COLLAPSING RIEMANNIAN MANIFOLDS
WHILE KEEPING THEIR CURVATURE BOUNDED. I

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0. Introduction

Let $Y^n$ be a complete connected riemannian manifold, and $p \in Y^n$. The injectivity radius, $i_p$, of the exponential map at $p$ is defined to be the smallest $r$ such that $\exp_p B_r(p)$ fails to be a diffeomorphism onto its image. The present paper is the first of two which are concerned with the situation in which the size of injectivity radius is "small" relative to the curvature.

In this part I, we show that if a smooth manifold $X^n$ admits a certain topological structure called an $F$-structure of positive rank, then $X^n$ also admits a family of metrics, $g_\delta$, such that as $\delta \to 0$, $i_\delta$ converges uniformly to zero at all points $p$, but the curvature, $K_\delta$, stays bounded (independent of $p$ and $\delta$). Such a family of metrics is said to collapse with bounded curvature (by rescaling, one can assume $|K_\delta| \leq 1$).

In part II we prove a sort of strengthened converse to this collapsing result. A riemannian manifold $Y^n$ is said to be $\varepsilon$-collapsed if $i_\varepsilon < \varepsilon$ for all $p$. Intuitively, such a manifold appears to have dimension $< \varepsilon$ if one examines it on a scale $\gg \varepsilon$. We show that in each dimension, there exists a critical radius, $\varepsilon(n)$, such that if $Y^n$ is $\varepsilon(n)$-collapsed and $|K| \leq 1$, then $Y^n$ admits an $F$-structure of positive rank. Thus, if $Y^n$ admits a metric which is sufficiently collapsed, it actually admits a family of metrics which collapse with bounded curvature.

An $F$-structure on a space, $X$, is a natural generalization of a torus action. Different tori (possibly not all of the same dimension) act locally on finite covering spaces of subsets of $X$. These local actions satisfy a compatibility condition, which insures that $X$ is partitioned into disjoint "orbits." The $F$-structure is said to have positive rank if all orbits are of positive dimension.

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The existence of an $F$-structure of positive rank is a definite constraint on the topology of a space. In the compact case, it implies, for example, that the Euler characteristic is zero (compare however Example 1.6 of [4]). This vanishing phenomenon does not carry over to Ptolemaic numbers, except in the presence of further hypotheses; see Example 1.9 (we will show elsewhere that the Ptolemaic numbers of $X^{st}$ vanish if it admits a so-called "pure structure of positive rank, with amenable holonomy"). However, there is a strong interaction between $F$-structures, characteristic numbers, and secondary geometric invariants; see [3], [5], [13].

The collapsing family of metrics $g_b$, associated to an $F$-structure of positive rank is obtained roughly as follows. Start with a metric $g$ which is invariant for the structure in the sense that the local torus actions are isometric. Then shrink $g$ in certain directions tangent to the orbits. In some cases, it is also necessary to expand $g$ in directions orthogonal to the orbits, in order to keep the curvature bounded. Thus, the diameter $\text{diam}(Y, g_b)$, and volume, $\text{Vol}(Y, g_b)$, may go to infinity as $(Y, g_b)$ collapses; it may also happen that they stay bounded or converge to zero.

The following examples (although they are presented informally) should serve to give some feeling for the concepts mentioned so far; see §1 for the precise definition of an $F$-structure (which is somewhat technical).

**Example 0.1 (The Klein Bottle).** View the Klein bottle as the total space of a circle bundle $S^1 \to K^2 \to S^1$. For each interval, $I$, in the base space, there are two canonical fiber preserving circle actions on $p^{-1}(I)$, which differ by the automorphism $x \to x^{-1}$; one of these local actions is continued around the base circle, the opposite action is obtained. As a consequence of this holonomy phenomenon, no global action is possible. (See Example 1.2 for further examples.)

**Example 0.2 (Graph Manifolds).** Take a finite collection of surfaces, $S^2_i$, with $\partial S^2_i = \bigcup_{j \neq i} S^1_i$, a disjoint union of circles. The product manifolds, $S^2_i \times S^1$, have boundary components which are tori, $S^1_i \times S^1_i$. Form a manifold with empty boundary, $Y$, by identifying these tori in pairs by elements of $SL(2, \mathbb{Z})$.

On each piece $S^2_i \times S^1_i \subset Y^2$, $S^1_i$ acts by rotation of the factor $S^1_i$. At boundary components which have been identified, the corresponding circle actions need not agree. But if not, they generate an action of a 2-torus, which extends both of them. Thus, in this example, the torus which acts locally is of dimension 2 near such identified boundary components, and of dimension 1 elsewhere.

**Example 0.3 (Compact Flat Manifolds).** If $X^*$ is compact and flat, by the Bieberbach Theorem there is a finite normal covering $X^*$, which is isometric to a torus. Since the action of this torus on itself is transitive, the induced orbit structure on $X^*$ consists of a single orbit, $X^*$. 

Example 0.4 (Collapse by scaling). If \((Y\gamma, g)\) is a complete manifold with injectivity radius uniformly bounded from above, then the family \((Y\gamma, \delta^2 g)\) collapses (intuitively, to a single point, in the compact case)\). However, for this collapse, the sectional curvature does not remain bounded unless \(Y\gamma\) is flat.

In view of Example 0.4, from now on the word "collapse" will be taken to mean "collapse with bounded curvature."

The following is the most transparent and in a sense the most basic collapse with bounded curvature.

Example 0.5 (Generalized warped products). Start with a surface of revolution, \(M^2\), obtained by revolving an arc in the upper half plane about the \(x\)-axis. Thus, \(M^2\) is diffeomorphic to \(S^1 \times I\). The obvious isometric circle action on \(M^2\) lifts to an isometric \(R\) action on the infinite cyclic covering \(\tilde{M}^2 = R \times I\). Let \((\mathbb{Z}) \subset R\) denote the subgroup generated by a translation of size \(\delta\). Then the family \(\tilde{M}^2 / (\mathbb{Z})\) collapses, but the curvature remains unchanged (we have unrolled \(M^2\) and then rolled it up more tightly).

To extend the above example to higher dimensions, take \(\tilde{M}^{k-1} = X' \times R',\) with (generalized) warped product metric

\[
(0.1) \quad g = g_1(x) + \sum_{i,j=1}^{k} a_{ij}(x) dy_i dy_j.
\]

Then \(\tilde{M}^{k-1} = X' \times R'/\mathbb{Z}\) collapses to \(X'\).

Note that the orbits of the \(\mathbb{Z}\)-structure on \(\tilde{M}^{k-1}\) have constant dimension, 1.

In such cases (as above) the collapse can always be performed so that the diameter remains bounded. In particular, the volume goes to zero.

Example 0.2 (continued). The collapse associated to the \(F\)-structure on the graph manifold, \(Y'\), is particularly easy to describe if the identifications of the boundaries simply interchange the coves of the two circles. In this case, choose a "cusp-like" metric on \(S^2\) which near the boundary is isometric to the product of an interval and a circle, \(S^1 \times I\), of length \(\delta\). The curvature and volume can be chosen bounded independent of the size of \(\delta\) for such a metric.

Now form \(\Sigma\times S^2\) with the product metric, and identify corresponding boundary components. The resulting manifold, \((Y', g_0)\), has injectivity radius everywhere \(= \delta\). In fact, \(\text{Vol}(Y', g_0) \times e\delta\). However, the orbits of the \(F\)-structure are not of constant dimension and \(\text{diam}(Y', g_0) \to \infty\).

