

# TV- $L^1$ and the flat norm for shape signatures and distance functions

Simon Morgan

Los Alamos National Laboratory

(Thesis Adviser: Robert Hardt)

# Outline

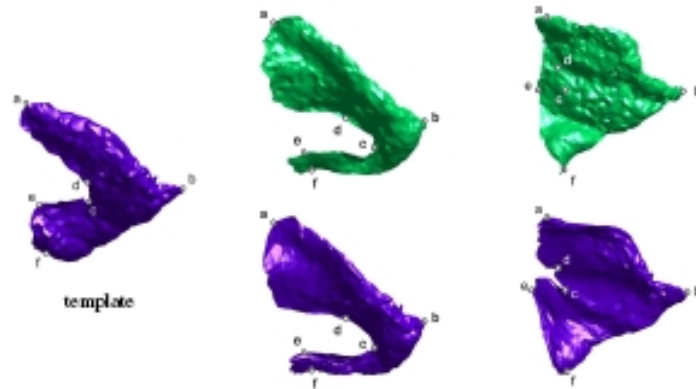
- Previous Applications
- TV- $L^1$  model and Currents and the flat norm
- Shape Questions
- Multiscale analysis
- Regularity and Non Uniqueness
- Currents mod 2 and  $(M, \varepsilon, \delta)$  minimal sets vs diffeomorphisms
- Implementation Schemes

# Application: TV- $L^1$ for small scale feature detection

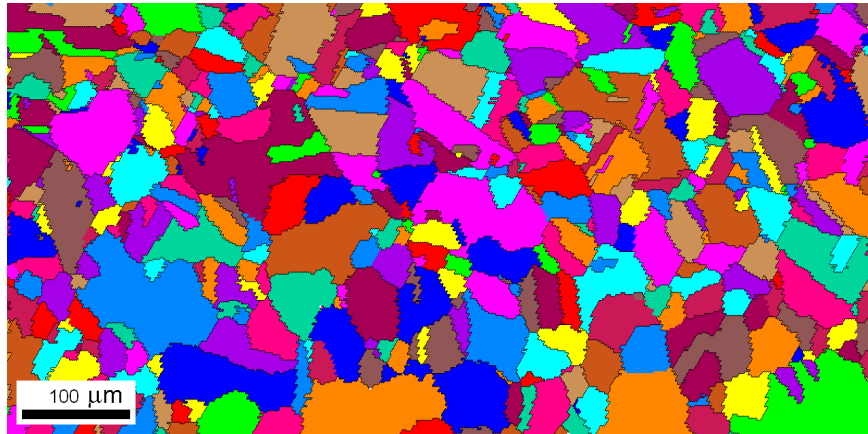


from: The TV- $L^1$  Model For Multiscale Decomposition.  
Yin, Goldfarb and Osher (2007)

# Application: Flat norm on currents for brain surface matching



from: Currents for Surface Matching Vaillant and  
Glaunes (2005)



Austenitic steel with engineered grain boundaries (Image from scanning electron microscope. Todd Allen, University of Wisconsin at Madison)

grain boundary triple/quadruple junctions

artifacts from hexagonal grid

larger scale curvatures from non-planar grain boundaries

## The $TV - L^1$ model

For functions  $g \in BV$  in the plane

$$TV(g) = \int_{\text{Domain}} |\nabla g| da + \int_{\text{Jump discontinuities}} |\Delta g| dl$$

where  $\Delta g$  is the step size of a jump discontinuity

In the  $TV - L^1$  model, for an  $L^1$  function  $f$  representing an image in the plane,  $g \in BV$  is a de-noised version of  $f$  and minimizes

$$TV(g) + \gamma \int |f - g| da$$

regularization

fidelity

If  $\gamma$  gets smaller in

$$E(f) = \min_g \left\{ TV(g) + \gamma \int |f - g| da \right\}$$

then the function  $|f - g|$  takes on larger values and can represent geometric information at different scales, depending on  $\gamma$ , such as edges and features of the face, rather than just noise, independently of luminance level.



$f$        $f/g$

## TV-L<sup>1</sup> on level sets

Let level set  $G_t = \{x : g(x) \geq t\}$

for  $g \in BV$ , the length of the boundary  $H^1(\partial G_t)$  is defined and finite.

