Analysis Qualifying Dry Run #2
Solutions

1.

a. Define *subset of* $\mathbf{R}^n$ *of measure zero.*
Solution: A subset C of $\mathbf{R}^n$ has measure zero iff for any $\epsilon > 0$ there exists a sequence $\{A_n : n \in \mathbf{Z}_+\}$ of rectangles so that (1) $C \subset \cup_{n\in\mathbf{Z}_+} A_n$, and (2) $\sum_{n\in\mathbf{Z}_+} v(A_n) < \epsilon$ [5 pts]

b. Define *subset of* $\mathbf{R}^n$ *of content zero.*
Solution: A subset C of $\mathbf{R}^n$ has content zero iff for any $\epsilon > 0$ there exists $N \in \mathbf{Z}_+$ and a set $\{A_n : 1 \leq n \leq N\}$ of N rectangles so that (1) $C \subset \cup_{n=1}^N A_n$, and (2) $\sum_{n=1}^N v(A_n) < \epsilon$[5 pts]

c. Show that a compact set of measure zero has content zero. [10 pts]
Solution: Denote by C the compact set of measure zero, and suppose that $\epsilon > 0$, and $\{A_n : n \in \mathbf{Z}_+\}$ are as in the definition. For each A_n, let B_n be an open rectangle with sides twice as long as $A_n's$ and the same center. Then $\{B_n : n \in \mathbf{Z}_+\}$ is an open cover of C. Since C is compact, there exists a finite sub-cover, of (say) $N \in \mathbf{Z}_+$ open rectangles; after renumbering, these are $\{B_1, ..., B_N\}$. Note that

$$\sum_{n=1}^{N} v(B_n) \leq \sum_{n=1}^{\infty} v(B_n) = 2^n \sum_{n=1}^{\infty} v(A_n) < 2^n\epsilon.$$

d. Suppose that $C \subset \mathbf{R}^n$ is a bounded Jordan-measureable set. and $f : C \to \mathbf{R}$ is an integrable function. Show that the functions $f_+, f_- : C \to \mathbf{R}$ defined by

$$\begin{aligned} f_+(x) &= \max(f(x), 0), \\ f_-(x) &= \min(f(x), 0) \end{aligned}$$

are also integrable. [10 pts]
Solution: Let $\epsilon > 0$, and R be a rectangle containing C, and $\mathcal{P}$ a partition of R for which

$$U(\chi_C f, \mathcal{P}) - L(\chi_C f, \mathcal{P}) < \epsilon.$$

Denote by $\mathcal{S}$ the set of sub-rectangles $S \in \mathcal{P}$ for which $\chi_C f(x) > 0$ for some $x \in S$. Since $\chi_C f_+(x) = 0$ if $x \in S \in \mathcal{P} \setminus \mathcal{S}$,

$$U(\chi_C f_+, \mathcal{P}) = \sum_{S\in\mathcal{S}} M(\chi_C f_+, S) = \sum_{S\in\mathcal{S}} M(\chi_C f, S),$$

$$L(\chi_C f_+, \mathcal{P}) = \sum_{S\in\mathcal{S}} m(\chi_C f_+, S) \geq \sum_{S\in\mathcal{S}} m(\chi_C f, S),$$

so

$$U(\chi_C f_+, \mathcal{P}) - L(\chi_C f_+, \mathcal{P}) \leq \sum_{S \in \mathcal{S}} M(\chi_C f, S) - \sum_{S \in \mathcal{S}} m(\chi_C f, S)$$
$$\leq U(\chi_C f, \mathcal{P}) - L(\chi_C f, \mathcal{P}) < \epsilon.$$

The argument for f_- is exactly analogous, or you can use $f_- = f - f_+$ and the sum property.

e. Suppose that $C \subset \mathbf{R}^n$ is a bounded Jordan-measurable set, and that $f : C \to \mathbf{R}$ is a bounded function. Suppose also that for any $\epsilon > 0$, there exist integrable functions $g : C \to \mathbf{R}$ and $h : C \to \mathbf{R}$ so that (1) $g(x) \leq f(x) \leq h(x)$ for every $x \in C$, and (2)

$$\int_C h - \int_C g < \epsilon$$

Show that f is integrable. [30 pts]
Solution: As in part (d), let $\epsilon > 0$, R be a rectangle, and $\mathcal{P}$ be a partition for which

$$U(\chi_C g, \mathcal{P}) - L(\chi_C g, \mathcal{P}) < \epsilon, \; U(\chi_C h, \mathcal{P}) - L(\chi_C h, \mathcal{P}) < \epsilon.$$