As far as we are aware, the first example of collapse (apart from warped products and scaling) was discovered by M. Berger in about 1962. He considered the collapse of the unit sphere \(S^3\), obtained by shrinking the circles of the Hopf fibration. It is clear that the "limit" of this collapse should be \(S^1\) (in fact, with a metric of curvature \(= 4\)). The notion of the limit of a collapse can
be made precise by introducing the concept of Hausdorff limit: see [9] and §2. Berger's interest in the collapse of $S^1$ stemmed from his observation that it provides a counterexample to a specific conjecture concerning a lower bound for the injectivity radius on odd-dimensional manifolds of positive curvature.

Another significant collapse (with variable topology) was discovered in the context of manifolds of positive curvature by Abad and Wallach [1]. They exhibited an infinite sequence of pairwise nonisomorphic, homogeneous 7-manifolds with uniformly pinched positive curvature. By the finiteness theorem in riemannian geometry (see [2], [4], [9], [11]) such a sequence must collapse.

The remainder of this paper is divided into five sections and one appendix as follows.

1. $\Gamma$-structures and $P$-structures
2. Pure polarized collapses with bounded diameter
3. Polarized volume collapse
4. Nonpolarized collapses
   (a) Introduction
   (b) Main computation
   (c) Construction of slice polarizations
5. $P$-structures and complete metrics on open manifolds
   (a) Introduction
   (b) Construction of a complete metric, $g_1$
   (c) Expansion of $g_0$
   (d) Collapse of the expanded metric

Appendix: Pure polarized structures on essential manifolds.

In §1, we define and give examples of generalized group actions called $\mathcal{G}$-structures. Essentially, an $\mathcal{F}$-structure is a $\mathcal{G}$-structure for which all the groups which act locally are tori. In §2, we consider the case of a pure structure. Basically, this means that a single connected group acts locally, up to automorphism, on a finite covering space. We assume, moreover, that all orbits are of the same positive dimension; compare Examples 0.1, 0.4, and 0.5. This second condition defines what is called a pure polarized structure. For such structures, by shrinking a compatible metric in the direction of the orbits while leaving it unchanged in the orthogonal directions, we obtain a collapse for which the diameter stays uniformly bounded.

In §3 we consider the polarization for which the groups which act locally are not all of the same dimension. In this case, we can collapse in such a way that $\text{Vol}(Y, g_0) \to 0$, but $\text{diam}(Y, g_0) \to \infty$; compare Example 0.2.
As already indicated, there exist manifolds, $M^p_\omega$, admitting $F$-structures (which are in fact pure) of positive rank, but for which some characteristic number is nonzero. By the Chern-Weil theory, these manifolds admit no collapse with bounded curvature such that $\text{Vol}(M^p_\omega, g_0) \to 0$. In particular, $M^p_\omega$ does not carry any polarized $F$-structure. However, in §4, we show that any $F$-structure of positive rank admits what we call a slice polarization. This can be used to collapse in such a way that the volume behavior is controlled by the geometry of the orbit structure. For example, the manifolds $M^5_\omega$ can be collapsed so that the volume stays bounded. But it can also happen that the volume goes to infinity or to zero (even though the slice polarization is not an honest polarization: compare [10]).

In §5 we consider open manifolds which carry an $F$-structure outside a compact set. On such manifolds, we obtain complete metrics of bounded curvature with properties analogous to those of the metrics constructed in §2–4.

In the Appendix we exhibit a class of manifolds with the property that if a pure $F$-structure exists, it must be polarized. As a consequence, many of these manifolds can be shown to admit no pure $F$-structure of positive rank, although they do not admit such structures which are not pure.

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1. $\hat{g}$-structures and $F$-structures

In this section, we discuss certain generalizations of the concept of a group action.

A partial action, $A$, of a topological group, $G$, on a Hausdorff space, $X$, is given by the following data.

(i) A neighborhood $\mathcal{B} \subset G \times X$ of $e \times X$, where $e \in G$ is the identity element. This $\mathcal{B}$ is called the domain (of definition) of the action.

(ii) A continuous map $A : \mathcal{B} \to X$, also, writes $(g, x) = g \cdot x$, such that $(g_1 g_2) \cdot x = g_1 (g_2 \cdot x)$ whenever $(g_1, g_2 \cdot x)$ and $(g_2 \cdot x, x)$ lie in $\mathcal{B}$, and such that $e \cdot x = x$ for all $x \in X$.

Two partial actions $(A, \mathcal{B}_1), (A_2, \mathcal{B}_2)$ are called (locally) equivalent if there is a domain $\mathcal{B} \subset \mathcal{B}_1 \subset \mathcal{B}_2$ containing $e \times X$, such that $A_1 | \mathcal{B}_2 = A_2 | \mathcal{B}_2$. A
local action. \( (A) \), is defined as the equivalence class of a partial action \( A \) of \( G \) on \( X \). Every global action \( A \) defines an obvious local action \( A_{loc} \). A local action which can be obtained in this way is called complete.

**Remark 1.1.** An elementary connectedness argument shows that \( A_{loc} \) determines \( A \) uniquely, in case \( G \) is connected.

In the smooth case, the category of local actions is equivalent to the category of infinitesimal actions: these are continuous homomorphisms of the Lie algebras of \( G \) to the Lie algebra of vector fields on \( X \). For example, if \( G = R \), then a local action is given by a vector field on \( X \), and completeness amounts to the integrability of the field.

From now on, we assume that \( G \) is connected.

A subset \( X_0 \subset X \) is called (locally) \( (A) \)-invariant if for some representative \( (A, \delta) \subset (A) \), one has \( gx \in X_0 \) for all \( (g, x) \in \delta \). Since the intersection of \( (A) \)-invariant sets is \( (A) \)-invariant, it follows that each point \( x \in X \) is contained in a unique minimal \( (A) \)-invariant subset called the orbit \( \mathcal{O} = \mathcal{O}_x \subset X \), and that the orbits partition the space \( X \). Moreover, if \( A_{loc} \) is complete, the orbits of \( A_{loc} \) and \( A \) coincide.

A local action, \( (A) \), on \( X \) can be restricted to any open subset, \( U \subset X \), by taking an open subset \( \delta' \subset G \times X \), which contains \( x \times U \) and such that \( gx \in U \) for all \( (g, x) \in \delta' \) with \( x \in U \). Furthermore, if \( Y \to X \) is a local homeomorphism, then \( (A) \) pulls back to a local action, \( f^*(A) \), on \( Y \), in a similar way.

Now consider a sheaf \( \mathcal{P} \) of connected topological groups over \( X \). Let \( \mathcal{P}(U) \) denote the group of sections over \( U \). An action of \( \mathcal{P} \) on \( X \) is given by a local action of the group \( \mathcal{P}(U) \) on \( U \) for every connected open set \( U \subset X \), such that the structure homomorphisms \( \mathcal{P}(U) \to \mathcal{P}(U') \) (for \( U' \subset U \)) agree with the restrictions of the local actions from \( U \) to \( U' \).

A set \( S \) is called invariant if for all open sets \( U \), the intersection \( S \cap U \) is invariant for \( \mathcal{P}(U) \). Again, \( X \) is partitioned into minimal invariant sets called orbits. A set which is the disjoint union of orbits is called saturated.

**Example 1.1.** In the smooth case, an action of \( \mathcal{P} \) on \( X \) amounts to a homomorphism of the Lie algebra sheaf associated to \( \mathcal{P} \) into the sheaf of germs of vector fields on \( X \). As a specific example, let \( X \) be an affine flat manifold, infinitesimally (and hence locally) acted on by the Lie algebra sheaf of parallel vector fields.

Let \( G \) denote the stalk of \( \mathcal{P} \) at \( x \). If \( f: Y \to X \) is a locally homeomorphic map, let \( f^*(\mathcal{P}) \) denote the pullback sheaf.

