Lecture 1: Introduction

- Prerequisites, mechanics, etc…
- Examples
- What we will/won't do
- How many problems are convex?

Prerequisites, mechanics, etc…

- Elementary linear algebra and analysis
- Exposure to engineering and operations research problems
- Matlab programming or willingness to learn

- Office hours: Tue 9-10:50am or by appt., 3086 Duncan Hall

- TA: TBA (volunteers?)

- Credits: 3 credit hours
- Grading: homework (approx. 6 sets), a midterm exam (take home), a few Matlab codes, a group project that requires reading papers

What fraction of "real" problems are convex

- By no means all
- Many more than are recognized
- Convex optimization plays important role in nonconvex optimization (more later)

Analogy: linear programs

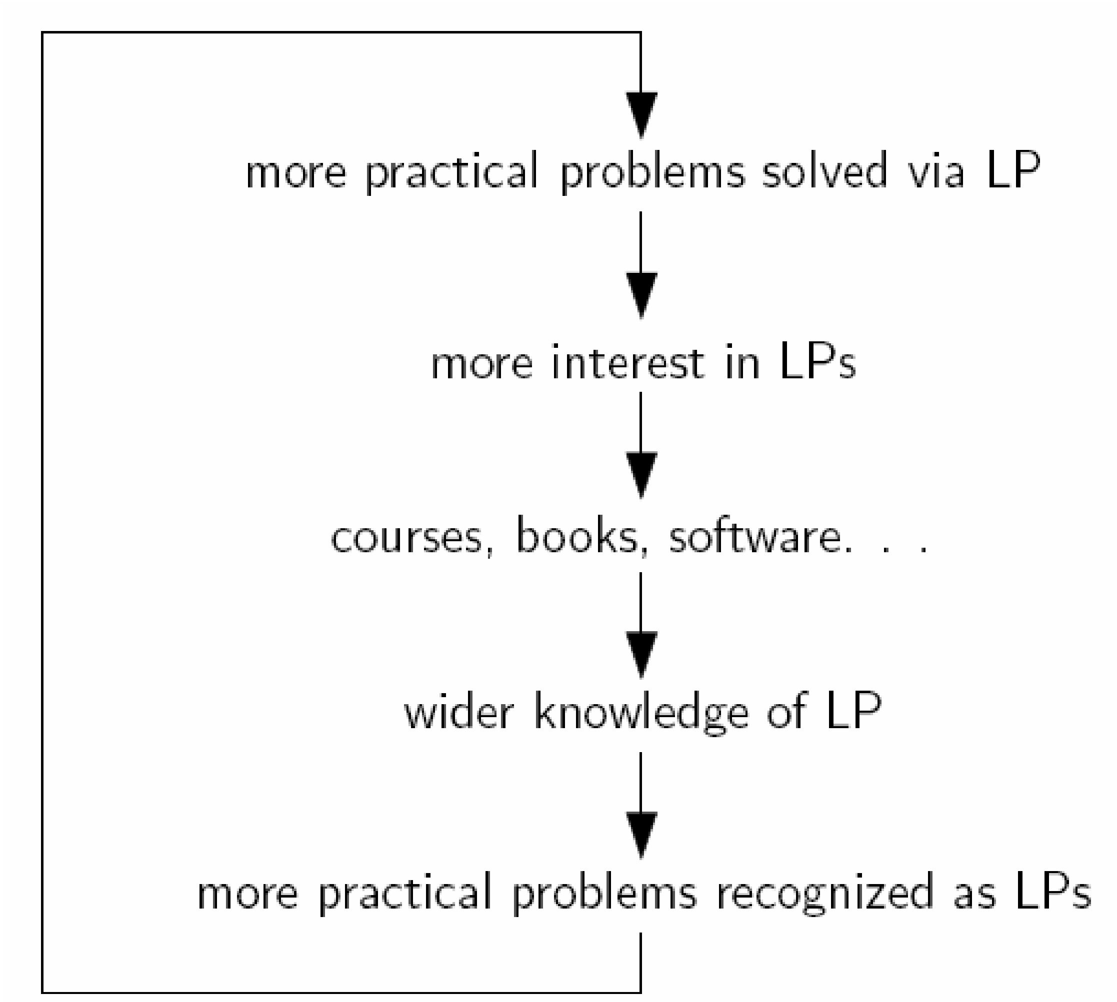
- no "closed form" solution
- very large LPs solved very quickly in practice
- extensive, useful theory

how many problems are LPs?

