Integration on Surfaces: Crib Sheet

**Definition 1:** Suppose $U \subset \mathbb{R}^n$ is open, and $f : U \to \mathbb{R}$ is a function. Then the support of $f = \text{supp } f = \text{closure of } \{x \in U : f(x) \neq 0\}$.

**Definition 2:** $\mathbb{R}^n$ a normed vector space, then $C_0^\infty(U, \mathbb{R}^n) = \{f \in C^\infty(U, \mathbb{R}^n) : \text{supp } f \text{ is compact}\}

**Definition 3:** $\Omega_k^0(U) = C_0^\infty(U, \Lambda^k(\mathbb{R}^n)), 0 \leq k \leq n$.

**Definition 4:** Suppose $\omega \in \Omega_0^0(U)$, $\omega = gdx_1 \wedge ... \wedge dx_n$, $g \in C_0^\infty(U, \mathbb{R})$. Then
\[
\int_U \omega = \int_U g.
\]

**Remark:** To make sense out of this, you must convince yourself that there is a bounded Jordan-measurable set $C$ with $\text{supp } g \subset C \subset U$, and define the integral on the right to mean $\int_C g$. Then you must see that you get the same result no matter which Jordan measurable set $C$ you pick that satisfies these conditions.

Recall that for any differentiable map $\phi : U \to \mathbb{R}^n$, open $V$ with $\phi(U) \subset V$, and $\omega \in \Omega^n(V)$,
\[
\phi^* \omega = (\omega \circ \phi) \det D\phi.
\]

**Theorem 1:** If $\phi$ is $C^\infty$-invertible, $\det D\phi > 0$, and $\omega \in \Omega_0^n(V)$,
\[
\int_U \phi^* \omega = \int_V \omega
\]

**Proof:** In view of all the definitions above, this is merely a restatement of the Change of Variables Theorem.

**Definition 5:** $S \subset U$ is a smooth surface iff there exists a $C^\infty$-invertible $\phi : U \to \mathbb{R}^n$ so that $S = \{x \in U : \phi_n(x) = 0\}$.

**Example 1:** Take $U = \mathbb{R}^n$, $S = S_n \equiv \{x : x_n = 0\}$, and $\phi = \text{the identity map}$. This is the prototype smooth surface!

**Nomenclature:** $\phi$ as in the definition of smooth surface is a chart for $S$ in $U$. 

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Definition 6: Suppose that $\phi$ is a chart for $S$ in $U$, $x \in S$. Then

$$T_x^\phi S = \{ v \in \mathbb{R}^n : (D\phi(x)v)_n = 0 \}.$$

For $w \in \mathbb{R}^n$, write $w = (w', w_n)$, $w' \in \mathbb{R}^{n-1}$.

Lemma 1: $T_x^\phi S = \{ D\phi^{-1}(\phi(x))(w', 0) : w' \in \mathbb{R}^{n-1} \}$

Proof: $D\phi(x)D\phi^{-1}(\phi(x))(w', 0) = (w', 0)$, so $\{ D\phi^{-1}(\phi(x))(w', 0) : w' \in \mathbb{R}^{n-1} \} \subset T_x^\phi S$. On the other hand, if $v \in T_x^\phi S$, then $D\phi(x)v = (w', 0)$ for some $w' \in \mathbb{R}^{n-1}$, so $v = D\phi^{-1}(\phi(x))D\phi(x)v = D\phi^{-1}(\phi(x))(w', 0)$.

Proposition 1: Suppose that $\phi$ and $\psi$ are charts for $S$ in $U$. Then for every $x \in S$,

$$T_x^\phi S = T_x^\psi S$$

Definition 7: The tangent space $T_x S$ to $S$ at $x$ is the subset $T_x^\phi S$ for any chart $\phi$ for $S$ in $U$.

Remark: You can identify the tangent space at $x$ with the tangent vectors of smooth curves in $S$ passing through $x$.

Definition 8: a couple of useful linear maps: $\Pi \in L(\mathbb{R}^n, \mathbb{R}^{n-1})$: $\Pi(w', w_n) = w'$. $\Pi^T \in L(\mathbb{R}^{n-1}, \mathbb{R}^n)$: $\Pi^T w' = (w', 0)$.

Lemma 2: If $\phi$ is a chart for $S$ in $U$, then $\Pi(\phi(S)) = \Pi(S_n \cap \phi(U)) \subset \mathbb{R}^{n-1}$ is open.