The following is a significant generalization of the concept of completeness introduced previously.
Definition 1.1. An action of a sheaf $\mathcal{F}$ of connected groups on $X$ is called complete if for all $x \in X$, there exists an open neighborhood $V(x)$ and a locally homomorphic map $\pi: V(x) \to V(x) \times (V(x) \times \mathbb{R})$ such that

(i) If $x \in \pi(x)$, then for any open neighborhood $W \subset V(x)$ of $x$, the structure homomorphism $\pi(x)(W) \to \pi(x)$ is an isomorphism.

(ii) The local action of $\pi(x)$ on $V(x)$ is complete.

Example 1.1. (continued). For affine flat manifolds, this agrees with the usual definition of completeness.

Note that the orbic of $\pi(x)(V)$ on $V(x)$ project to orbits of $\mathcal{F}$ on $V(x)$.

Suppose $\pi: V(x) \to V(x)$ is a normal covering. Then the group, $G$, of covering transformations of $\pi: V(x) \to V(x)$ has a natural (holonomy) action on $\pi(x)$. It follows that there is a sheaf $\xi$ on $V(x)$ such that the stack of $\pi(x)$ at $x \in V(x)$ is the module of the structure homomorphism $\pi(x)(V(x))$ and by Remark 1.1, for $x \in X$, $g \in \pi(x)(V(x))$, $x \in V(x)$, we have

$\gamma(g') = \gamma(g)\gamma(x).

Definition 1.2. A $\mathcal{F}$-structure, $\mathcal{F}$, on $X$ is a sheaf, $\mathcal{F}$, of connected topological groups on $X$ and a complete local action of $\mathcal{F}$ on $X$ such that the sets $V(x)$ can be chosen to satisfy the following conditions.

(i) $\pi: V(x) \to V(x)$ is a normal covering.

(ii) For all $x$, $V(x)$ is saturated.

(iii) For all $\xi$, if $x, y \in \xi$, then $V(x) = V(y)$.

It follows from (iii) that $\pi(x)$ is a locally constant sheaf, i.e., $\pi(x)$ is cohomologically isomorphic to the sheaf of locally constant maps of $\xi$ to the group $G$. Put otherwise, $\pi(x)$ is a flat bundle such that each fiber is a group and the holonomy acts by automorphisms of the fiber. However, it need not be the case that $\xi$ is locally constant on some neighborhood of $\xi$, since the structure homomorphisms need not be injective; see Example 1.4 and Remark 1.2.

Definition 1.3. If $\mathcal{F}$ is locally constant on $V(x)$ for all $x$, then $\mathcal{F}$ is called pure.

Suppose $\mathcal{F}$ is pure. Let $x \in X$, and fix $x \in V(x)$ with $\pi(x) = x$. Since $\pi(x)(V(x)) = G$, it follows that $V(x) \times \pi(x)(V(x)) = V(x)$, where $\pi(x)(V(x))$ is the holonomy covering of $V(x)$ with base point $\pi(x)$. As a consequence of (1.1), the action of $G$ descends to $V(x)$ and (1.1) continues to hold there.

For $x \in X$, let $\mathcal{F}(x)$ denote the holonomy covering of the locally constant sheaf $\mathcal{F}(x)$ with canonical basepoint. Let $\pi: \mathcal{F}(x) \to X$ be the projection. Given $\gamma \in X$ and a curve, $c$, from $x$ to $y$, the space $\mathcal{F}(x)(y)$, $x \in \mathcal{F}(x)$, is naturally identified with a component of $\pi^{-1}(V(x)) \times (V(x))$. Also, parallel translation along $c$ includes an isomorphism $G \to G$. Thus, the action of $G$ on $\mathcal{F}(x)(y)$, $x \in \mathcal{F}(x)$, is defined.
induces an action of $G_x$ on the corresponding component of $\pi^{-1}(V(x))$ and it is immediate from (1.1) and the definition of $(X^x, x)$ that these actions give rise to a global action of $G_x$ on $(T^x, x)$. Thus,

**Proposition 1.1.** If a $\mathcal{E}$-structure is pure, then the local action of $\rho(x, x)$ is complete.

**Example 1.2 (Flat bundle).** A basic example of a pure $\mathcal{E}$-structure is the following. Let $E$ be the total space of a locally constant sheaf, $\rho$, of connected groups, and $\pi: E \to X$ the projection. Then for $x \in X$ and $y \in \rho^{-1}(x)$, there is an obvious action of the stalk $G_y$ of $\rho^*(y)$ on $\rho^{-1}(U)$, provided $U$ is chosen so that $\rho(U)$ is trivial. In particular there is a pure $\mathcal{E}$-structure on $E$, where the sheaf which acts is $\rho_x^*(\mathcal{E})$.

**Definition 1.4.** A $\mathcal{E}$-structure is called an $F$-structure if for all $x$, the group $G_x$ is isomorphic to a torus, and the sets $V(x)$ (of Definition 1.2) can be chosen so that the coverings $\tilde{V}(x)$ are finite.

**Definition 1.5.** If one can always choose $\tilde{V}(x) = V(x)$, then the $F$-structure is called a $T$-structure.

**Example 1.3 (Example 0.3 reformulated).** Let $X^x$ be a compact flat riemannian manifold. By the Bieberbach theorem, for each $x \in X$ the holonomy covering $(X^x, x)$ has the natural structure of a torus, $T^x$. The torus $T^x$ acts on itself by the left translation. Hence, it acts canonically on any $(X^x, x)$ as well. The holonomy transformations act on $T^x$ by conjugation. The set $\bigcup_x T^x$ has the natural structure of a locally constant sheaf $\rho$ (with stalk $T^x$) and the action of $T^x$ on any fixed $(X^x, x)$ induces the local action of $\rho$ on $X$.

**Example 1.4 (Structure homomorphism not injective).** Let $X$ be the space formed as follows. Take $S^1 \times [0, 1]$ and attach $S^1 \times [0, 1]$ to $S^1 \times [0, 1]$ by a covering map of degree 2. The image of $S^1 \times [0, 1]$ in $X$ is a Möbius band $B_1$. The image $S^1 \times [0, 1]$ in $X$ is a half open cylinder, $C_1$. Moreover, $C_1 \cap B_1 = S$, where $S$ is the central circle in $\lambda$, and $C_1$ is the closure of $C_1$.

The orbise, $\tilde{V}$, in $X$ will be the images of circles, $S^1 \times A$. For each connected open set $U \subset X$, put

$$T(U) = \{ \mathcal{E} \in U \} \cap U \subset \mathfrak{g}.$$ 

Let $\rho$ be the sheaf associated to the presheaf which assigns to each $U$ the identity component of the isometry group of $\mathcal{T}(U)$. By definition, there is a complete action of $\rho$ on $X$, which defines a $T$-structure.

Note that for all $U = X$, $\rho(U)$ is isomorphic to a circle. However, if, for example, $T(U_1) = B_1 \cup C_1$ and $T(U_2) = C_1$, then the restriction map $\rho(U_1) \to \rho(U_2)$ is a 2-fold covering. As a consequence the total space of $\rho$ is not Hausdorff at points lying over $S$, and the local action of $\rho$ is not locally isomorphic to a pure structure in a neighborhood of $S$. 


Remark 1.2. Observe that since a non-trivial local isometry defined on a connected subset of a riemannian manifold is not equal to the identity on any nonempty open subset, examples like the one above do not occur for effective local actions of compact groups in the smooth category. From now on, we will restrict attention to \( g \)-structures of this type. For such structures, the restriction maps \( g(U_i) \to g(U_i) \) are injective.