1940s: "the real world is nonlinear, hence LP silly"

a few example known (in planning)

many examples found **after** LP became widely known…



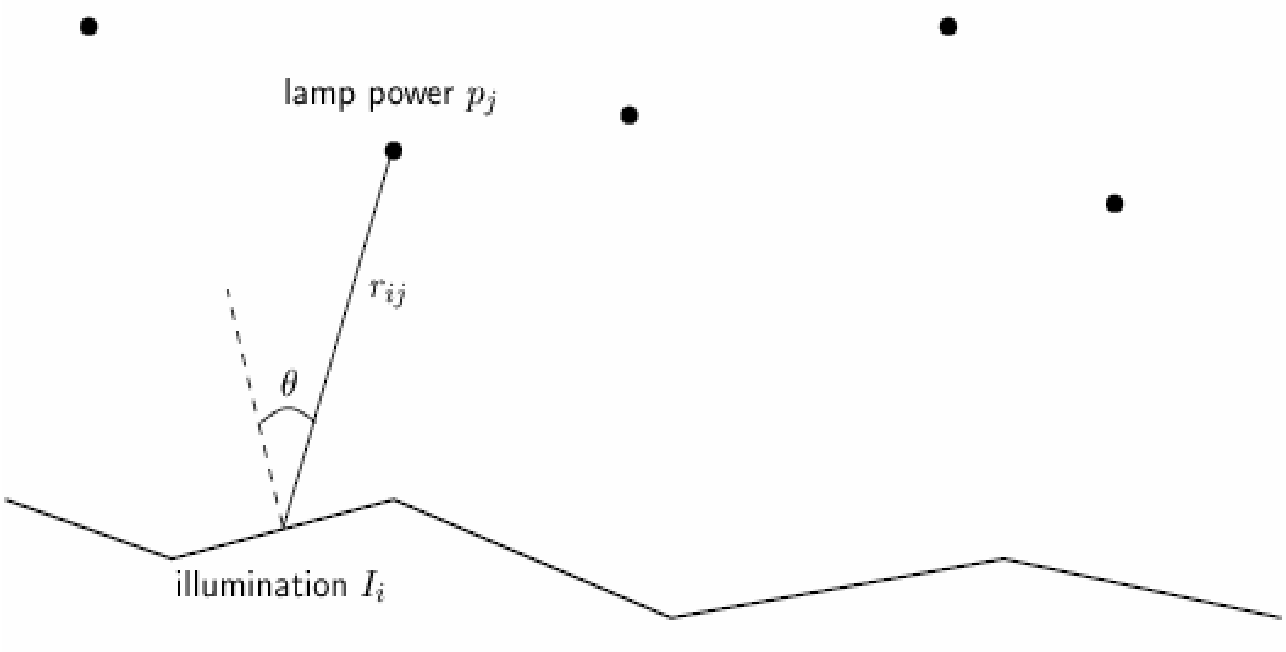
(we guess) same story
for convex optimization

Convex optimization problem

- An example: portfolio optimization
  - Invest some capital in a set of n assets
  - x_i represents the investment in the ith asset, so $x \in R^n$ describes the overall portfolio allocation
  - r_i, which is given, represents the rate of the expected return from Asset i
  - The constraints represent
    - a limit on the budget
    - investments are nonnegative (short positions are not allowed)
    - a minimum acceptable expected return for the whole portfolio
  - The objective represents the overall risk (or variance) of the portfolio return