So we can integrate up the boundary to get the regularity term

$$TV(g) = \int_0^{T \geq \max(g)} H^1(\partial G_t) dt$$

and by Fubini we can obtain the fidelity term by integrating the area of the symmetric differences of the level sets

$$\int |f - g| dA = \int_0^{T \geq \max(g)} H^2(Ft \cup Gt) - H^2(Ft \cap Gt) dt$$

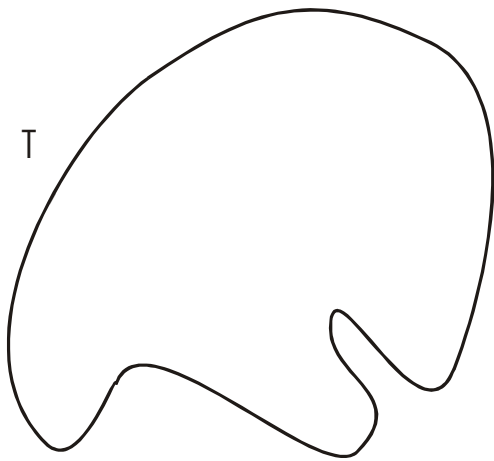
This gives: The Layer cake formula

$$\begin{aligned}
 E(f) &= \min_g \left\{ TV(g) + \gamma \int |f - g| da \right\} \\
 &= \min_g \int_0^{T \geq \max(g)} \left( \begin{array}{c} H^1(\partial G_t) + H^2(Ft \cup G_t) \\ -H^2(Ft \cap G_t) \end{array} \right) dt \\
 &= \int_0^{T \geq \max(g)} \min_{G_t} \left( \begin{array}{c} H^1(\partial G_t) + H^2(Ft \cup G_t) \\ -H^2(Ft \cap G_t) \end{array} \right) dt
 \end{aligned}$$

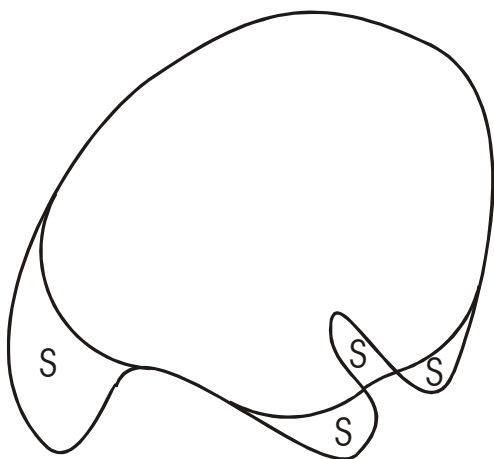
because  $\forall$  minimizing  $G_t$ ,  $\exists$  minimizing  $G_{t'}$  (for  $t' < t$ ) such that  $G_t \subset G_{t'}$ . Proof by contradiction using comparison.

# Definition of the flat norm on 1-currents (oriented curves with integer density) in $\mathbb{R}^2$

$$F(T) = \min_S \left( H^1(T - \partial S) + H^2(S) \right), \quad S \text{ is an oriented region with integer density}$$



Corresponds to a level set of  $f$ .



$T - \text{boundary}(S)$  corresponds to a level set of  $g$  (rounded off version of  $T$ )

Weak topology:

$T_n \rightarrow T$  is defined as  $F(T_n \rightarrow T) \rightarrow 0$

Dual formulation:

$F(T) = \sup_{\phi} (T(\phi)), |\phi| \leq 1, |d\phi| \leq 1, \phi$  is a smooth compactly supported 1-form

$T(\phi) = \int_T \phi$  (line integral)

Cancellation



Noise and artifact resilience;

Staircasing due to grids and noise is removed

by flat norm by cancellation on curves

by  $TV-L^1$  on greyscale images and regions

## Multiscale flat norm

What is the ratio of the area and perimeter of my hand?

