Drag Lecture 4: Flow at Small Reynolds Number

▶ LANDAU & LIFSHITZ, §20

The last lecture concluded with the incompressible Navier–Stokes equations, written in the form

$$\mathrm{Re}(\mathbf{v}\cdot\operatorname{grad})\mathbf{v} + \mathrm{Re}(\operatorname{grad} p) + \Delta\mathbf{v} = \mathbf{0}$$

(dropping the hats from $\mathbf{v}$ and p). Taking the curl of this equation, we have

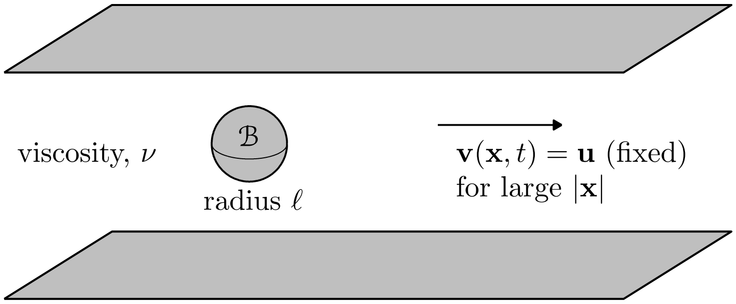
$$\mathrm{Re}(\operatorname{curl}(\mathbf{v}\cdot\operatorname{grad})\mathbf{v}) + \Delta\operatorname{curl}(\mathbf{v}) = \mathbf{0}.$$

When the Reynolds number

$$\mathrm{Re} = \frac{|\mathbf{u}|\ell}{\nu}$$

is small, we can neglect the first term in this last differential equation to obtain

$$\Delta\operatorname{curl}(\mathbf{v}) = \mathbf{0}.$$



We have two goals.

- Solve the equation $\Delta\operatorname{curl}(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v}$.
- Find the force on the sphere $\mathcal{B}$, known as drag, given that $\mathbf{v} = \mathbf{0}$ on the boundary of the sphere, $\partial\mathcal{B}$.

Since we are in an incompressible flow regime, we have $\mathrm{div}(\mathbf{v}) = 0$. Since $\mathbf{u}$ is constant, it also has zero divergence, so

$$0 = \mathrm{div}(\mathbf{v}) = \mathrm{div}(\mathbf{v} - \mathbf{u}).$$

We seek a form for $\mathbf{v} - \mathbf{u}$. An object with zero divergence can always be written as the curl of some quantity, which we shall call $\mathbf{A}$:

$$\mathbf{v} - \mathbf{u} = \mathrm{curl}(\mathbf{A}).$$

We make the judicious guess that

$$\mathbf{A} = (\mathrm{grad} f(|\mathbf{x}|)) \times \mathbf{u}$$

for some scalar-valued function f.

We now need a vector calculus identity, $\mathrm{curl}(f\mathbf{u}) = f\,\mathrm{curl}(\mathbf{u}) + \mathrm{grad}\,f \times \mathbf{u}$ for general f and $\mathbf{u}$. Since $\mathbf{u}$ is constant in our particular case, this identity reduces to $\mathrm{curl}(f\mathbf{u}) = \mathrm{grad}\,f \times \mathbf{u}$. Hence

$$\begin{aligned} \mathbf{v} &= \mathbf{u} + \mathrm{curl}(\mathrm{grad}\,f \times \mathbf{u}) \\ &= \mathbf{u} + \mathrm{curl}(\mathrm{curl}(f\mathbf{u})). \end{aligned}$$

Taking a curl, and applying another vector calculus identity, we arrive at

$$\begin{aligned} \mathrm{curl}(\mathbf{v}) &= \mathrm{curl}(\mathrm{curl}(\mathrm{curl}(f\mathbf{u}))) \\ &= (\mathrm{grad}\ \mathrm{div} - \Delta)\mathrm{curl}(f\mathbf{u}) \\ &= -\Delta\,\mathrm{curl}(f\mathbf{u}). \end{aligned}$$

Take a curl of both sides of this last equation and again apply $\mathrm{curl}(f\mathbf{u}) = \mathrm{grad}\,f \times \mathbf{u}$ to obtain

$$0 = \Delta\mathrm{curl}(\mathbf{v}) = -\Delta^2\mathrm{grad}\,f.$$

This implies that

$$\Delta^2 f = \text{constant}.$$

At large $|\mathbf{x}|$, we require that $\mathbf{v}(\mathbf{x}, t) \to \mathbf{u}$, and hence it must be that

$$\Delta^2 f = 0.$$

Thus, in spherical coordinates, we can write

$$\Delta^2 f = \Delta\Delta f = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Delta f.$$