Example

m lamps illuminating n point on small and flat areas



$$I_i = \sum_{j=1}^{m} a_{ij} p_j, \qquad a_{ij} = r_{ij}^{-2} \max\{\cos\theta_{ij}, 0\}$$

lamp power limits: $0 \leq p_j \leq p_{\max}$

problem: minimize $\max_{i=1,\ldots,n} \left| \log I_i - \log I_{\text{des}} \right|$

How to solve

- Least squares: minimize $\sum_j (I_j - I_{\text{des}})^2$
  closed form, widely available and reliable software, fast
  handle constraints by penalty

- Linear programming

… of course, these are approximate 'solution'

In fact, the problem can be formulated as a convex optimization problem, hence is readily solved

By the end of term, you will be able to

1. Develop, in a few hours, code for problems with 10s of lamps, 100s of patches

2. Develop, in one week, code that quickly solves problems with 100s of lamps, 1000s of patches

3. Characterize optimal power distribution, give limits of performance, etc.

Two additional constraints

1. no more than half total power is provided by the 10 lamps
2. no more than half of the lamps are on

does adding 1. or 2. complicate the problem?

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does adding 1. or 2. complicate the problem?

with 1., still easy to solve

with 2., **extremely difficult** to solve

Moral:

without the proper background, very easy problems can appear quite similar to very difficult problems

(untrained) institution doesn't always work

- What we will cover
  - elementary convex analysis
  - recognizing & exploiting convexity
  - duality theory and its applications
  - semidefinite and second-order cone programming
  - a few (widely used) algorithms

- What we won't do
  - details of convex analysis (ref. "Convex Analysis" by Rockafella)
  - details of optimization theory (regularity conditions, constraint qualifications, …) (check out CAAM 460)
  - encyclopedia of algorithms e.g. will not cover the quasi-Newton, proximal-point, bundle methods …

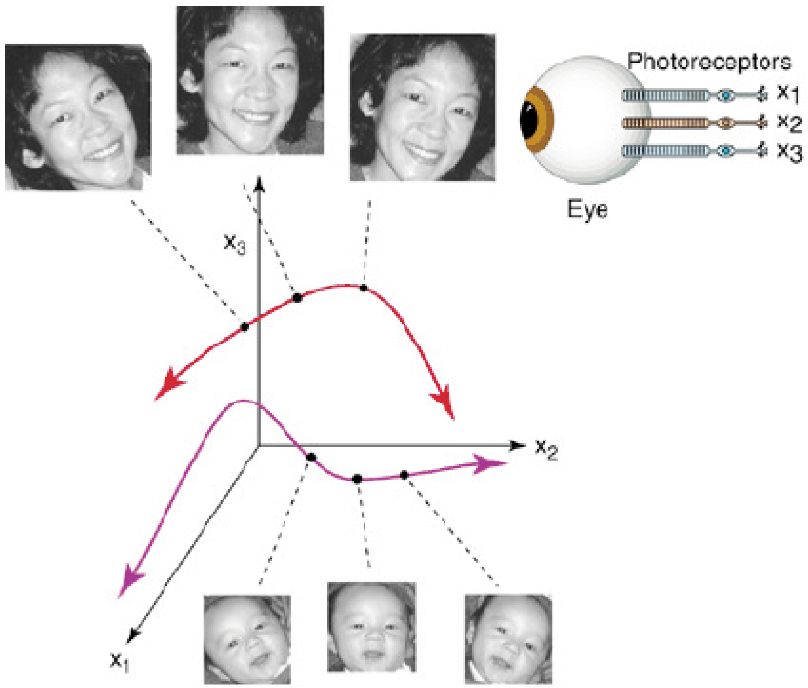
Manifold Learning (Dimension Reduction)

- Weinberger and Saul, 2004, CVPR "Best Paper"

