Drag Lecture 5b: Stokes Far From the Sphere

In Stokes flow we have made the following assumption: the inertial forces (convection term $(\mathbf{v} \cdot \mathrm{grad})\mathbf{v}$) are negligible because

$$(\mathbf{v} \cdot \mathrm{grad})\mathbf{v} \sim (\mathrm{velocity})^2.$$

This implies that pressure forces ($(\mathrm{grad}\, p)/\rho$) balance viscous forces ($\nu \Delta \mathbf{v}$).

Recall the steady incompressible Navier–Stokes equations,

$$\begin{aligned} (\mathbf{v} \cdot \mathrm{grad})\mathbf{v} &= -\frac{\mathrm{grad}\, p}{\rho} + \nu \Delta \mathbf{v} \\ \mathrm{div}\, \mathbf{v} &= 0. \end{aligned}$$

In the limit of small Reynolds number, we obtained the Stokes equation

$$\nu \Delta (\mathrm{curl}\, \mathbf{v}) = \mathbf{0}.$$

There arises a problem at large distances: the solution for $\mathbf{v}$ and p that follow from the Stokes problem do not satisfy the Navier–Stokes equations!

To see this, we need to manipulate the Stokes solution a bit. Recall the Stokes formula for $\mathbf{v}(\mathbf{x})$ derived in Lecture 4:

$$\mathbf{v}(\mathbf{x}) = -\frac{3}{4}\frac{\ell}{|\mathbf{x}|}\Big(\mathbf{I} + \frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\Big)\mathbf{u} - \frac{1}{4}\frac{\ell^3}{|\mathbf{x}|^3}\Big(\mathbf{I} - \frac{3\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\Big)\mathbf{u} + \mathbf{u}.$$

Taking a partial derivative, we see that

$$\frac{\partial}{\partial x_i}\mathbf{v} = \mathcal{O}\Big(\frac{|\mathbf{u}|\ell}{|\mathbf{x}|^2}\Big).$$

(Here $a = \mathcal{O}(b)$ means that a is of the same order as b.) From this we conclude that

$$(\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = \mathcal{O}\Big(\frac{|\mathbf{u}|^2\ell}{r^2}\Big),$$

where $r = |\mathbf{x}|$.

We also see that

$$\nu \Delta \mathbf{v} = \mathcal{O}\Big(\nu\, \ell \frac{|\mathbf{u}|}{r^3}\Big).$$

Can you verify these computation for the leading order behavior for $(\partial/\partial x_i)\mathbf{v}$), and hence $(\mathbf{v}\cdot \mathrm{grad})\mathbf{v}$ and $\nu\Delta\mathbf{v}$?

In polar coordinates, we also found that

$$p = p_\infty - \frac{3}{2}\eta\frac{\ell}{r^2}|\mathbf{u}|\cos\theta,$$

so that

$$\mathrm{grad}\, p = 0 - \frac{3}{2}\eta\ell|\mathbf{u}|\mathrm{grad}\Big(\frac{\cos\theta}{r^2}\Big).$$

After some work, we can show

$$\mathrm{grad}\, p = \mathcal{O}\Big(\frac{\eta\ell|\mathbf{u}|}{\rho r^3}\Big), \quad \text{as } r\to\infty.$$

But then the term on the left-hand side of the incompressible Navier-Stokes equation is of size

$$(\mathbf{v}\cdot\mathrm{grad})\mathbf{v} = \mathcal{O}\Big(\frac{|\mathbf{u}|^2\ell}{r^2}\Big),$$

while the right hand side is of size

$$-\frac{\mathrm{grad}\, p}{\rho} + \nu\Delta\mathbf{v} = \mathcal{O}\Big(\nu\,\ell\frac{|\mathbf{u}|}{r^3}\Big).$$

The two sides of the equation behave differently as $r\to\infty$! Thus, the solution to the Stokes equation is not consistent with the Navier–Stokes equations at points far from the sphere.

How can one obtain a better approximation? Here's an idea: Replace the $(\mathbf{v}\cdot\mathrm{grad})\mathbf{v})$ term with $(\mathbf{u}\cdot\mathrm{grad})\mathbf{v}$, which is called the *Oseen approximation*, to get better accuracy far away from the sphere. We could then look for a way to match up the Stokes approximation (accurate near the sphere) with the Oseen approximation (accurate far from the sphere).

We shall pursue this lead in the coming lectures.

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