Analysis Qualifying Dry Run #3

due 14 April 2016

Time limit is 3 hours. Please write begin and end times on your paper. Each problem is worth the number of points indicated. To receive full credit, you must name all major theorems and state definitions used in your arguments. Do not consult texts, notes, old homeworks, or (of course) anyone else for the duration of the exam. Hints may provided for some problems; if so, please note that these are merely hints, not full solutions, and many details are left for you to fill in.

Grading: grades will be assigned based on (1) correct and complete statements of major theorems and definitions used, (2) correct and ordered logic, and (3) readability.

1. Suppose that f is differentiable on $(0,1)$ and that f' is unbounded. Show that f is *not* Lipshitz continuous. [30 pts]

2. (a) For $\alpha \geq 0$, define $\|\cdot\|_\alpha : C^0[0,1] \to \mathbf{R}$ by

$$\|f\|_\alpha = \sup\{e^{-\alpha x}|f(x)| : x \in [0,1]\}.$$

Show that $\|\cdot\|_\alpha$ is a norm, and that $C^0[0,1]$, equipped with the norm $\|\cdot\|_\alpha$, is a complete normed vector space. [You may quote without proof the fact that $C^0[0,1]$ is a complete normed vector space with the norm $\|\cdot\|_0$, that is, $\alpha = 0$, which is the ordinary sup norm.] [10 pts]

(b) Suppose that $k : [0,1] \times [0.1] \to \mathbf{R}$ is a continuous function, fixed for the remainder of this problem. Denote by V_α the complete normed vector space defined in part (a), and by $K : V_\alpha \to V_\alpha$ the map defined by

$$(Ku)(x) = \int_0^x k(x,y)u(y)dy.$$

Prove that this map is well-defined, that is, that Ku is continuous if u is, and that K is a continuous linear map. [10 pts]

(c) Show that the operator (or uniform) norm $\|K\|_\alpha$ of $K : V_\alpha \to V_\alpha$ satisfies

$$\|K\|_\alpha \leq \frac{1}{2}$$

if α is sufficiently large. [10 pts]

(d) Prove that for any $f \in C^0[0,1]$, there is a $u \in C^0[0,1]$ for which

$$f(x) = u(x) + \int_0^x k(x,y)u(y)dy.$$

[10 pts]

3. Suppose that $U \subset \mathbf{R}^n$ is open, $f \in C^\infty(U, \mathbf{R}^m)$, and $Df(x)$ is surjective at every $x \in U$. Deduce that (a) $m \leq n$ [10 pts], and (b) $f(U)$ is open [20 pts].