If the action of \( g \) on \( X \) defines a \( g \)-structure, \( g \), and \( g' \subset g \) is a substructure, then \( g' \) defines a \( g \)-structure, \( g' \), called a substructure. We write \( g' \subset g \). Note that the stalks of \( g' \) are not required to be closed subgroups. The substructure \( g' \) whose stalks are the closures of these of \( g \) is written \( \overline{g'} \), the closure of \( g' \).

For a \( g \)-structure \( g \) as in Remark 1.2 above, for each \( x \in X \) the neighborhood \( V(x) \) of Definition 1.2 can be chosen such that there is a (unique) pure substructure, \( g_x \) of \( g(V(x)) \), with stalk \( G_{x,n} = G_x \).

The stalks of \( g' \) at \( x \) is defined as \( \dim g_x \). We say that \( g' \) has positive rank if the rank is positive for all \( x \in X \).

Let \( g' \subset g \) (with \( g \) as above). Let \( X = \bigcup U_x \) be a locally finite covering by connected open sets \( U_x \). For each \( x \), let \( g_{x,n} \subset g' \) be a pure substructure with stalk \( g_{x,n} \subset g' \), \( x \in U_x \).

Definition 1.6. The collection \( \{ (U_x, g_{x,n}) \} \) is called an atlas for \( g' \) if

(i) Each \( U_x \) is saturated for the orbits of \( g' \).

(ii) For each \( x \), there exist \( U_x \ni x \), such that \( G_{x,n} = G_x \).

Definition 1.7. A substructure \( g' \subset g \) is called a polarization if it has an atlas \( g' \), such that for all \( x \), the rank of \( g_{x,n} \) is the same positive number at all \( x \in U_x \) (although \( G_{x,n} \) might depend on \( x \)).

A polarization is called pure if \( g' \) is a pure substructure (in which case it suffices to take a single \( U_x = X \)). The notions of atlas and polarization play an important role in the collapsing constructions of §§2, 3, and 4.

If \( g' \) has positive rank, one way to find a substructure, \( g' \subset g \), of positive rank which possesses an atlas is the following. Take a locally finite open covering by sets \( U_x = V(x) \) and pure a substructure \( g_{x,n} \) on each \( U_x \) such that \( G_{x,n} = G_x \) and the rank of \( g_{x,n} \) is positive. Enlarge this covering by adding all nonempty intersections, \( U_{x,n} = U_x \cap \cdots \cap U_n \), and assign to \( U_{x,n} \) (the pure substructure whose stalk at \( x \in U_{x,n} \) is the smallest subgroup \( G_{x,n} \) containing \( V(U_{x,n}) \)). Then \( (U_x, g_{x,n}) \) is an atlas for the substructure \( g' \) determined by the condition \( G_{x,n} = G_{x,n'} \), where \( U_{x,n} \) is the intersection of all those \( U_x \) containing \( x \). The rank of \( g' \) is positive and it is easy to see that in fact we can choose \( g' \) such that rank \( g' = \text{rank} \ g' \).

The following lemma is convenient for the constructions of §5 and provides a simple picture of structures which possess an atlas.
Lemma 1.2. Let \((\{U_{\alpha}\}, \mathcal{A})\) be an atlas for \(X\) (an (possibly empty) open) manifold \(X\). Then there is an atlas \((\{U_{\alpha}, \mathcal{A}\})\) for \(X\) with the following properties:

1. the sets \(U_{\alpha}\) have compact closure.
2. \(U_{\alpha} \neq \emptyset \) for some ordering \(\{\alpha\}\), \(\mathcal{A}_\alpha = \mathcal{A}_{\gamma}\), \(\mathcal{A}_\alpha \subseteq \mathcal{A}_{\gamma}\).
3. For all \(\alpha\) and \(\beta \neq \alpha\) with \(\alpha \notin \mathcal{A}_{\gamma}\) and \(\mathcal{A}_\alpha = \mathcal{A}_{\gamma}\).

Proof. (1) Clearly, we can assume \(\{U_{\alpha}\}\) itself has this property.

(2) Let \(U_{\alpha}\), \(U_{\beta}\), satisfy \(x \in U_{\alpha} \cap U_{\beta}\), but neither \(\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\beta}\), nor \(\mathcal{A}_{\beta} \subseteq \mathcal{A}_{\alpha}\).

Take saturated open sets \(U_{\alpha}^x, \tilde{U}_{\alpha}\), \(U_{\beta}^x, \tilde{U}_{\beta}\), with \(U_{\alpha}^x \cap \tilde{U}_{\alpha} = \emptyset\) and \(U_{\beta}^x \cap \tilde{U}_{\beta} = \emptyset\). \(U_{\alpha} \setminus U_{\beta}^x \subseteq \tilde{U}_{\alpha} \subseteq \tilde{U}_{\beta} \setminus U_{\beta}^x\). Since, clearly, \(\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\beta}\), \(\mathcal{A}_{\gamma} \subseteq \mathcal{A}_{\alpha}\), \(\mathcal{A}_{\beta} \subseteq \mathcal{A}_{\alpha}\), \(\mathcal{A}_{\gamma} \subseteq \mathcal{A}_{\beta}\), it follows that \(U_{\alpha}^x, \tilde{U}_{\alpha}\), \(U_{\beta}^x, \tilde{U}_{\beta}\), together with the remaining \(U_{\alpha}, \tilde{U}_{\alpha}\) cover \(X\).

(3) Let \(\{U_{\alpha}\}, \mathcal{A}_{\alpha}\) be maximal subcollection of \(\{U_{\alpha}\}\) such that \(\{U_{\alpha}\}\) is connected and \(\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\gamma}\) whenever \(x \in U_{\alpha} \cap U_{\beta}\). Put \(\mathcal{A}_{\alpha} = \mathcal{A}_{\gamma}\). Let \(\{U_{\alpha}\}, \mathcal{A}_{\alpha}\) be the \(\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\gamma}\) whose intersection with \(\mathcal{A}_{\alpha}\) is nonempty. Let \(\{U_{\alpha}\}, \mathcal{A}_{\alpha}\) be the connected components of \(\{U_{\alpha}\}, \mathcal{A}_{\alpha}\). By proceeding in this way and repeating the process for all subcollections as above, we obtain the required covering.

An atlas satisfying the properties of Lemma 1.2 is called regular.

Remark 1.3. Let \(\mathcal{A} = (\{U_{\alpha}\}, \mathcal{A}_{\alpha})\) be an atlas for \(X\). Let \(\{U_{\alpha}\}\) be an open covering by saturated subsets. \(U_{\alpha} \subseteq U_{\gamma}\). Then by restricting \(\mathcal{A}_{\gamma}\) to \(U_{\alpha}\) we obtain a regular atlas, \(\mathcal{A}^\prime\), for a substructure \(\mathcal{A}^\prime \subseteq \mathcal{A}\). We write \(\mathcal{A}^\prime \subseteq \mathcal{A}\).

Remark 1.4. If the atlas \((\{U_{\alpha}\}, \mathcal{A}_{\alpha})\) of Lemma 1.2 is a polarization, then the regular atlas \((\{U_{\alpha}\}, \mathcal{A}_{\alpha})\) is a polarization as well.

Remark 1.5. By dropping (1) in Lemma 1.2, we can also drop (2) and strengthen (2) to read \(\mathcal{A}_{\alpha} \subseteq \mathcal{A}_{\gamma}\).

A Riemannian metric g is called invariant for \(\mathcal{A}\) if the local action of the sheaf g is isometric.