Answer depends upon scale of unit

Define Flat norm with scale

$$F_\gamma(T) = \min_S \left( H^1(T - \partial S) + \gamma H^2(S) \right)$$

# Shape questions

Multiscale geometry

Shape signature

Shape description and regularity measures

geomasures

curvature

how  $S$  varies with scale

e.g. Faces, Hearts and Placentas

How close are two curves in  $\mathbb{R}^2$ ?

statistics

object recognition

registration

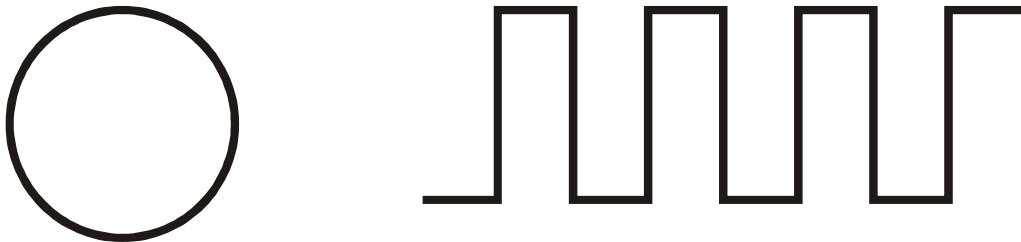
# Regularity and non-uniqueness on $S$ and Boundary $S$

Constant curvature away from  $T$ , depends on  $\gamma$ ,

Intersection conditions on  $T$  and  $\partial S$

Agree on  $T$ ,

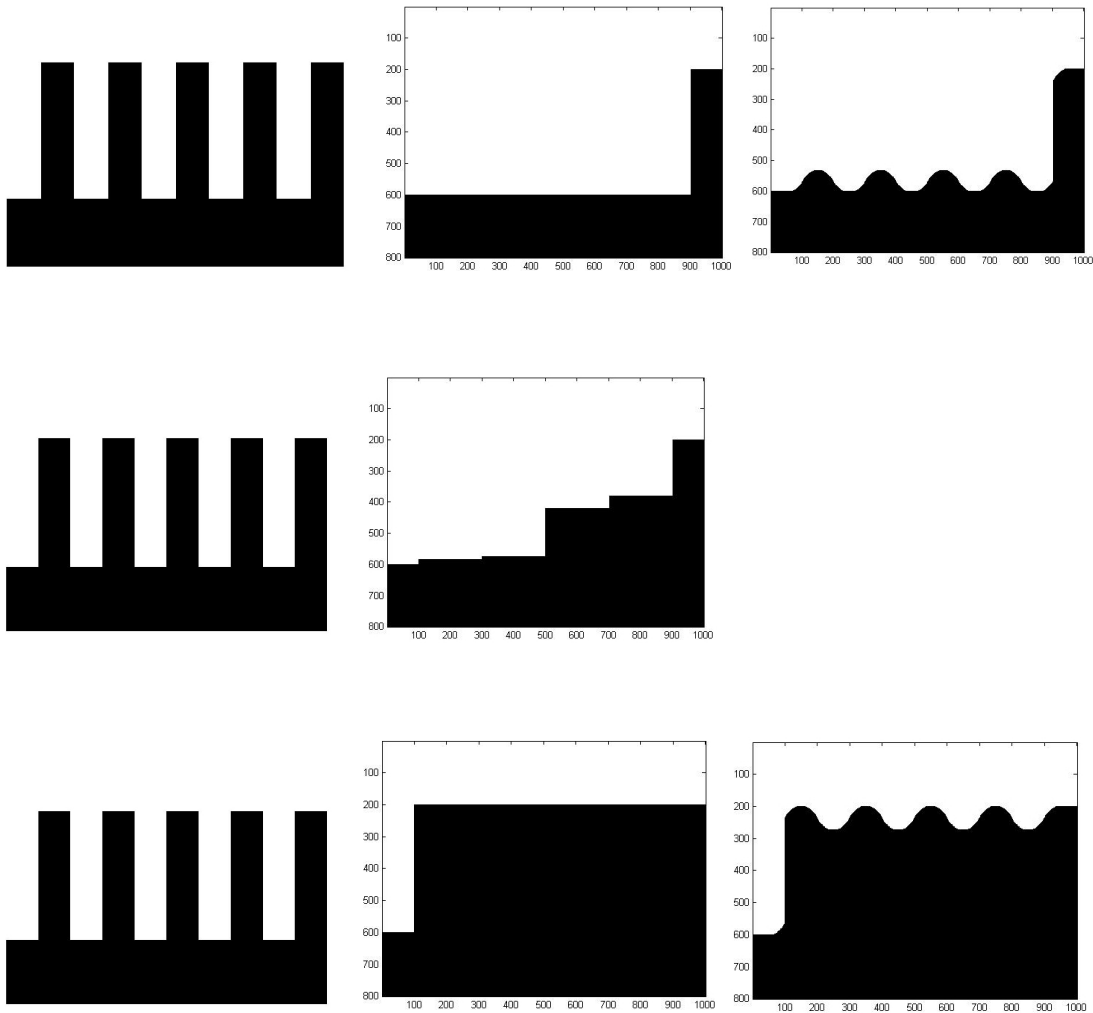
Tangential joining



Examples of discrete and continuous 1-parameter family of non-uniqueness of  $S$

affects robustness

Graph cut implementation showing instability around configurations at critical  $\gamma$  with non-unique  $S$ .



