

The Bregman Methods: Reviews and New Error Cancellation Results

Wotao Yin
(joint work with Stan Osher)

Department of Computational and Applied Mathematics
Rice University

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2. Fast, accurate iterations for *constrained* ℓ_1 -like minimization.

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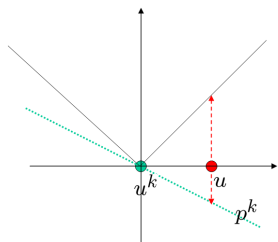
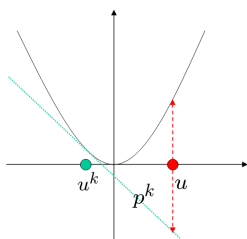
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Bregman Distance

- ▶ Original model: $\min J(u) + f(u)$. Regularizer $J(\cdot)$
- ▶ Given $u^k, p^k \in \partial J(u^k)$
- ▶ Bregman distance:

$$D(u, u^k) := J(u) - (J(u^k) + \langle p^k, u - u^k \rangle)$$



- ▶ New model: $u^{k+1} \leftarrow \min \alpha D(u, u^k) + f(u)$. E.g.: $\alpha = 5$. p^k is obtainable from previous iteration.

Bregman = Add Back Residuals

- ▶ Original model:

$$u \leftarrow \min \mu J(u) + \frac{1}{2} \|Au - b\|_2^2.$$

- ▶ Bregman original form:

$$u^{k+1} \leftarrow \min \mu [J(u) - (J(u^k) + \langle p^k, u - u^k \rangle)] + \frac{1}{2} \|Au - b\|_2^2$$
$$p^{k+1} \leftarrow p^k + A^\top (b - Au^{k+1}).$$

- ▶ Add-Back form:

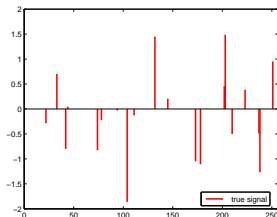
$$u^{k+1} \leftarrow \min \mu J(u) + \frac{1}{2} \|Au - b^k\|_2^2$$
$$b^{k+1} \leftarrow b + (b^k - Au^{k+1}).$$

Each subproblem has the same form of the original problem.

Bregman Regularization: Better Solution Quality

Example: Compressive Sensing Reconstruction from Noise Input

Original signal u , sparse



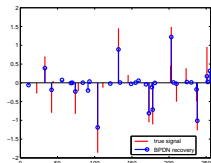
Noisy Gaussian measurements: $b = Au + \omega$, where A : 100×250 .

Compare:

1. Basis pursuit: $u \leftarrow \min \mu \|u\|_1 + \frac{1}{2} \|Au - b\|_2^2$
2. Bregman: $u^{k+1} \leftarrow \min \bar{\mu} \|u\|_1 + \frac{1}{2} \|Au - b^k\|_2^2$, $b^{k+1} \leftarrow$ add back

Basis Pursuit (Non-Bregman) vs Bregman

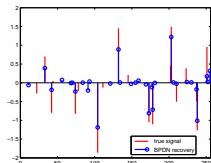
1. Recover u by: $\min \mu \|u\|_1 + \frac{1}{2} \|Au - b\|_2^2$



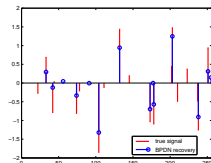
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Not sparse
 μ too small

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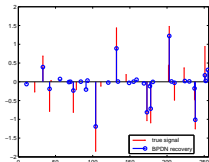
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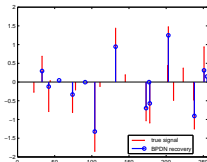
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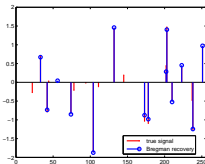


