Analysis Qualifying Dry Run #1

due 4 Feb 2016

Time: *this time only, no time limit!* [The usual time limit is 3 hours.] Please write begin and end times on your paper. Each problem is worth the number of points indicated. To receive full credit, you must name all major theorems and state definitions used in your arguments. Do not consult texts, notes, old homeworks, or (of course) anyone else for the duration of the exam. Hints may provided for some problems; if so, please note that these are merely hints, not full solutions, and many details are left for you to fill in.

Grading: For this assignment, I will give particular weight to clarity and organization of your solutions. Are all hypotheses stated clearly? Are the major definitions and theorems used in your proofs identified and used correctly? Is the flow of logic well-defined and easy to follow? You have seen many examples of proofs in your mathematical studies. Here's your chance to write a few!

1. Suppose that $f : \mathbf{R}^n \to \mathbf{R}^{\mathbf{m}}$ is *positively homogeneous of degree 1*, that is,

$$f(tx) = tf(x) \text{ for any } t \geq 0, x \in \mathbf{R}^n.$$

Note that all linear functions have this property.
(a) Show that f need not be linear, by producing a counterexample. [15 pts]
(b) Suppose in addition that f is differentiable at $0 \in \mathbf{R}^n$. Show that f is linear. [20 pts]

2. Define $\lambda : C^0[0,1] \to C^0[0,1]$ by

$$(\lambda u)(x) = u(x) + \int_0^x dt\, tu(t),\ u \in C^0[0,1].$$

Prove that $\lambda \in \mathrm{Inv}(C^0[0,1], C^0[0,1])$. Note: in case there is any doubt, $C^0[0,1]$ is the Banach space of continuous functions on $[0,1]$, equipped with the sup norm. [35 pts]

3. Suppose $a < 0 < b$ and $A \in C^1((a,b), \mathbf{R}^{n\times n})$, $A(0) = I$. Show that

$$\left(\frac{d}{dt} \det A\right)(0) = \mathrm{tr}\left(\frac{dA}{dt}(0)\right).$$

Hint: One possible approach is to use the expansion of the determinant by row or column and inducion on n, for example:

$$\det A = \sum_{i=1}^{n} (-1)^{i-1} A_{i,1} \det \mathrm{co}A[i,1].$$

Here $\mathrm{co}A[i,j]$ is the *(i,j) cofactor matrix* of A = the $(n-1)\times(n-1)$ matrix obtained by removing the ith row and jth column from A. [30 pts]