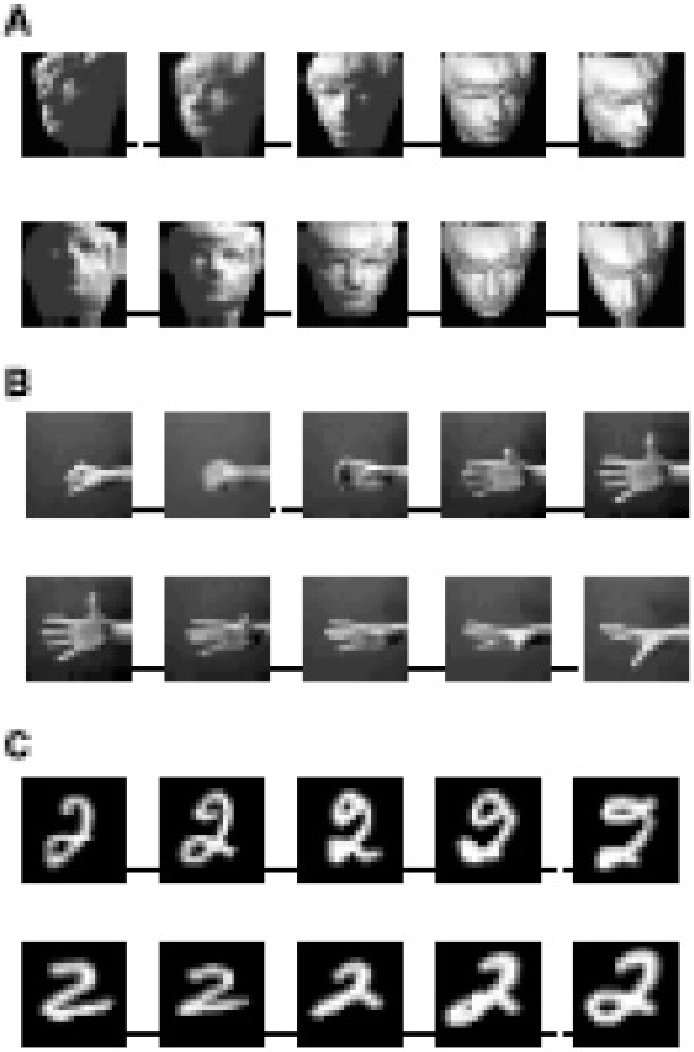
Question:

Given data sampled from a manifold, how to determine the dimension of the manifold and embed it to a lower dimensional space?