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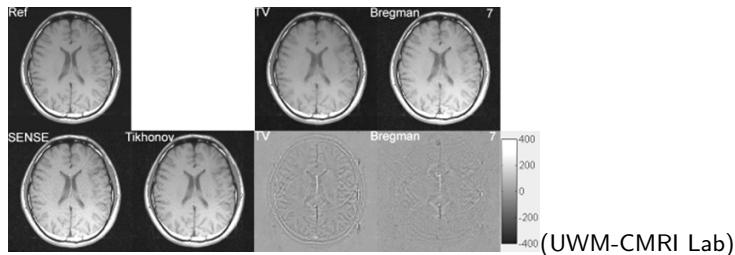
2. Recover u by Bregman: set $\bar{\mu} = 150$, after 5 iterations



Sparse, better fitting

Example: image deblurring and/or denoising

- ▶ $J(u) = \mu TV(u)$
- ▶ $f(u) = \frac{1}{2} \|Au - b\|_2^2$
- ▶ Stop when $\|Au^k - b\|_2^2 \approx \text{est.} \|Au^{true} - b\|_2^2$



Less signal in the residual.

- ▶ For ℓ_1 , Bregman gives *sparser, better fitted* signals
- ▶ For TV, Bregman gives *less staircasing, higher contrast*

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- ▶ Reasons: *iterative boosting*
 1. For small k , u^k is over-regularized yet correctly captures larger nonzeros/edges.
 2. Minimizing $D(u, u^k)$ does not penalize the nonzeros/edges in u^k .

Bregman Regularization vs Prox-Point

The proximal point algorithm: (Rockafellar'76, Censor-Zenios'92, Kiwiel'97, etc.)

$$u^{k+1} \leftarrow \min_u f(u) + D(u, u^k) \quad (1)$$

Differences (Bregman Regularization vs Prox-Point):

1. Goal: Bregman – not to minimize $f(u)$; Prox – to minimize $f(u)$;
2. J : Bregman – nonsmooth, convex; Prox – smooth, strongly convex;
3. Stop: Bregman – prior convergence; Prox – near convergence.

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 - ▶ Work for noisy data
 - ▶ Start with over-regularization
 - ▶ $f(u^k) \downarrow$, stop $f(u^k) \approx f(\text{real } u)$ est.

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Applied to Constrained Minimization

Y.-Osher-Goldfarb-Burger 07

- ▶ Purpose: $u_{real} \leftarrow \min\{J(u) : Au = b\}$, constrained
- ▶ Bregman: $u^{k+1} \leftarrow \min D_J(u, u^k) + \frac{1}{2}\|Au - b\|_2^2$, unconstrained
- ▶ Properties:
 - ▶ $u^k \rightarrow u_{real}$
 - ▶ Fast, finite convergence for ℓ_1 -like $J(u)$
 - ▶ Even if subproblems are solved inexactly, under some conditions, solution is accurate and error converges to machine precision.

Convergence results

Assumption: J is convex, f is convex & differentiable, subproblem solutions exist; p^k starts from 0 (equivalent, b^k starts from b).

Theorem (General convergence)

Under the Assumption, $\{u^k\}$ of (??) satisfies

1. *Monotonic decrease:* $f(u^{k+1}) \leq D(u^{k+1}) + f(u^{k+1}) \leq f(u^k)$.
2. *Convergence:* if u^* minimizes f and $J(u) < \infty$, then $f(u^k) \leq f(u^*) + J(u^*)/k$ and, thus, $f(u^k) \rightarrow f(u^*)$.
3. *Denoise b , noise reduction:* let $f(\cdot) = f(\cdot; b)$ (e.g., $f(\cdot) = \frac{1}{2} \|A \cdot - b\|_2^2$) and suppose $f(\bar{u}, \bar{b}) = 0$ (\bar{b} and \bar{u} are noiseless input and signal, resp); then $D(\bar{u}, u^{k+1}) < D(\bar{u}, u^k)$ as long as $f(u^{k+1}; b) > f(\bar{u}; b)$.