Hence

$$\Delta f = \text{constant}_1 + \frac{\text{constant}_2}{r} = \frac{2a}{r},$$

i.e.,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Delta f = \frac{2a}{r}.$$

This implies that

$$(r^2 f')' = 2ar,$$

which we integrate to obtain

$$r^2 f' = ar^2 + b.$$

Rearrange and integrate to get

$$f = ar - \frac{b}{r}.$$

(What about the constant of integration? Notice that f was introduced in the expression for $\mathbf{A}$, where it appears in the form grad f: the constant of integration plays no role, so we set it to zero.)

We conclude that

$$\begin{aligned} \mathbf{v} &= \mathbf{u} + \operatorname{curl}(\operatorname{curl}(f\mathbf{u})) \\ &= \mathbf{u} - \frac{a}{|\mathbf{x}|}\Big(\mathbf{I} + \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\Big)\mathbf{u} + \frac{b}{|\mathbf{x}|^3}\Big(\frac{3\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2} - \mathbf{I}\Big)\mathbf{u}. \end{aligned} \tag{1}$$

Students registered for multiple credits should show, in detail, how to derive this final expression for $\mathbf{v}$.

To determine a and b we set $\mathbf{v}(\mathbf{x}) = \mathbf{0}$ if $|\mathbf{x}| = \ell$. Thus

$$\begin{aligned} \mathbf{0} &= \mathbf{u} - \frac{a}{\ell}\Big(\mathbf{I} + \frac{\mathbf{x}\mathbf{x}^T}{\ell^2}\Big)\mathbf{u} + \frac{b}{\ell^3}\Big(\frac{3\mathbf{x}\mathbf{x}^T}{\ell^2} - \mathbf{I}\Big)\mathbf{u} \\ &= \Big(1 - \frac{a}{\ell} - \frac{b}{\ell^3}\Big)\mathbf{u} + \Big(-\frac{a}{\ell^3} + \frac{3b}{\ell^5}\Big)\mathbf{x}\mathbf{x}^T\mathbf{u} \end{aligned}$$

for all $\mathbf{x}$ with $|\mathbf{x}| = \ell$. Now if $\mathbf{x}^T\mathbf{u} = 0$, then

$$1 - \frac{a}{\ell} - \frac{b}{\ell^3} = 0$$

so we can solve for

$$a = \frac{3}{4}\ell, \qquad b = \frac{1}{4}\ell^3.$$

Thus, we arrive at

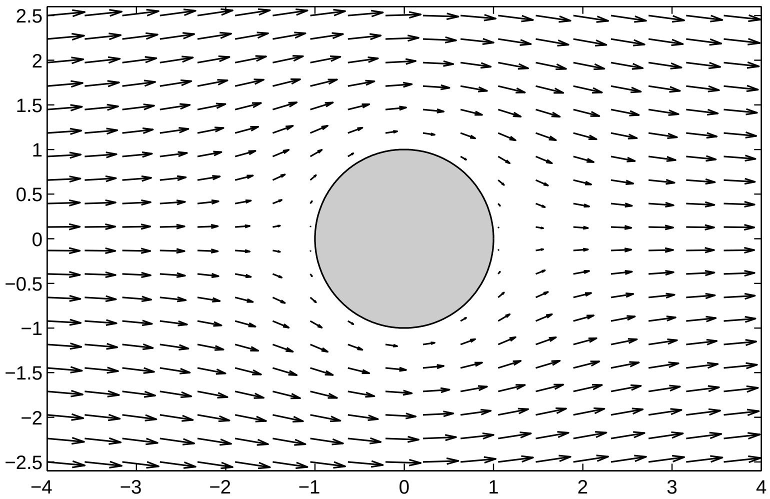
$$f = \frac{3\ell r}{4} - \frac{\ell^3}{4r}.$$