Lemma 1.3. Let \(\mathcal{A} = (\{U_{\alpha}\}, \mathcal{A}_{\alpha})\) be a regular atlas for \(X\) and let \(\mathcal{A}^\prime = (\{U_{\alpha}^\prime\}, \mathcal{A}_{\alpha}^\prime)\), where \(U_{\alpha}^\prime \subseteq U_{\alpha}\) Suppose \(\mathcal{A}\) has the property that all coverings \(V(x) \to V(x)\) (Definition 1.2) can be chosen finite. Then there is an invariant metric for \(\mathcal{A}^\prime\).

Proof. The relation of Lemma 1.2(2) indicates a natural partial ordering on the \(\{U_{\alpha}\}\). Start with some maximal \(U_{\alpha}\) and take a finite covering of \(U_{\alpha}\) by sets of the form \(V(x_1) \times \cdots \times V(x_n) \subset U_{\alpha}\). Take any metric \(g_{\alpha}\) on \(V(x_1)\), pull it back to \(V(x)\), and average over the action of \(G_{\alpha}\) and over the finite group of covering transformations. Extend the resulting metric to any smooth metric \(g_{\alpha}\) on \(V(x)\) by repeating the process for \(g_{\beta}\). By continuing in
this way we get a metric on $U'\cup U''$ with $U'$ maximal and in the same way we get the required invariant metric for $\mathcal{G}$.

By using (invariant) smoothings of distance functions for an invariant metric, we can construct invariant smooth functions, say $f_\alpha : U' \rightarrow [0, 1], \alpha \in A'$ with $\bigcup f_\alpha'^{-1}(1/2) = \mathcal{X}$. Then a standard application of Thom's Theorem gives the following result.

**Lemma 1.4.** For almost all $e_\alpha \in (1/2, 1)$ the sets $f_\alpha'^{-1}(e_\alpha)$ are smooth closed codimension 1 submanifolds which intersect transversally (and define an atlas $\mathcal{M} \subset \mathcal{A}'$).

We can now verify the result on the vanishing of the Euler characteristic mentioned above.

**Proposition 1.5.** Let $X$ be a compact manifold which carries an $F$-structure $\mathcal{M}$ of positive rank, then $\chi(X) = 0$.

**Proof.** Let $\mathcal{A}'$ be an atlas for a substructure of positive rank with $U_\alpha = \mathcal{X}(x_\alpha)$. Let $\mathcal{A}' \subset \mathcal{A}'$ be as in Lemma 1.4. Then for every intersection $U_\alpha \cap \cdots \cap U_\beta$ there is a finite covering, say $U_\alpha \subset \bigcup_{\gamma \in K_\alpha} \mathcal{X}(x_\gamma)$, on which a torus $G_{x_\gamma}$ acts with no common fixed points. By a well-known argument almost all elements of $G_{x_\gamma}$ are fixed point free. Thus, by the Lefschetz fixed point theorem, $\chi(U_{\alpha \gamma}) = 0$, and $\chi(X) = 0$ as well.

**Remark 1.6.** In Proposition 1.5 it is not essential that $X$ is a manifold.

Here are some further examples of $F$-structures.

**Example 1.5 (S^2 and R^4).** View $R^4$ as $C^1 = (z_1, z_2)$. There is an obvious $T$-action $(T^2 = (\theta_1, \theta_2))$ given by

$$\theta_1(z_1, z_2) = e^{i\theta_1}z_1, \quad \theta_2(z_1, z_2) = e^{i\theta_2}z_2,$$

with orbits of dimensions 0, 1, 2. Since there exist orbits of dimension 0, the corresponding $T$-structure admits no polarization.

There is also an induced $T$-structure on the unit sphere, $S^3$. All orbits of dimension 2, with the exception of the circles $S^1 \times \{(z_1, 0)\}$ and $S^1 \times \{(0, z_2)\}$. Any choice of 1-parameter subgroup, $S^1$, with $0 < \theta_1, \theta_2 < \gamma < \infty$, gives rise to a pure polarization, $\mathcal{M}$, for which all orbits are 1-dimensional.

We can define another $T$-structure on $S^3$ which is not pure by picking $\gamma$, with $1/\sqrt{2} < \gamma < 1$, and setting

$$U_{\gamma} = \{(z_1, z_2) \in S^3 | |z_1| < \gamma\}, \quad j = 1, 2.$$

For $x \in U_{\gamma} \cup U_{\gamma} \cup U_{\gamma}$ we let $G_x = S^1 \times \{(z_1, 0)\}$ and $\psi(x) = U_{\gamma}(V(x) = U_{\gamma})$. For $x \in U_{\gamma} \cap U_{\gamma} \cup U_{\gamma}$, $G_x = T^2$ and $V(x) = U_{\gamma}$. Note that for $\gamma < 1$, the orbits $S^1 \times \{(z_1, 0)\}$ and $S^1 \times \{(0, z_2)\}$ are never principle orbits of $\mathcal{M}$, i.e., their isotropy groups, while discrete, are not minimal.

Finally, observe that this example generalizes to higher dimensions.
Example 1.2 (continued, submanifolds). Let $A \in \text{Sl}(2, \mathbb{Z})$ be an automorphism of the torus $T^2$ with two real distinct eigenvalues. Thus, if

$$(1.5) \quad A = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

then $|a + d| > 2$. The mapping torus $M^1$, of $A$, is by definition the affine flat bundle $T^2 \to M^1 \to S^1$, with holonomy $\alpha$ (as is well known, $M^1$ is a submanifold).

As above, there is a pure $T$-structure on $M^1$ whose orbits are the fibers. This $T$-structure has a natural pure polarization (of rank 2). It also admits exactly two pure polarizations of rank 1, the orbits of which correspond to the eigen-directions of $A$ (and are not closed).

Example 1.6 (The flat bundle $\delta^2_1$). Let $R^2 \to \delta^2_1 \to S^1$ denote the trivial $R^2$ bundle over $S^1$, equipped with the connection whose holonomy is given by rotation through an angle $2\pi \theta$. A point in $\delta^2_1$ is denoted by $(t, w)$ where $t \in R/\mathbb{Z}$ (not $S^1$) and $w \in R^2$. Then parallel translation $v$ units along the base is given by

$$(1.6) \quad P(v)(t, w) = (t + v, R(v)w),$$

where $R(v)\theta$ denotes rotation through an angle $2\pi v \theta$.

Observe that $\delta^2_1$ carries the structure of a complete flat Riemannian manifold whose isometry group $I(\delta^2_1)$ is the torus, $S^1 \times S^1$, generated by

$$(1.7) \quad T(u)(t, w) = (t + u, w), \quad R(v)(t, w) = (t, R(v)w).$$

The full group $I(\delta^2_1)$ defines a pure $T$-structure which is of rank 2 everywhere except along the zero section of $\delta^2_1$. Any 1-parameter subgroup other than $R(v)$ defines a pure polarization of rank 1.

Example 1.7 (The space $\mathcal{A}^4 = \mathcal{U}_9 \mathcal{O}_7$). Consider the family $[0, 1] \times \delta^2_1$ consisting of pairs $(\theta, \delta^2_1)$. The spaces $\delta^2_1$ and $\delta^1_1$ are abstract isometrically. Their parallel translations are given by

$$(1.8) \quad P(\theta)(t, w) = \begin{cases} (t + \theta, w), & \theta = 0, \\ (t, R(\theta)v(w)), & \theta = 1. \end{cases}$$

The map

$$(1.9) \quad f(0, t, w) = (1, t, R(\theta)w)$$

provides an isometry between $\delta^2_1$ and $\delta^2_1$. It induces the isomorphism of isometry groups given by the matrix

$$(1.10) \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The space formed from $[0, 1] \times \delta^2_1$ by identifying $(0, E)$ with $(1, E)$ via $f$ will be denoted by $\mathcal{A}^4$. The flat $T^2$-bundle (locally constant sheaf) over $S^1$ with holonomy given by (1.10) is a nilmanifold. Its pullback to $\mathcal{A}^4$ defines a
pure $T$-structure on $\mathcal{M}^4$. As above, this structure has rank 2 everywhere except along the zero sections of the various $\mathcal{D}_i^2$, where it has rank 1. Moreover, unlike the matrix $A$ in (1.5), the matrix in (1.10) has only a single eigenvector. It corresponds to the circle $S^1$, the action of which fixes the zero sections of the $\mathcal{D}_i^2$. From this it is clear that the pure $T$-structure on $\mathcal{M}^4$ has positive rank but admits no polarization. In fact, it is the most basic example of an $F$-structure with this property. There are no such examples in dimension 3 and any example in dimension 4 contains an orbit, a neighborhood of which looks like (perhaps a finite covering of) this example.