(P is a common refinement of partitions that do the job for g and h separately.) Since the integral of an integrable function lies between the upper and lower sums for any partition, and $h \geq g$,

$$\int_C g - \epsilon \leq L(\chi_C g, \mathcal{P}) \leq U(\chi_C g, \mathcal{P}) \leq U(\chi_C h, \mathcal{P}) \leq \int_C h + \epsilon$$

Since the integrals of h and g are within ϵ of each other,

$$U(\chi_C h, \mathcal{P}) - L(\chi_C g, \mathcal{P}) \leq 3\epsilon$$

Finally, since $g \leq f \leq h$,

$$U(\chi_C f, \mathcal{P}) - L(\chi_C f, \mathcal{P}) \leq U(\chi_C h, \mathcal{P}) - L(\chi_C g, \mathcal{P}) \leq 3\epsilon$$

As ϵ was arbitrary, conclude that f is integrable on C.

2. Suppose that $U \subset \mathbf{R}^n$ is open, and $f \in C^1(U, \mathbf{R}^m)$. Define $g : U \times U \to \mathbf{R}$ by

$$g(x, y) = \begin{cases} \frac{|f(x) - f(y) - Df(y)(x-y)|}{|x-y|} & \text{if } x \neq y, \\ 0 \text{ else.} \end{cases}$$

Show that $g \in C^0(U \times U, \mathbf{R})$. [40 pts]
Solution: The task is to prove that g is continuous at every $(x_0, y_0) \in U \times U$. If $x_0 \neq y_0$, the result follows from standard facts about continuous functions (the quotient is continuous if numerator and denominator are, and denominator is non-zero). So the only thing to

prove is that g is continuous at $(x_0, x_0), x_0 \in U$. Since $g(x_0, x_0) = 0$ by definition and g is nonnegative, it amounts to showing that for every $x_0 \in U$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $|(x, y) - (x_0, y_0)| < \delta$ (sup norm!) then $g(x, y) < \epsilon$.

Here is an argument for this, presented not in the standard proof manner but backwards, as you might discover it. The definition of derivative says that for any y, $g(x, y) \to 0$ as $x \to y$. However, that's not what is required: instead, we need to show that $g(x, y) \to 0$ as $(x, y) \to (x_0, x_0)$. If for example we were to redefine $g(x, y) = 1 - x/y$ if $y \neq 0$, $g(x, 0) = 0$ (with $U = \mathbf{R}^2$) you see that $g(x, y) \to 0$ for $x \to y$ does not imply that $g(x, y) \to 0$ for $(x, y) \to (0, 0)$. You need some sort of uniformity. Looking again, at the definition, you see $f(x) - f(y)$ as part of the numerator. You should have a nearly automatic reaction to seeing this and knowing that $f \in C^1$: use the Mean Value Theorem! For $x \neq y$,

$$\frac{f(x) - f(y) - Df(y)(x - y)}{|x - y|} = \frac{\left(\int_0^1 dt Df(tx + (1 - t)y)\right)(x - y) - Df(y)(x - y)}{|x - y|}$$

Since $Df(y)$ is constant as far as the integration is concerned, you can move it under the integral sign, so

$$= \frac{\left(\int_0^1 dt[Df(tx + (1 - t)y) - Df(y)]\right)(x - y)}{|x - y|}$$

Now it should be coming clear how you proceed: choose $\delta > 0$ so that $B_\delta(x_0) \subset U$. If $x, y \in B_\delta(x_0)$, then the entire line segment between x and y is in $B_\delta(x_0)$, and in particular $|(tx + (1 - t)y) - y| < \delta$. You might have to make δ even smaller to obtain two additional properties: (1) $B_{\bar{\delta}}(x_0) \subset U$, and (2) for $z, w \in B_{\bar{\delta}}(x_0)$, $|Df(z) - Df(w)| < \epsilon$. Taking $z = tx + (1 - t)y, w = y$ you see that the integrand above is $< \epsilon$ in norm for every $t \in [0, 1]$, and therefore using the integral triangle inequality and the operator norm property,

$$g(x, y) \leq \frac{\int_0^1 dt|Df(tx + (1 - t)y) - Df(y)||x - y|}{|x - y|} < \epsilon.$$

So does this do it? If $|(x, y) - (x_0, x_0| < \delta$, that means $\max(|x - x_0|, |y - x_0|) < \delta$, so $x, y \in B_\delta(x_0)$ as required.

Q. E. D.

PS: So apparently you could not replace the hypothesis "$f \in C^1$" with "f differentiable" - else the MVT argument would not work, not to mention the use of the local uniform continuity of Df. Do you think it is just the argument, or can you think of a counterexample?