Finally, we arrive at

$$\mathbf{v}(\mathbf{x}) = -\frac{3}{4}\frac{\ell}{|\mathbf{x}|}\Big(\mathbf{I} + \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\Big)\mathbf{u} - \frac{1}{4}\frac{\ell^3}{|\mathbf{x}|^3}\Big(\mathbf{I} - \frac{3\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\Big)\mathbf{u} + \mathbf{u}.$$

You can use this last formula to visualize the velocity field $\mathbf{v}$ *in two dimensions. Pick some constant* $\mathbf{u}$ *and radius* $\ell > 0$*, then use MATLAB's* `quiver` *command to plot the vector field* $\mathbf{v}(\mathbf{x})$ *for* $|\mathbf{x}| \geq \ell$*. Alternatively, experiment with the* `vfield.m` *code on the class website.*

The plot below shows the velocity field $\mathbf{v}(\mathbf{x})$ for $\ell = 1$ and $\mathbf{u} = [1, 0]^T$.



In polar coordinates, we have

$$\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \begin{bmatrix} |\mathbf{u}|\cos(\theta)(1 - 3\ell/(2r) + \ell^3/(2r^3) \\ -|\mathbf{u}|\sin(\theta)(1 - 3\ell/(4r) - \ell^3/(4r^3) \end{bmatrix}.$$

What about the pressure, which we curled away earlier? We have

$$\begin{aligned}
\operatorname{grad} p &= \eta\Delta\mathbf{v} \\
&= \eta\Delta\operatorname{curl}(\operatorname{curl}(f\mathbf{u})) \\
&= \eta\Delta(\operatorname{grad}\operatorname{div}(f\mathbf{u}) - \mathbf{u}\Delta f) \\
&= \eta\Delta\operatorname{grad}\operatorname{div}(f\mathbf{u}) \\
&= \operatorname{grad}(\eta\Delta\operatorname{div}(f\mathbf{u})) \\
&= \operatorname{grad}(\eta\mathbf{u}\cdot\operatorname{grad}\Delta f).
\end{aligned}$$

Letting p_∞ denote the pressure far from the sphere, we have

$$p = \eta\mathbf{u}\cdot\operatorname{grad}\Delta f + p_\infty = p_\infty - \frac{3}{2}\eta\frac{\ell}{|\mathbf{x}|^3}\mathbf{u}\cdot\mathbf{x}.$$

In polar coordinates,

$$p = p_\infty - \frac{3}{2}\eta\frac{\ell}{r^2}|\mathbf{u}|\cos(\theta).$$

Next time we will compute the drag force

$$\mathbf{F} = \int_{\partial\mathcal{B}} \mathbf{\Pi}\mathbf{n}\,\mathrm{d}S = \int_{\partial\mathcal{B}} (p\mathbf{n} - \boldsymbol{\sigma}'\mathbf{n})\,\mathrm{d}S,$$

and we will see that it satisfies

$$|\mathbf{F}| = 6\pi\ell\eta|\mathbf{u}|.$$

[Steve Cox, 3 February 2009]

The MATLAB code `vfield.m` for computing the velocity is given below.

```
%
% vfield.m
%
% Plot velocity field for low Reynolds flow around a sphere
% Based on analysis in Landau and Lifshitz, Section 20

 ell = 1;                                % radius of the sphere
 u = [1;0];                              % velocity field as |x| -> oo
 m = 20;                                 % grid density for quiver plot
```

```
x1 = linspace(-4,4,m);                 % position, horizontal grid
x2 = linspace(-2.5,2.5,m);             % position, vertical grid
v1 = zeros(length(x2),length(x1));   % velocity, horizontal grid
v2 = zeros(length(x2),length(x1));   % velocity, vertical grid

I = eye(2);
for k=1:length(x1), for j=1:length(x2)
   x = [x1(k);x2(j)];
   if norm(x)<ell, vv = [0;0];
   else
      vv = -.75*(ell/norm(x))*(I+(x*x')/(norm(x)^2))*u ...
           -.25*((ell/norm(x))^3)*(I-3*x*x'/(norm(x)^2))*u + u;
   end
   v1(j,k) = vv(1);
   v2(j,k) = vv(2);
end, end


figure(1), clf
quiver(x1,x2,v1,v2,'k-');               % quiver plot
z = ell*exp(linspace(0,2i*pi,500));
hold on
fill(real(z),imag(z),.8*[1 1 1])        % plot sphere
axis equal
axis([min(x1) max(x1) min(x2)-.1 max(x2)+.1])
```