**Example 1.8** (A nonpolarizable structure on $T^1 \times R^4$). The space $\mathcal{M}^4$ in Example 1.7 can be regarded as the total space of the complex line bundle with first Chern number 1, over $T^1$. If we take the Whitney sum of this bundle with the bundle of Chern number $-1$, we obtain the trivial bundle with total space $T^2 \times R^4$. Now, by a simple modification of the previous example, we find a pure $T$-structure on $T^2 \times R^4$ which is of rank 2 except at $T^2 \times 0$ where it is of rank 1. Moreover, this structure admits no polarization.

**Example 1.9** (Pure $T$-structure on $M^2$ with of $M^2$ = $2$). The previous example of a pure $T$-structure of positive rank which admits no polarizations can be sharpened. These exist closed manifolds which carry a pure $T$-structure of positive rank, but which have nonzero signature. Hence, as noted in §9, they admit no polarized $T$-structure whatsoever.

The following particularly nice family of such examples is due, essentially, to T. Januszkiewicz. To describe them. Let

$$(1.11) \quad T^{2n+1} = (e^{\theta_0}, \ldots, e^{\theta_{2n+1}}), \quad D = (e^{\phi}, \ldots, e^{\phi})$$

and let $S_i$ denote the image of

$$(1.12) \quad (1, \ldots, e^{\theta_{i+1}}, \ldots)$$

in $T^{2n+1}/D$. Then $T^{2n+1}/D$ acts on

$$(1.13) \quad CP(2l) = (z_{2n+1}/D), \quad \sum |z_i|^2 = 1,$$

with $2l + 1$ fixed points.

$$(1.14) \quad p_l = \left(0, \ldots, \hat{1}, 0, \ldots \right).$$

If we use the product structure

$$(1.15) \quad S^1 \times \cdots \times S^1 \times \cdots \times S^1$$

on $T^{2n+1}/D$ and identify the tangent space to $CP(2l)$ at $p_l$, with

$$(1.16) \quad \left(z_{2n+1}, \ldots, z_{2n+1} \right).$$
then $T^{2i+1}/D$ acts by the standard representation of $T^D$. Let $D_i$ denote the diagonal of $T^{2i+1}/D$ with respect to the product structure in (1.15). Then the action of $S^1$ on the tangent space at $p_k$ is given by

$$a_k(z_1, \ldots, z_{2i+1}) = D_k(z_1, \ldots, z_i, \ldots, z_{2i+1}).$$

Now choose normal coordinate systems at all points $p_i$ and from each of these delete a ball, $B^j_i$, about the origin. Take two copies $\Sigma^I_i$, $\Sigma^J_i$ of the resulting manifold with boundary and form a closed manifold, $M^I_i$, as follows. Let

$$f_j : [1, \ldots, 2i + 1] \to \{0, 1\}$$

be any function which takes the value 1 an odd number of times, $j = 1, \ldots, 2i + 1$. Put

$$F = (f_1, \ldots, f_{2i+1}).$$

To obtain $M^I_i$, glue corresponding boundary components of $\Sigma^I_i$ and $\Sigma^J_i$ by the identifications

$$(z_1, \ldots, z_i, \ldots, z_{2i+1}) \sim (\tau_j(z_1, \ldots, z_{i-1}, \ldots, z_i, \ldots, z_{2i+1}),$$

where $i_j, \ldots, i_{2i+1}$ are the integers at which $f_j$ takes the value 1. The torus action on $\Sigma^I_i \sqcup \Sigma^J_i$ gives rise to a pair $T$-structure on $M^I_i$. To describe the holonomy of the corresponding flat bundle, $E$, it suffices to consider loops $\tau_1, \ldots, \tau_j$, where $\tau_j$ passes from $\Sigma^I_i$ to $\Sigma^J_i$ through $\partial B^j_i$, and returns to $\Sigma^I_i$ through $\partial B^j_i$. Then using (1.12)–(1.17), it follows that the holonomy around $\tau_j$ is given by the matrix

$$\begin{pmatrix}
(-1)^{5i/3} a_j(1) & 0 \\
0 & (-1)^{5i/3} a_j(1)
\end{pmatrix}$$

where

$$\tau_k(\ell) = f_j(\ell) + f_{2i+1}(\ell),$$

and for $k \neq j$,

$$a_j(k) = \begin{cases} 0, & f_j(k) + f(2i + 1) \equiv 0 \mod 2, \\
0, & f_j(k) + f(2i + 1) \equiv 1 \mod 2,
\end{cases}$$
Finally, since the identifications on the boundary components are orientation reversing, if $X^*_2$, $Z^*_2$ are both given the orientation induced from $C(2)$, then $M^*_1$ also acquires an orientation. Moreover, the signature, $\sigma(M^*_2)$, is given by

$$\sigma(M^*_2) = 2\sigma(X^*_2) = 2\left[\sigma(C(2)) - (21 + 1)\sigma(\mathbb{R}^3)\right] = 2\sigma(C(2)) = 2.$$

2. Pure polarized collapses with bounded diameter

In this section, we discuss the collapse associated to a pure polarization, $\mathcal{P}$, of an $F$-structure, $\mathcal{S}$, on a manifold $Y$. Let

$$g = g + h$$

be a metric which is invariant for $\mathcal{S}$; see Lemma 1.2. Here $h$ vanishes on vectors tangent to the orbits of $\mathcal{P}$, and $g$ vanishes on vectors normal to these orbits. Put

$$g_\delta = h^\perp + h.$$

Theorem 2.1. As $\delta \to 0$, the family $(Y, g_\delta)$ collapses. Moreover,

$$\text{dist}_{g_\delta}(p, q) \leq \text{dist}_{g}(p, q), \quad \delta_1 \leq \delta_2,$$

and for each compact set $U$, there is a constant $\epsilon(U)$ such that

$$\sup_{p, q \in U} |K_p| \leq \epsilon(U).$$

Proof. Observe that (2.3) is obvious and the main point is, of course, (2.4). We claim that when the appropriate coordinates are introduced, (2.4) is obvious as well.

Let $p \in Y$. Take a basis of Killing fields, $\{X_i\}$, tangent to the orbits of $\mathcal{P}$ in a neighborhood of $p$.

Let $N^{n-1}$ be a local transversal to the orbits $\{\delta\}$ with $p \in N^{n-1}$. Choose local coordinates $(x_1, \ldots, x_{n-1})$ on $N^{n-1}$ with $p$ at the origin. Since $[X_i, X_j] = 0$, there is a unique coordinate system $(x_1, \ldots, x_n)$ on each orbit $\delta$, with $\delta \cap N^{n-1} = (0, \ldots, 0)$ and $X_j = \partial_i / \partial x_i$. By projecting onto $N^{n-1}$, i.e. $\delta \to \delta \cap N^{n-1}$, we obtain coordinates $(x_1, \ldots, x_n, \ldots, x_{n-1})$ in a neighborhood $B_1(O) \times B_1(O)$ of $p$ with, say, $\Sigma x_i^2 \leq r^2$, $\Sigma x_i^2 \leq \epsilon^2$.