Image Manifold

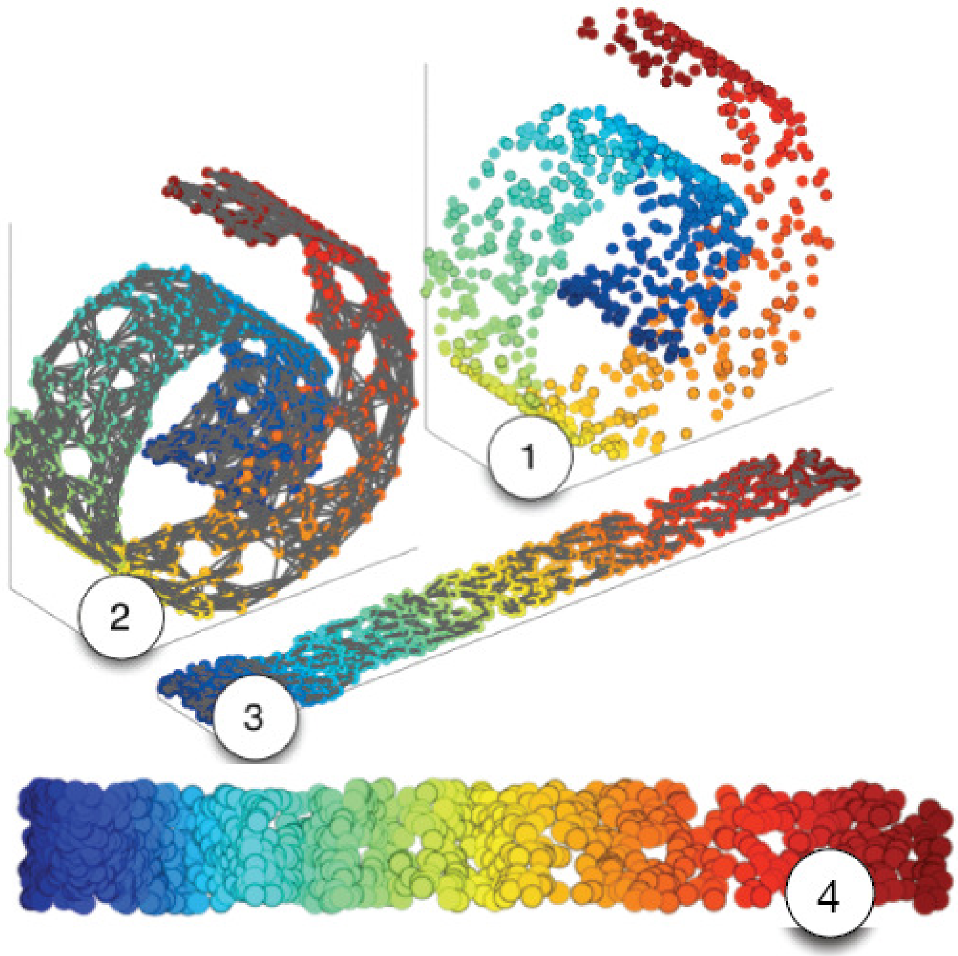


(Seung & Lee, 2000)
(Tenenbaum et al, 2000)



Manifold Learning Approach

- Preserve local geometric relationships (Constraints)
- Maximize the sum of squared pair-wise distances (Objective)



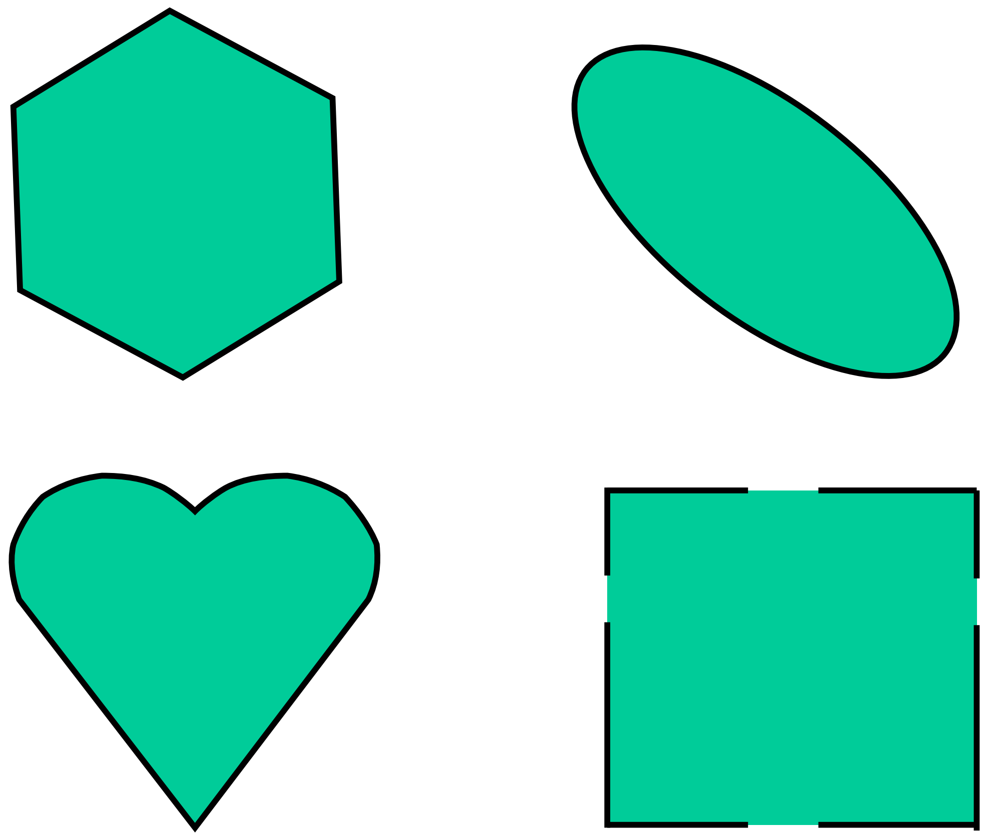
After some work, the problem can be formulated as a semi-definite program (a special class of cvx opt)