Region  
cut

4 neighbor graph cut

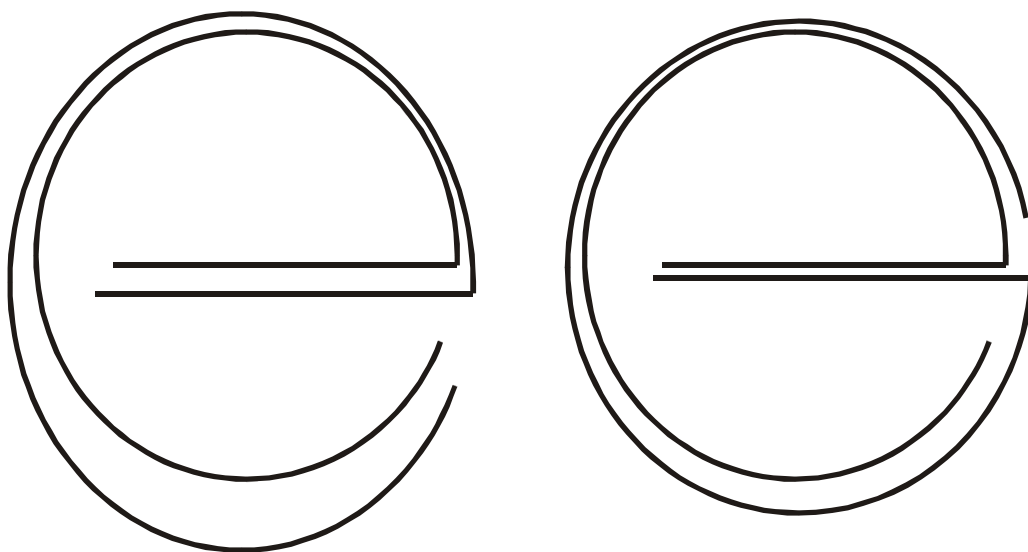
16 neighbor graph cut

If the curves are boundaries

TV- $L^1$  on characteristic functions of regions computes the flat norm for their boundary.

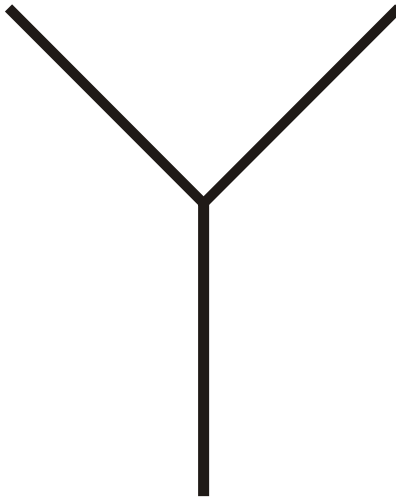
If the curves are oriented but not boundaries

The flat norm generalizes this to sets of curves which are not boundaries of regions.



If the curves are not oriented

Sometimes need currents mod 2

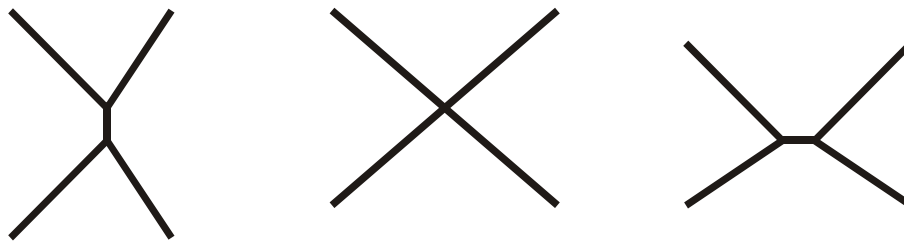


Currents mod 2 can be too flexible topologically

In Currents mod 2,  $T\text{-Boundary}(S)$  may be disconnected

The triple junction problem need to restrict candidates to  $\varepsilon/\delta$  minimal sets or diffeomorphisms,

but then minimize among them with flat norm mod 2.