Convergence results

Lemma

If $f(\cdot) = \frac{1}{2}\|A \cdot -b\|_2^2$ and an u^k satisfies $f(u^k) = 0$, then u^k is a solution of $\min\{J(u) : f(x) = 0\}$. This holds even if subproblems are inexactly solved.

Theorem (Finite convergence for ℓ_1)

Let $J(\cdot) = \mu\|\cdot\|_1$ and $f(\cdot) = \frac{1}{2}\|A \cdot -b\|_2^2$. If $Ax = b$ is consistent, then there exists K such that any u^k , $k > K$, is a solution of $\min\{J(u) : f(x) = 0\}$.

The theorem extends to any piece-wise linear J .

Error Cancellation

- ▶ Error cancellation is a happy result due to *adding back!*

$$b^k \leftarrow b + (b^{k-1} - Au^k) \quad (2)$$

$$u^{k+1} \leftarrow \min J(u) + \frac{1}{2} \|Au - b^k\|_2^2. \quad (3)$$

- ▶ Suppose we computed $u_{inexact}^k = u^k + w^k$. w^k is error.
- ▶ (??) becomes $b_{inexact}^k \leftarrow b + (b^{k-1} - Au_{inexact}^k) = b^k - Aw^k$.
- ▶ (??) becomes

$$u^{k+1} \leftarrow \min J(u) + \frac{1}{2} \|A(u + w^k) - b^k\|_2^2$$

which includes the *model error* w^k .

Let w be a model error, and consider

$$\min J(u) + f(u + w). \quad (4)$$

Define

- ▶ u_{exact} : exact solution of (??)
- ▶ $u_{inexact} = u_{exact} + \epsilon$: computed inexact sol of (??)
- ▶ u_{real} : exact solution of $\min J(u) + f(u)$.

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Theorem (General case)

If u_{exact} and $u_{exact} - w$ are on the same face of $\text{graph}(J)$, then

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Corollary (J is ℓ_1)

If u_{exact}^{k+1} and $u_{exact}^{k+1} - w^k$ have no opposite signs, then

$$u_{inexact}^{k+1} - u_{real}^{k+1} = \epsilon^{k+1} - w^k.$$

Theorem Proof:

$$\begin{aligned}u_{inexact} &= u_{exact} + \epsilon \\&= \left[\underset{u}{\operatorname{argmin}} J(u) + f(u + w) \right] + \epsilon \\&\quad \text{(introduce } v := u + w\text{)} \\&= \left[\underset{v}{\operatorname{argmin}} J(v - w) + f(v) \right] - w + \epsilon \\&\quad \text{(use the condition)} \\&= \left[\underset{v}{\operatorname{argmin}} J(v) + f(v) \right] - w + \epsilon \\&= u_{real} - w + \epsilon.\end{aligned}$$

In other words, $u_{inexact} - u_{real} = \epsilon - w$.

Error Cancellation Example

- ▶ u_{real} : 500 entries, 25 nonzero, sparse
- ▶ $b = Au_{real}$: 250 linear projections, A has Gaussian random entries
- ▶ Recover u_{real} by solving $\min\{\|u\|_1 : Au = b\}$
- ▶ Run Bregman, each subproblem inexactly solved with the same tol $\equiv 1e-6$
by: FPC, FPC-BB, GPSR, GPSR-BB, or SpaRSA

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| Itr k | 1 | 2 | 3 | 4 | 5 |
|---|--------|--------|---------|---------|----------|
| $\frac{\ u_{real} - u_{inexact}^k\ }{\ u_{real}\ }$ | 6.5e-2 | 2.3e-7 | 6.2e-14 | 7.9e-16 | 5.6e-16. |

Relative error converged to the machine precision!

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Relative error converged to the machine precision!

- ▶ Classical results require *diminishing tolerances* for convergence, but they are *not* needed for ℓ_1 and above solvers. Why?