In terms of these coordinates, the matrix $g((x, y, \delta))$, representing the metric $g_\delta$, can be calculated as follows. Note that translation in the direction of $x_i$ preserves (coordinate fields and) inner products, since $\partial_i / \partial x_i$ is a Killing field.

$\footnote{The notation $\epsilon(\cdot)$ will always mean a constant which depends only on the quantities within the parentheses.}$
Thus, if we put

\[ \frac{\partial}{\partial y_i} = X_i + V_i, \]

where \( X_i \) is tangent to orbits and \( V_i \) is normal to orbits,

\[ \langle X_i, V_j \rangle_{g_s} = 0, \]

and the following matrices are independent of \( y_1, \ldots, y_n \).

\[ A(y_1, \ldots, y_n) = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{pmatrix}, \]

\[ B(y_1, \ldots, y_n) = \begin{pmatrix} \frac{\partial}{\partial x_n} & \frac{\partial}{\partial y_n} \\ \frac{\partial}{\partial x_{n+1}} & \frac{\partial}{\partial y_{n+1}} \end{pmatrix}, \]

\[ C(y_1, \ldots, y_n) = \begin{pmatrix} X_1 & X_2 \\ \vdots & \vdots \\ X_n & X_{n+1} \end{pmatrix}, \]

\[ D(y_1, \ldots, y_n) = \begin{pmatrix} V_1 & V_2 \\ \vdots & \vdots \\ V_n & V_{n+1} \end{pmatrix}. \]

Here \( \langle , \rangle_{g_s} \) denotes the inner product for \( g_s \). It follows that

\[ (g(x, y, \delta)) = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{pmatrix} - \begin{pmatrix} \delta^2 A(y) & \delta^2 B(y) \\ \delta^2 C(y) + D(y) \end{pmatrix} \]

As \( \delta \to 0 \), the matrix in (2.11) becomes singular. But if we make the change of coordinates

\[ u_i = \delta x_i, \quad du_i = \delta dx_i, \quad \frac{\partial}{\partial u_i} = \frac{1}{\delta} \frac{\partial}{\partial x_i}, \]

then

\[ (g(u, y, \delta)) = \begin{pmatrix} \frac{\partial}{\partial u_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial u_2} & \frac{\partial}{\partial y_2} \end{pmatrix} - \begin{pmatrix} \delta B(y) \\ \delta^2 C(y) + D(y) \end{pmatrix} \]
The family \((g(u, y, \delta))\) can be regarded as being defined on all of \(R_+^1 \times B_r(0)\). As \(\delta \to 0\), it converges smoothly to the generalized warped product metric 

\[
(g(x, y, 0)) = \begin{pmatrix}
A(y) & 0 \\
0 & D(y)
\end{pmatrix}
\]

on \(R_+^1 \times B_r(0) \subset R_+^1 \times R_+^{n-1}\), where for each fixed \(y\), the induced metric on \(R_+^1\) is flat; compare Example 0.5. Since \((g(x, y, \delta))|_{B_r(0) \times B_r(0)} \subset R_+^1 \times R_+^{n-1}\) is isometric to \((q(u, y, \delta))|_{B_r(0) \times B_r(0)} \subset R_+^1 \times R_+^{n-1}\), it is clear that \(|K_\delta|\) is uniformly bounded independent of \(\delta\) on compact subsets. This gives (2.4).

To see that \((Y_\delta, g_\delta)\) collapses, consider the closure \(\overline{\mathcal{P}}\) of \(\mathcal{P}\). The orbits (\(\delta\)) of \(\overline{\mathcal{P}}\) are compact flat manifolds. Since \(g_\delta\) restricted to the normal space of any \(\delta\) is independent of \(\delta\), it follows easily that the distance between any two such orbits is bounded below independent of \(\delta\). If \(Y\) is not complete, the same holds for \(\text{inf}_{\partial B_r} (\overline{\mathcal{P}}^* \setminus Y_r)\). In particular, the closed tubular neighborhood \(T_r(\overline{\mathcal{P}})\) is compact, independent of \(\delta\), for \(r < d_{\overline{\mathcal{P}}}\). If \(B_r(q, \delta)\) denotes the ball of radius \(r\) about \(q\), with respect to \(g_\delta\), clearly

\[
B_r(q, \delta) \subset T_r(\overline{\mathcal{P}}),
\]

and by (2.2),

\[
\lim_{\delta \to 0} \text{Vol}_{g_\delta}(B_r(q, \delta)) = \lim_{\delta \to 0} \text{Vol}_{g_\delta}(T_r(\overline{\mathcal{P}})) = 0.
\]

Let \(V(r, s)\) denote the volume of the ball of radius \(s\) on the sphere of curvature \(c = c(T_r(\overline{\mathcal{P}}))\). The constant \(c\)? It follows that for any \(s < r\), we have

\[
0 < \text{Vol}_{g_\delta}(B_r(q, \delta)) < V(r, s),
\]

if \(\delta\) is so small that

\[
\text{Vol}_{g_\delta}(B_r(q, \delta)) < \text{Vol}(B_r(q, \delta)) < V(r, s).
\]

Thus, \((Y_\delta, g_\delta)\) collapses.

A metric space \(X\) is said to be the Hausdorff limit (as \(\delta \to 0\)) of the family of metric spaces \(X_\delta\), if for all \(e_1, e_2\) there exists \(\delta(e_1, e_2)\) such that for \(\delta < \delta(e_1, e_2)\) there are \(\epsilon_i\) dense sets \((p_\delta(e_i, \epsilon, \delta)), p_\delta(e_i, \epsilon, \delta))\) in \(X_\delta\) and \((p_\delta(e_i, \epsilon, \delta))\) in \(X\) with

\[
(1 + \epsilon_2)^{-1} p_\delta(e_2, \epsilon_2, \delta), p_\delta(e_1, \epsilon_1, \delta) \leq p_\delta(e_2, \epsilon_2, \delta), p_\delta(e_1, \epsilon_1, \delta)
\]

\[
\leq (1 + \epsilon_2) p_\delta(e_1, \epsilon_1, \delta), p_\delta(e_2, \epsilon_2, \delta).
\]
see [9] for further discussion. Clearly, the Hausdorff limit of the family \( g(x, y, \delta) \) on \( B(0) \times B(0) \subset X \times R^2 \) is \( B(0) \), equipped with the metric corresponding to \((D, r))\). By definition, this is the metric on \( N^{\infty} \) for which the length of a vector is the length with respect to \( g \) of its projection orthogonal to \( \delta \). Up to isometry, it is independent of the choice of transversal, and is the unique metric on the local quotient space defined by pieces of the orbits of \( \mathcal{P} \), for which the projection is a renormalized submersion.

If the orbits \( \{ \delta \} \) are not closed, the global quotient spaces \( X/\mathcal{P} \) is not Hausdorff. But we can still look at the quotient space, \( X/\mathcal{P} \), for the orbits of \( \mathcal{P} \). Since \( \delta \) is dense in \( \mathcal{P} \) it follows from (2.23) that

\[
\lim_{\delta \to \mathcal{P}} \text{diam}_\mathcal{P}(\mathcal{E}_\delta) = 0.
\]

Thus, the Hausdorff limit of a compact subset \( U \) of \( Y \), which is saturated for \( \mathcal{P} \), is \( U/\mathcal{P} \), with the obvious quotient metric. Of course, this is not a smooth manifold near exceptional orbits.