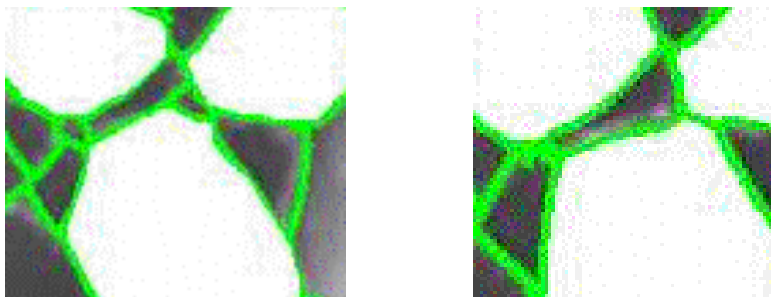


$(M, \varepsilon, \delta)$  minimal sets and diffeomorphisms affect 4-junctions differently.

$(M, \varepsilon, \delta)$  minimal sets are topologically flexible

diffeomorphisms are topologically rigid.

Some physical processes such as vascular morphology require topological flexibility.



Implementations

PDE

Mesh methods

Linear solvers using primal and dual

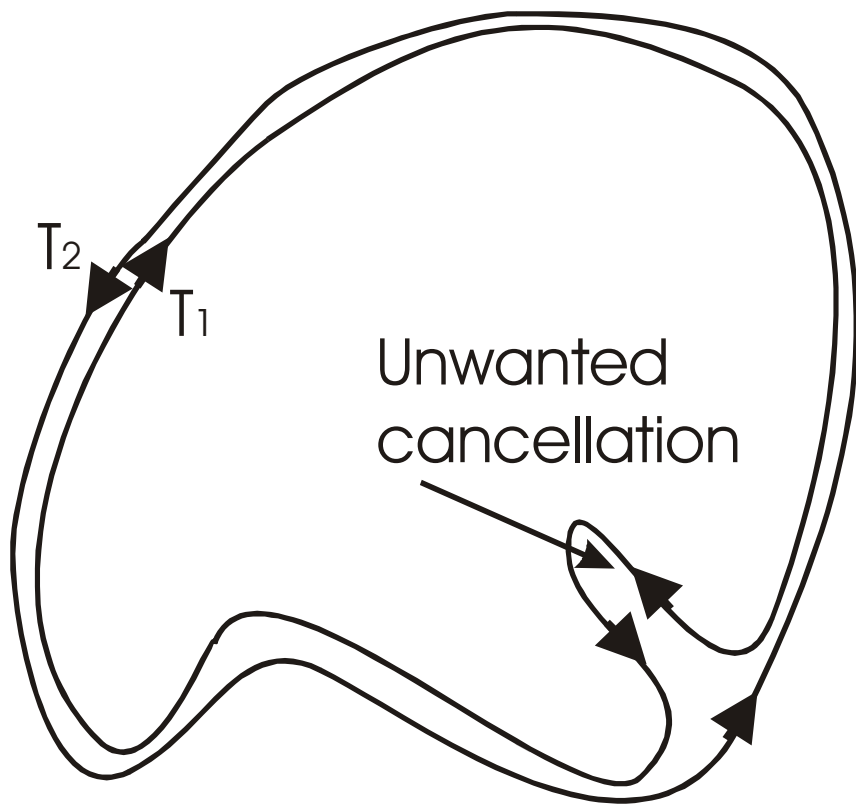
Graph cuts

Adaptive mesh with barycentric sub-  
division

## Reproducing kernel methods (Vaillant and Glaunes)

$$F(T) = \sqrt{\sum_i \sum_j v_i \cdot v_j e^{-d(x_i, x_j)^2 / \sigma^2}}$$

-more flexible, can selectively remove cancellation within and object and keep cancellation between (Some new kind of GMT object)



$$d(T_1, T_2) = \sqrt{\begin{aligned} & \sum_{i \in T_1} \sum_{j \in T_1} |v_i \cdot v_j| e^{-d(x_i, x_j)^2 / \sigma^2} \\ & + \sum_{i \in T_2} \sum_{j \in T_2} |v_i \cdot v_j| e^{-d(x_i, x_j)^2 / \sigma^2} \\ & - 2 \sum_{i \in T_1} \sum_{j \in T_2} v_i \cdot v_j e^{-d(x_i, x_j)^2 / \sigma^2} \end{aligned}}$$