Short Answer:

In $u_{inexact}^{k+1} - u_{real}^{k+1} = \epsilon^{k+1} - w^k$, ϵ^{k+1} almost cancels w^k for all k large.

A Slightly Long Answer:

- 1. Finiteness.** With enough accuracy, Bregman u^k reaches the optimal face in finitely many iterations (denoted by K) and stays.
 - The Theorem applies for $k \geq K$.
- 2. Error forgetting.** For any $k > K$, two *exact* Bregman iterations yields the global solution. In other words, errors before K can be forgotten.
 - For any $k \geq K$, $u_{real}^{k+1} = u_{real}$, the global solution.
 - $w^k = \epsilon^k$ and, thus, $u_{inexact}^{k+1} - u_{real} = \epsilon^{k+1} - \epsilon^k$.
- 3. Convergence.** Use a first-order solver with a fixed tol. Given $\epsilon^{k+1} - \epsilon^k$ is small enough, $\|\epsilon^{k+1} - \epsilon^k\| \rightarrow 0$ geometrically in k .
 - $u_{inexact}^k$ converges to u_{real} , the global solution, geometrically in k .

Generalizations

- ▶ Inverse scale space (Burger, Gilboa, Osher, Xu, etc.)
- ▶ Linearized Bregman (Yin, Osher, Mao, etc.)
- ▶ Logistic Regression (Shi, et al. Rice CAAM 08-08)
- ▶ Split Bregman (Goldstein, Osher, UCLA CAM08-29)
- ▶ A unified primal-dual framework, BOS (X.Zhang, et al. UCLA CAM09-99)
- ▶ More ... People use the words “Bregmanize”

Linearized Bregman

Idea: Linearize the fidelity term at u^k

Work: Y.-Osher-Goldfarb-Darbon 07, Osher-Mao-Dong-Y. 08, Cai-Osher-Shen 08, Y. 09

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▶ Example: data fitting = $\frac{1}{2}\|Au - b\|_2^2$

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- ▶ Application: non-negative least-squares, matrix completion

Linearized Bregman, Cont'd

Properties:

- ▶ gradient-ascend the dual of $\min\{\mu\|u\|_1 + \frac{1}{2\delta}\|u\|^2 : Au = b\}$
- ▶ Exact regularization: $\exists \bar{\delta}$: if $\delta > \bar{\delta}$, then solves $\min\{\|u\|_1 : Au = b\}$
- ▶ Empirically, # nonzeros of u^k grows monotonically in k

Yin, W. Analysis and Generalizations of the Linearized Bregman Method, Rice CAAM Report TR09-02. [link]

Operator Splitting ADM

Operator splitting + ADM gives Split Bregman (Goldstein–Osher 08)

- ▶ Operator splitting by (Wang–Yang–Y.–Zhang 07,08): Split $TV(u)$ to Du and $\sum_i \|(\cdot)_i\|$. Great payoff for many imaging problems.
- ▶ Apply the alternating direction of multipliers (ADM) method to above splitting.

Operator Splitting ADM

Alternating direction method: (Douglas–Rachford 60s, Glowinski–Marocco, Gabay–Mercier, 70s)

1. fix u , minimize w.r.t. v
2. fix v , minimize w.r.t. u
3. update λ

Example (Wang–Yang–Y.–Zhang 07,08) Compressed MRI, image debl

$$\min_u \mu TV(u) + \frac{1}{2} \|Au - b\|_2^2 \Leftrightarrow \min_u \{\mu \|w\|_1 + \frac{1}{2} \|Au - b\|_2^2 : w = Du\}$$

where A is partial Fourier or convolution. ADM extends to color images, duals, rank-minimization

Summary

1. Bregman improves ℓ_1 -like regularization quality for noisy data
2. Bregman applied to constrained ($Au = b$) minimization is not new but is fast and accurate due *adding back*
3. Various extensions take advantages of model structures

More details and solvers at *Rice L_1 -Related Optimization Project*