tions and priors for the latter, the variational approaches have their analytic advantage. In these models, images are presented as functions, and this allows us to define concepts like gradients, curvature, diffusion, level sets, perimeter, length of boundary, areas, etc. and to incorporate them in image processing. In addition, the variational models make substantial use of and benefit from the well-established areas of mathematics including PDEs, geometric measure theory, and optimization. One of the highlights of variational image models is that minimizing the total variations in various applications preserves sharp edges in images.

In what follows, we assume that our images are defined on an open set $\Omega \in \mathbb{R}^2$, which typically is a square region, and we present four image modeling approaches followed by a discussion of their relations.

1.1 *BV* Images and *G* Oscillation Patterns

BV is the space of functions with bounded variation. Intuitively, a function in the space *BV* is an L^1 function that gives a finite value when testable by a "limited" set of C_c^∞ functions in the form of $\operatorname{div}(\vec{g})$, where $\vec{g} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $\|\vec{g}(x)\|_\infty \leq 1$.

Definition 1.1.1. *[72] Let $u \in L^1$, and define*

$$\mathbf{TV}(u) = \|Du\| := \sup\left\{ \int u \operatorname{div}(\vec{g})\, dx \ :\ \vec{g} \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n), \|\vec{g}(x)\|_{l^2} \leq 1 \textit{ for all } x \in \mathbb{R}^n \right\}, \tag{1.1.1}$$

and $\|u\|_{BV} = \|u\|_{L^1} + \|Du\|$, *where* $C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *denotes the set of continuously differentiable vector-valued functions that vanish at infinity. The Banach space of functions with bounded variation is defined as*

$$BV = \left\{u \in L^1 : \|u\|_{BV} < \infty\right\},$$

and is equipped with the $\|\cdot\|_{BV}$*-norm. We often call* $\|Du\|$ *the* **TV** *semi-norm.*

$BV(\Omega)$ with Ω being a bounded open domain is defined analogously to BV where in the above definition L^1 and $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ should be replaced by $L^1(\Omega)$ and $C_c^\infty(\Omega;\mathbb{R}^n)$, respectively.

It is helpful to understand that $\text{div}(\vec{g})$ can contain highly oscillating signal as illustrated in the following example. Let $n = 1$ and $g_t(x) = \sin(tx)$ with $|g_t(x)| = 1$. Then $\text{div}(\vec{g}) = \frac{\mathrm{d}g_t(x)}{\mathrm{d}x} = t\cos(tx)$. If t is large, then $\text{div}\vec{g}$ is highly oscillating. Taking the inner product with an oscillating function, a function $u \in BV$ has a small **TV** semi-norm if u is not oscillating. To see this, imagine that u is piece-wise constant and has a few small jumps. Then, one can easily see that $\int u\text{div}(\vec{g})$ is small because the oscillations of $\text{div}(\vec{g})$ cancel with u where u is constant, so $\int u\text{div}(\vec{g})$ is close to the total variation of the jumps. For example, let $u(x)$ be a 1-dimensional piece-wise constant function that equals 1 for $0 \leq x < 1$ and 0 otherwise and $g_t(x) = \sin(tx)$. Even though $\text{div}(\vec{g}(x)) = t\cos(tx)$ has strong oscillations and huge amplification, $\int u\text{div}(\vec{g}) = \int_0^1 t\cos(tx)\mathrm{d}x = \sin(t) \leq 1$. In fact, $\int u\text{div}(\vec{g}) \leq \|Du\| = 2$ for any $\vec{g}$.

The use of BV in image modeling dates back to the early 90's when Rudin, Osher, and Fatemi published their celebrated work [55]. BV images have several advantages over L^p or H^p images. Since natural images often depict multiple objects which occlude one another, natural images contain object boundaries as sharp edges, which are discontinuous. Both the BV and the L^p spaces allow its functions to have discontinuities, but the Sobolev H^p space requires its functions f to have $\nabla^p f \in L^2$, therefore, no sharp edges. On the other hand, the L^p norm does not measure discontinuities, but the size of intensity blowup: the "layer-cake representation" from [41] gives

$$\int_\Omega |u(x)|^p \mathrm{d}x = p \int_0^\infty t^{p-1} |\{x : |u(x)| > t\}| \mathrm{d}t,$$

where $|S|$ denotes the Lebesgue measure of the set S.

To help understand the definition of BV, let us consider the case that $u \in C^1$, i.e., u is an edgeless image; then generalized derivative Du is the usual derivative vector ∇u, i.e.,

$$\|Du\| = \int_\Omega |\nabla u|_{l^2} = \int_\Omega \left| (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}) \right|_{l^2} dx$$

since

$$\int u \, \mathrm{div}(\vec{g}) \, dx = -\int \vec{g} \cdot \nabla u \, dx. \tag{1.1.2}$$

Also noting that $C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n; \mathbb{R}^n)$, we can define the following functional on $C_0(\mathbb{R}^n;\mathbb{R}^n)$: [11]

$$F_u(g) \equiv \int u \operatorname{div}(\vec{g})\, dx := \int u \operatorname{div}(\vec{g}_i)\, dx \tag{1.1.3}$$

for $g \in C_0(\mathbb{R}^n;\mathbb{R}^n)$ where $g_i \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$ uniformly converges to g (thanks to $\|u\|_{BV} < \infty$, the limit does not depend on the choice of the sequence).

More generally, if u is in the Sobolev space $W^{1,1}(\mathbb{R}^n)$, then Du is the generalized derivative of u. Therefore, from the definition (1.1.3) and using integration by parts, we have

$$u \in BV \cap W^{1,1}(\mathbb{R}^n) \Leftrightarrow \sup_{\vec{g} \in C_0(\mathbb{R}^n;\mathbb{R}^n)\ \|\vec{g}\|_{l^2} \leq 1} \int Du \cdot \vec{g} \leq \infty. \tag{1.1.4}$$

We can also see from (1.1.4) that each u defines a bounded linear functional $F_u(g)$ on $C_0(\mathbb{R}^n;\mathbb{R}^n)$ [11]. Using Riesz' representation theorem (also referred as the Riesz-Markov theorem) on the isomorphism between the dual of $C_0(\mathbb{R}^n;\mathbb{R}^n)$ and the set of bounded vector Radon measures, we immediately have the following equivalent definition:

$$BV = \{u : Du \text{ is a bounded Radon vector measure on } \mathbb{R}^n\}.$$

This definition of BV is used in much of the literature. When Du is considered a measure, $\|Du\|$ over a set $\Omega \subseteq \mathbb{R}^n$ equals the total variation of Du as the Borel positive measure over