$$\begin{aligned}
\max_{K}\ & \mathrm{Tr}(K) \\
\text{s.t.}\ & \left((\mathbf{e}_{ij})(\mathbf{e}_{ij})'\right) K \\
& = (G_{ii} + G_{jj} - G_{ij} - G_{ji}) \\
& \text{for } i, j \in \eta \\
& \left(\mathbf{1}\mathbf{1}'\right) K = 0 \\
& K \text{ p.s.d.}
\end{aligned}$$

where $\mathbf{e}_{ij} = [\ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots]'$

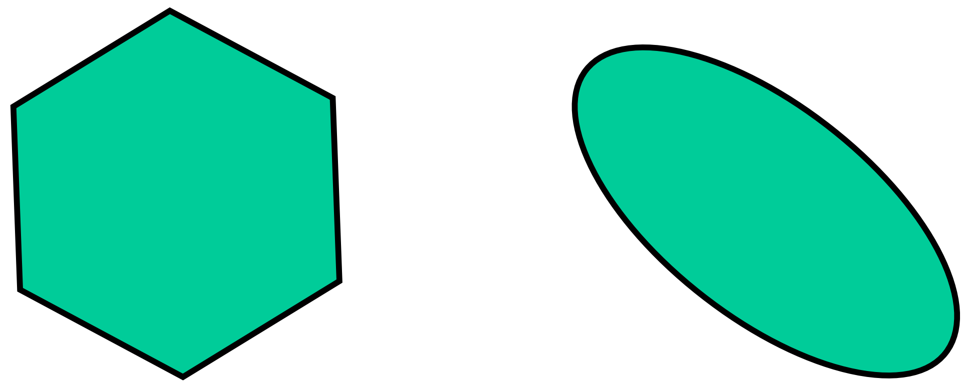
Convex set

- Convex
- Not convex



Convex set

- Convex



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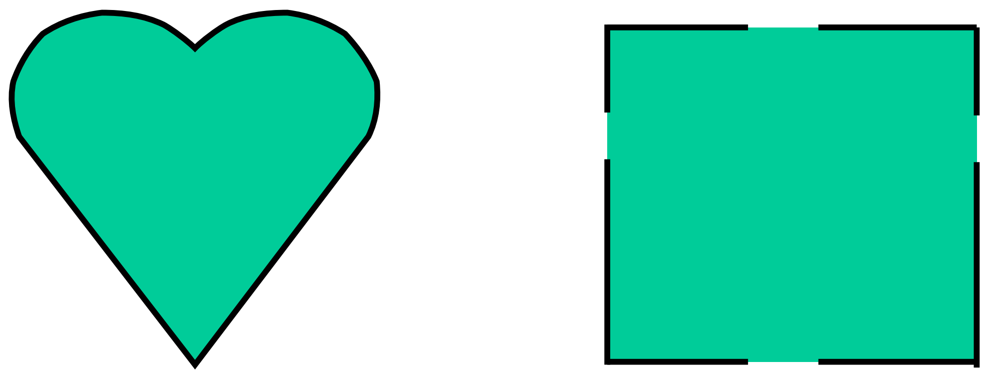


