Introduction

- Optimization problems governed by PDEs with uncertain coefficients arise in many applications.
- Optimization usually performed for average value of objective functional.
- I adopt risk-averse approach that accounts for possibility of large deviations from average value and rare events; efficient numerical solution more challenging.
- Illustrate importance of risk-averse approach.
- Discuss solution approach.

Stochastic Advection-Diffusion Problem

\[-\nabla \cdot (\kappa(x, \xi) \nabla y(x, \xi)) + v(x, \xi) \cdot \nabla y(x, \xi) = f(x, \xi) \quad x \in D\]
\[y(x, \xi) = g(x) \quad x \in \Gamma_D, \text{ a.a. } \xi\]
\[\frac{\partial}{\partial n} y(x, \xi) = 0 \quad x \in \Gamma_N, \text{ a.a. } \xi\]

where
- \(y(x, \xi)\) - a state variable (e.g. concentration or pressure)
- \(\kappa(x, \xi)\) - a random diffusion coefficient (e.g. permeability)
- \(v(x, \xi)\) - a random velocity field
- \(f(x, \xi)\) - a randomly located source term
- \(\partial\) - a deterministic control

Given \(z(x)\) the solution \(y(x, \xi) \equiv y(x, \xi; z)\) is a random variable.

Objective of Optimization

Find \(z(x)\) to “minimize” exceedance of target value by PDE solution \(y(x, \xi)\) in red region \(\Gamma_c\).

'Minimize' \(s(y(x, \xi)) = \max\{y(x, \xi) - 1, 0\}\) (depends on \(\xi\))

Risk-neutral vs. Risk-averse Optimization

Optimization under uncertainty problem:
\[
\min_{z \in Z} R[s(y(x, \xi))]
\]
where \(R[\cdot]\) is a measure of uncertainty in \(s(y(x, \xi))\).

Risk-neutral approach: optimize on average

\[\mathbb{E}[s(y(x, \xi))] = \int_{\xi} s(y(x, \xi)) d\xi\]
\[\approx \frac{1}{N} \sum_{j=1}^{N} s(y(x, \xi_j))\]

This approach treats all realizations equally.

Risk-averse approach: optimize mean plus semideviation

\[R[X] = \mathbb{E}[X] + \mathbb{E}[(X - \mathbb{E}[X])^+]\]
where \((\cdot)^+ = \max\{\cdot, 0\}\)
\[\mathbb{E}[(X - \mathbb{E}[X])^+] \approx \frac{1}{N} \sum_{k=1}^{N} (X^{(k)})^+\]

Here \(X^{(k)} = s(y(x, \xi_k))\)

This approach penalizes realizations that fall above average.

Risk-averse approach: optimize Conditional Value-at-Risk

Value-at-Risk (VaR)_z:
\[\text{VaR}_z(X) = \inf_{\xi \in \mathbb{R}} \{\Pr[X < \xi] > \beta\} = \text{VaR}_z(X) = \inf_{\xi \in \mathbb{R}} \{\Pr[X > \xi] < 1 - \beta\}\]

Conditional Value-at-Risk (CVaR)_z:
\[\text{CVaR}_z(X) = \frac{1}{1 - \beta} \int_{\beta}^{1} \text{VaR}_z(X) d\tau = \text{VaR}_z(X) + \frac{1}{1 - \beta} \mathbb{E}[(X - \text{VaR}_z(X))^+]\]
\[\approx \inf_{\xi \in \mathbb{R}} \{t + \frac{1}{1 - \beta} \sum_{j=1}^{N} (X^{(k)} - t)^+\}\]

Efficient Sampling

“Good” solutions potentially require a large number of samples. Each sample requires solving a PDE.
Adapt variance reduction methods to decrease effective number of samples, such as
- Control variates: for \(C\) such that \(E[C] = 0\)
  \[E[s(y(x, \xi))] = \frac{1}{N} \sum_{j=1}^{N} s(y(x, \xi_j)) \approx \frac{1}{N} \sum_{j=1}^{N} s(y(x, \xi_j)) + \alpha C^{(j)}\]
- Multi-level approximations: using spatial grids of varying coarseness to solve the PDE.
- Importance sampling:
  \[\int_{\xi} s(y(x, \xi)) d\xi = \int_{\xi} s(y(x, \xi)) \frac{d\nu(\xi)}{d\mu(\xi)} q(\xi) d\xi\]

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