Topology: simplicial methods

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Topological complexes: the idea

Use combinatorics to define the space structure

Think of a space as of an “assembly”, a “complex”
- a composition of “elementary pieces”

Use algebraic methods for the analysis

What can serve as an “elementary piece”? 
Simplexes

1-simplex $\sigma_{12}^{(1)}$

$\sigma_{12}$

2-simplex $\sigma_{123}^{(2)}$

3-simplex $\sigma_{1234}^{(3)}$

0-simplex $\sigma_1^{(0)}$
Simplicial complex $K$
Face of the simplex

Face of a $n$-simplex is a $n-1$ simplex

Simplex contains all of its faces
Simplicial complex $K$
Simplicial complex $K$: a union of simplexes

Overlap of any 2 simplexes produces a simplex from $K$

Simplicial complex $K$ is defined by:

1. the list of its simplexes
2. the list of simplex “incidences”
Simplex orientation
Boundary of the simplex

\[ \partial \sigma^{(2)}_{123} = \sigma^{(1)}_{12} + \sigma^{(1)}_{23} + \sigma^{(1)}_{31} \]

\[ \partial \sigma^{(2)}_{123} \neq \sigma^{(1)}_{12} - \sigma^{(1)}_{23} + \sigma^{(1)}_{31} \]
Boundary of a polyhedron
Boundary of the simplicial complex
Boundary of an *oriented* simplicial complex
Boundary of an *oriented* simplicial complex
Boundary of an *oriented* simplicial complex
Boundary of an oriented simplicial complex
Simplicial complex $K$, orientation

\[ \partial \approx \]
What is the use of simplicial complexes?

**Topology** → **Algebra** → **Computation**
Paths, equivalence classes, $[\gamma]$  

Topological index $m$ of the path $[\gamma]_m$ produced the fundamental group $\pi_1(X) = \mathbb{Z}$
Simplicial complex

Space is approximated by the complex $K$
Discretization of paths

**Space is approximated by the complex $K$**

Paths $\gamma$ are defined over $K$

Paths $\gamma \rightarrow$ cycles, $z$
A cycle $z$ can be deformed over the simplex $K$
Cycle deformations $\Delta z$'s are snapped over the boundaries of 2D simplexes: $z_1 = z_2 + \partial \sigma^{(2)}$
Cycle deformations $\Delta z$’s are snapped over the boundaries of 2D simplexes: $z_1 = z_2 + \partial \sigma^{(2)}$
Homologous cycles:

\[ z_1 \sim z_2 \] (homotopic paths)
Topological analysis with cycles

How many classes of homologous cycles are there?
Non-homologous cycles: $z_1 \sim z_2$

What feature makes these cycles different?
Classes of cycles

Cycles $z_2$ can be contracted to a point, because it is a boundary of a \textit{contractible} 2D “surface”
Classes of cycles

1. Contractible cycles, e.g. $z_1$
2. Non-contractible cycles, e.g. $z_2$

Non-homologous cycles: $z_1 \not\sim z_2$
Homologies

\[ H_1 = \left( \text{classes of homologous cycles} \right) \]

"\( z_1 \) is homologous to \( z_2 \)" =

= "\( z_1 \) is equal to \( z_2 \) modulo a boundary cycle"

\[ H_1 = (Cycles) / (Boundaries) \]
"$z_1$ is homologous to $z_2$" =

= "$z_1$ is equal to $z_2$ modulo a boundary cycle"

$$H_1 = (Cycles)/(Contractible cycles)$$

= (non-contractible cycles and their multiples)
"$z_1$ is homologous to $z_2$" =

= "$z_1$ is equal to $z_2$ modulo a boundary cycle"

$$H_1 = \frac{(\text{Cycles})}{(\text{Contractible cycles})} = \mathbb{Z}$$

First homology group
Fundamental vs. homological group

\[ \pi_1(X) = \mathbb{Z} \]

\[ H_1(X) = \mathbb{Z} \]
**Theorem 1:** Homological groups do not depend on simplicial subdivision of polyhedron*

**Theorem 2:** Homological groups are topologically invariant

* For fine enough subdivisions
What we ultimately want with homologies

How many classes of homologous cycles are there, in every dimension?
Betti numbers – number of cycles in every dimension

Circle

(1, 1, 0, 0, ...
Topological properties, examples

Cycle connectedness: \(3\) \(\emptyset\)-cycles, and \(3\) pieces
Betti index – base cycles in every dimension

Circle

Sphere

Torus

“Topological barcode”

(1, 1, 0, 0,...)

(1, 0, 1, 0,...)

(1, 2, 1, 0,...)
How to build simplexes in practice?

How to build a triangulation of a surface?

http://www.cgal.org
Čech complex
Čech complex
Čech complex
Čech complex
Čech complex
Čech complex
If the cover is fine enough, the homologies of the complex $K$ are the same as the homologies of the original space.
A manifold and its cover
A cover generates simplex
Simplex produces full topological information

Homologies, etc.

Test: what is the “Topological barcode” of this space?

Sphere

(1, 0, 1, 0, ...)

Topology from sensor networks

V. de Silva, Homological sensor networks, (2007)
Hole in sensor coverage area
What is the wireless topology of the US?

Hole in sensor coverage area
Point cloud data

(a) Surface
(b) Molecule
(c) Universe
The ideas of topological persistence

Points

$\varepsilon$-Balls

Čech Complex
The unfolding of the topological information

Example: Sphere
Topological barcode of a sphere

Topological barcode \((1, 0, 1, 0, 0 \ldots)\)
The unfolding of the topological information

$0D$

$1D$

$2D$
The unfolding of the topological information

"Topological barcode"

(1, 2, 1, 0,...)

↓

Torus

↓
“Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone...”

A. Solzhenitsyn, “The First Circle” (1955—58)

Homology: An Idea Whose Time Has Come

Summary

1. Simplexes and simplicial complexes
2. Boundaries and orientations
3. Homologous cycles
4. Homological group

Next: Neuroscience applications...

jPlex, computational topology software, Stanford University