Definition 9: Suppose that $\phi$ is a chart for $S$ in $U$, and $\omega \in \Omega^n_0(U)$. Then

$$\int_S^\phi \omega = \int_{\Pi(\phi(S))}^{\phi^{-1} \circ \Pi^T) \omega}.$$

Remark: The integral on the right is the $n - 1$-dimensional version of the $n$-form integral defined above, so makes sense. The next thing to do is to show that the definition is actually independent of the choice of chart, as was the definition of tangent space - these two facts are related. To see that every choice of chart yields the same definition of the integral requires comparing the pull-back forms $(\phi^{-1} \circ \Pi^T)^* \omega$ and $(\psi^{-1} \circ \Pi^T)^* \omega$ for two (possibly) different charts $\phi$ and $\psi$. It would be natural to compose the inverse of one map with the other, and in fact that’s essentially what we will do, but since neither map is strictly speaking
invertible we have to be a bit more subtle. The idea is to examine the composition of the two pull-backs, with more or less the inverse of the one associated with $\phi$ in the middle:

$$(\psi^{-1} \circ \Pi^T)^*(\Pi \circ \phi)^*(\phi^{-1} \circ \Pi^T)^* \omega$$

You can associate this composition two different ways: first, as

$$(\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T)^*(\phi^{-1} \circ \Pi^T)^* \omega$$

(1)

The second is to isolate the $\psi$ pull-back:

$$(\psi^{-1} \circ \Pi^T)^*((\phi^{-1})^*(\Pi^T \circ \Pi)^* \phi^*) \omega.$$  (2)

**Lemma 3:** $\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T$ is a $C^\infty$-invertible map: $\Pi(\psi(S)) \rightarrow \Pi(\phi(S))$ (these are open sets in $\mathbb{R}^{n-1}$). Furthermore, if $\det D\phi > 0$ and $\det D\psi > 0$ on $U$, then $\det D(\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T) > 0$ on $\Pi(\psi(S))$.

This Lemma almost links $((\phi^{-1} \circ \Pi^T)^* \omega$ and $((\psi^{-1} \circ \Pi^T)^* \omega$ by pull-back by a $C^\infty$-invertible map, which because of the C-of-V theorem would mean their integral would be the same - but not quite, because it’s not $\omega$ that’s being pulled back in equation 2. However it’s just as good, because of the following two observations:

**Proposition 2:** Suppose that $\psi$ is a chart for $S$ in $U$, and $\omega \in \Omega^{n-1}_0(U)$. Then $(\psi^{-1} \circ \Pi^T)^* \omega = 0$ iff for every $x \in S$, $(v_1, v_2, \ldots v_{n-1}) \in (T_xS)^{n-1}$, $\omega(x)(v_1, \ldots, v_{n-1}) = 0$.

**Remark:** That is, as far as integration over $S$ is concerned, it’s only the action of $n-1$-forms on the tangent space of $S$ that matters. This leads naturally to a definition of forms on $S$, but we shall not have need of that abstraction - see Ch. 5 in Spivak if you want to know more about this.

**Proposition 3:** $\omega - ((\phi^{-1})^*(\Pi^T \circ \Pi)^* \phi^*) \omega$ vanishes on all $(v_1, v_2, \ldots v_{n-1}) \in (T_xS)^{n-1}$.

**Theorem 2:** Suppose that $\phi$ and $\psi$ are charts for $S$ in $U$, and that $\det D\phi > 0$, $\det D\psi > 0$ in $U$. Then

$$\int_S^\phi \omega = \int_S^\psi \omega$$

for any $\omega \in \Omega^0_0(U)$.

**Proof:**

$$\int_S^\psi \omega = \int_{\Pi(\psi(S))}^{\Pi^T} (\psi^{-1} \circ \Pi^T)^* \omega.$$
\[ \int_{\Pi(\phi(S))} \left( (\psi^{-1} \circ \Pi^T)^* (\omega - ((\phi^{-1})^*(\Pi^T \circ \Pi)^* \phi^* ) \omega) \right) \\
+ \int_{\Pi(\psi(S))} \left( (\Pi \circ \phi \circ \psi^{-1} \circ \Pi^T)^* (\phi^{-1} \circ \Pi^T)^* \omega \right) \]

using equations 1 and 2. However the first integrand is the pull-back by $\psi^{-1} \circ \Pi^T$ of a form that vanishes on every $T_xS$, according to Proposition 3, and by Proposition 2 this pull-back vanishes - here is where we use the invariance of the definition of $T_xS$ with respect to choice of chart (i.e. it didn’t matter whether we used $\phi$ or $\psi$ to define $T_xS$). That is, the first integral vanishes. The second is the integral of a pull-back by a $C^\infty$-invertible map with positive determinant (Lemma 3), so the C-of-V theorem (Theorem 1) finishes the proof.

This Theorem justifies the

**Definition 10:** For $\omega \in \Omega^{n-1}_0(U)$, define

\[ \int_S \omega = \int_{\Pi(\phi(S))} (\phi^{-1} \circ \Pi^T)^* \omega. \]

where $\phi$ is any chart for $S$ in $U$ with $\det D\phi > 0$. 
