1) \\ &= - \int_{V_0} \Big(\mathrm{div} \Big(\rho \mathbf{v} (\tfrac{1}{2} |\mathbf{v}|^2 + p/\rho) - \mathbf{v} \cdot \boldsymbol{\sigma'} \Big) - \sigma'_{ik} \frac{\partial v_i}{\partial x_k} \Big) \, \mathrm{d}V. \qquad (2) \end{aligned}$$

Applying the Divergence Theorem to this last equation gives a surface integral,

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = - \int_{\partial V_0} \Big(\rho v (\tfrac{1}{2} |\mathbf{v}|^2 + p/\rho) - \mathbf{v}^T \boldsymbol{\sigma'} \Big)^T \mathbf{n} \, \mathrm{d}S - \int_{V_0} \sigma'_{ik} \frac{\partial v_i}{\partial x_k} \, \mathrm{d}V,$$

but recall that on the surface ∂V_0, we imposed the boundary condition $\mathbf{v} = \mathbf{0}$, so the terms in this last integral that include $\mathbf{v}$ vanish, giving

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = - \int_{V_0} \sigma'_{ik} \frac{\partial v_i}{\partial x_k} \, \mathrm{d}V.$$

Exploiting the symmetry of $\boldsymbol{\sigma'}$, we can write this last integral as

$$\begin{aligned} \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= -\frac{1}{2} \int_{V_0} \sigma'_{ik} \Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \Big) \mathrm{d}V \\ &= -\frac{1}{2} \eta \int_{V_0} \Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \Big)^2 \mathrm{d}V. \end{aligned}$$

Students registered for multiple credits should show, in detail, how to derive equation (2) *from* (1). Hint: Work backwards.

▶ LANDAU & LIFSHITZ, §19: Reynolds number

We consider three important numbers, motivated by physical experience/intuition:

- *kinematic viscosity*, $\nu \equiv \eta/\rho \sim \mathrm{cm}^2/\mathrm{sec}$;
- *relevant length*, $\ell \sim \mathrm{cm}$;
- *relevant velocity*, $u \sim \mathrm{cm}/\mathrm{sec}$.

For example, in the flow past a sphere illustrated earlier, we would typically take ℓ to be the radius of the sphere.

There is one 'natural' way to combine these parameters, known as the *Reynolds number*,

$$\mathrm{Re} \equiv \frac{u\ell}{\nu}.$$

More precisely, define

$$\widehat{\mathbf{x}} \equiv \frac{1}{\ell}\,\mathbf{x}, \qquad \widehat{\mathbf{v}} \equiv \frac{1}{u}\,\mathbf{v},$$

giving

$$\widehat{\mathbf{v}}(\widehat{\mathbf{x}}, t) = \widehat{\mathbf{v}}(\mathbf{x}/\ell, t) = \frac{1}{u}\mathbf{v}(\mathbf{x}, t)$$

and

$$\widehat{p}(\widehat{\mathbf{x}}, t) = \frac{1}{\rho u^2}\,p(\mathbf{x}, t).$$

First differentiate with respect to x_i, using the chain rule:

$$\frac{\partial \widehat{\mathbf{v}}}{\partial \widehat{x}_i}\frac{1}{\ell} = \frac{1}{u}\frac{\partial v}{\partial x_i} \quad \Longrightarrow \quad \frac{\partial \mathbf{v}}{\partial x_i} = \frac{u}{\ell}\frac{\partial \widehat{\mathbf{v}}}{\partial \widehat{x}_i}.$$

Taking another derivative,

$$\frac{\partial^2 \mathbf{v}}{\partial x_i^2} = \frac{u}{\ell^2}\frac{\partial^2 \widehat{\mathbf{v}}}{\partial \widehat{x}_i^2}.$$

Consider *steady* viscous flows, that is, flows for which

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0},$$

in which case the incompressible Navier–Stokes equations simplify to

$$(\mathbf{v} \cdot \operatorname{grad})\mathbf{v} = -\frac{\operatorname{grad} p}{\rho} + \nu \Delta \mathbf{v}.$$

We bring all terms to the left and substitute in our length and velocity scales:

$$\mathbf{0} = (\mathbf{v} \cdot \operatorname{grad})\mathbf{v} + \frac{1}{\rho} \operatorname{grad} p + \nu \Delta \mathbf{v} = \frac{u^2}{\ell} (\widehat{\mathbf{v}} \cdot \operatorname{grad})\widehat{\mathbf{v}} + \frac{u^2}{\ell} \operatorname{grad} \widehat{p} + \frac{\nu u}{\ell^2} \Delta \widehat{\mathbf{v}}.$$

Substituting in the definition of the Reynolds number, we obtain

$$\mathrm{Re}(\widehat{\mathbf{v}} \cdot \operatorname{grad})\widehat{\mathbf{v}} + \mathrm{Re}(\operatorname{grad} \widehat{p}) + \Delta \widehat{\mathbf{v}} = \mathbf{0}.$$

[Steve Cox, 27 January 2009]