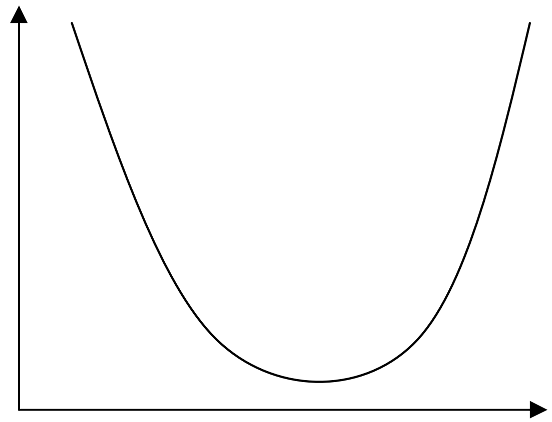
$C \subseteq \mathbf{R}^n$ is *convex* if

$$x, y \in C, \quad \lambda \in [0,1] \Longrightarrow \lambda x + (1-\lambda) y \in C$$

More later...

Convex function

- Convex



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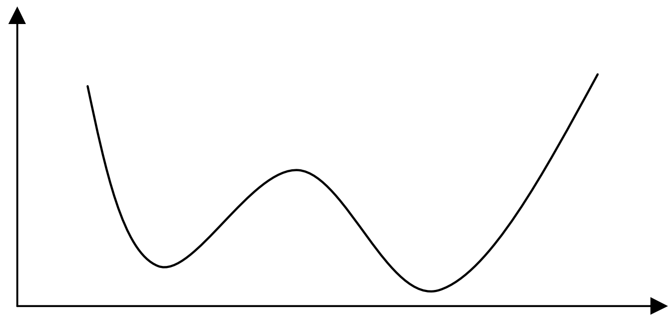

More later…

Convex function

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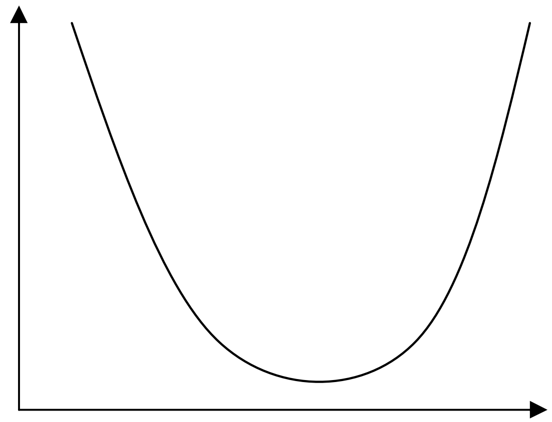
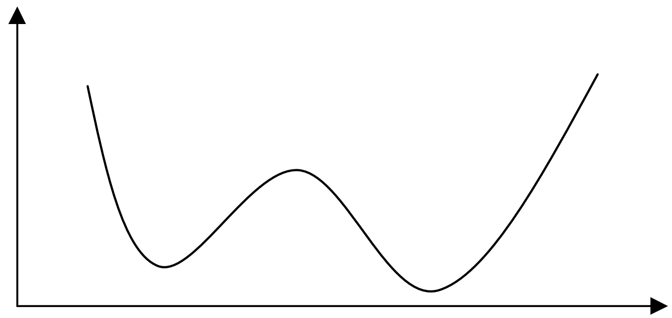


$f : \mathbf{R}^n \to \mathbf{R}$ is *convex* if

$$x, y \in \mathbf{R}^n, \quad \lambda \in [0, 1]$$
$$\Downarrow$$
$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- Not convex



More later…

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

$f(x)$ convex, C convex

Usually $C \equiv \{x \mid f_i(x) \leq 0,\ f_i \text{ convex func. } \forall\, i\}$

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- Convex optimization problems
  - can be solved numerically with great efficiency
  - occur often in engineering problems
  - have extensive, useful theory
  - often go *unrecognized*