Example 1.4 (continued). The polarization \( \mathcal{P} \), defined by the subgroup \( \mathcal{G} \), is closed if and only if \( \gamma \) is rational. If \( 1 + \gamma = p/q \) is rational, the Hausdorff limit \( S^2/\mathcal{P} \), is the surface of revolution, obtained by revolving the curve

\[
y = \frac{1}{2} \left( \frac{p^2}{q^2} \cos^2 x + \frac{q^2}{p^2} \sin^2 x \right)^{1/2},
\]

\[0 \leq x \leq \pi/2,\] about the \( x \)-axis. Thus it is a topological \( S^2 \) with two non-smooth points.

For \( \gamma \) irrational, \( S^2/\mathcal{P} \), is the interval \([0, \pi/2]\).

Example 2.1 (Tori). The pure polarizations \( \mathcal{P}(E^4) \) of the canonical \( T^4 \) structure on the standard torus, \( T^4 \), are parametrized by subspaces, \( E^4 \), of \( R^4 \) which pass through the origin. When \( T^4 \) is collapsed along \( E^4 \), of course

\[
\text{Vol}(T^4) = \gamma \text{Vol}(T^4).
\]

But if one looks at \((T^4, g_2)\) up to homothety (i.e. isometry and scaling) it is a classical fact that the family \((T^4, g_2)\) corresponds to the image in the moduli space \( \text{SO}(n, R) \setminus \text{SL}(n, R) / \text{SL}(n, Z) \) of a geodesic which goes to infinity in \( \text{SO}(n, R) \setminus \text{SL}(n, R) \).

For example, if \( n = 2 \) and \((\mathcal{P}, E)\) corresponds to a line of slope \( \gamma = p/q \), then \((T^4, g_2)\) goes to infinity in \( H^2 / \text{SL}(2, Z) \). However, if \( \gamma \) is irrational, the \((T^4, g_2)\) makes an infinite sequence of excursions which carry it successively further towards infinity, followed by returns to a fixed compact set. The precise behavior is determined by the continued fraction expansion of \( \gamma \). Thus, for \( \gamma \) rational, up to scaling, \((T^4, g_2)\) becomes arbitrarily thin as \( \delta \to 0 \). For \( \gamma \) irrational, there exist \( \delta \) for which \((T^4, g_2)\) is arbitrarily thin. But there are also arbitrarily small \( \delta \) for which it is fat.
Example 2.2 (Almost flat manifolds; see [6]). The simplest (but quite typical) almost flat manifolds arise very naturally in our context. Let $S^1 \to N^3 \to \mathbb{R}^2$ be a circle bundle with connection over $\mathbb{R}^2$. If this bundle is topologically nontrivial, then $N^3$ does not have the fundamental group of $\mathbb{R}^2$ flat manifold. In fact, $N^3$ is a quotient of the Heisenberg group, and as such, is a nilmanifold (the fiber $S^1$ corresponds to the center). If $\mathbb{T}^2$ is given a metric, the connection induces a metric on $N^3$, for which rotation through the angle $\theta$ in the fibers is an isometry. Choose the metric on $\mathbb{T}^2$ to be flat and note that all fibers have the same length. Then by (2.13) and (2.14), for the collapse $(N^3, g_k)$ along the fibers, $g_k$ converges locally to a flat metric. In fact, for the sequence $(N^3, \delta^k g_k, \ldots)$ (where $\epsilon > 0$), both the curvature and the diameter approach zero, so that the limit of this collapse is a point.

Remark 2.1. It is easy to see that the calculation of Theorem 2.1 can be generalized to the case in which the abelian Lie algebra of Killing fields is replaced by a nilpotent Lie algebra. The latter is collapsed as in Example 2.2, rather than by scaling.

3. Polarized volume collapses

For collapses associated to a polarization for which all orbits are set of the same dimension, we will need a slight generalization of the calculation of Theorem 2.1 Suppose that in (2.2) we replace $\delta$ by a function $\rho$ which is constant on orbits;

$$\rho = \rho(y_1, \ldots, y_{n-k}) > 0$$

We fix attention on the origin $(0, 0)$ in $(x, y)$-space and make the change of coordinates,

$$u_i = \rho(0)x_i.$$

Now we obtain

$$\begin{pmatrix} \rho^2(x) A(y) & \rho^2(x) B(y) \\ \rho^2(0) A(y) & \rho^2(0) B(y) \end{pmatrix} = \begin{pmatrix} \rho^2(y) C(y) + D(y) \\ \rho^2(0) C(y) + D(y) \end{pmatrix}.$$

It follows that for, say, $|\rho| \leq 1$,

$$|K^x| \leq c(A, B, C, D, |\rho'/\rho|, |\rho''/\rho|).$$
where \( \rho' \) and \( \rho'' \) denote typical first and second partials of \( \rho \) with respect to \( \gamma \).

**Theorem 3.1.** Let \( Y^n \) be compact, let \( \mathcal{P} \) be a polarisation of an \( F \)-structure, \( \mathcal{F}_n \) on \( Y^n \) and let \( g \) be an invariant metric. Then there exists a family of metrics, \( g_b \), which are invariant for the \( F \)-structure defined by \( \mathcal{P} \), such that for \( b \leq 1/2 \),

1. \((Y^n, g_b)\) is \( \mathcal{P} \)-collapsed,
2. \(\text{diam}(Y^n, g_b) \leq c \log b\),
3. \(\text{Vol}(Y^n, g_b) \leq e^{\log^2 b} \),
4. \(|K_g| \leq c.

**Proof.** Let \( \{ U_a \} \) be as in Definition 1.7 and let \( f_a : U_a \to [1/2, 1] \) be smooth functions such that \( f_a \equiv 1 \) near \( U_a \) and

\[
\bigcup_a f_a^{-1}(1/2) = Y^n.
\]

As in Lemma 1.4, we can assume that \( f_a \) is constant on every orbit \( \partial \) of \( \mathcal{P} \). Set

\[
\rho_a = \frac{\log f_a(\gamma)}{\log(1/2)}
\]

The metric \( g_a \) will depend on a choice of ordering, \( U_1, U_2, \ldots \) of the \( \{ U_a \} \) (although its essential properties are independent of this choice). We start with the metric \( \log^2 \cdot g \) and put

\[
\log^2 \cdot g = g_b + h_b
\]

on \( U_b \), where the decomposition is as in (2.2) and \( h_b \) vanishes on the orbits of \( \mathcal{F}_b \). We then define \( g_b \) by

\[
g_b = \left\{ \begin{array}{ll}
\rho_b \mathcal{F}_b + h_b & \text{on } U_b, \\
\mathcal{F}_b & \text{on } Y \setminus U_b,
\end{array} \right.
\]

where \( \rho_b \) is as in (3.6). Proceeding by induction, we put

\[
s_j = g_{j-1} + h_{j-1},
\]

where \( h_{j-1} \) vanishes on the orbits of \( \mathcal{F}_{j-1} \) and define \( g_{j+1} \) by

\[
g_{j+1} = \left\{ \begin{array}{ll}
g_j \mathcal{F}_j + h_{j+1} & \text{on } U_{j+1}, \\
g_j & \text{on } Y \setminus U_{j+1}.
\end{array} \right.
\]

We claim that

\[
g_b = g_a
\]

has the required properties. Note that (2) and (3) are obvious, and that (1) follows as in the proof of Theorem 2.1.

To see (4), let \( \rho \in Y^n \) and \( G_{j,k} \) be the stalk of \( \mathcal{P} \). Let \( (x_1, \ldots, x_k, y_{j+1}, \ldots, y_n) (j > k = \text{rank } \mathcal{P}) \) be coordinates as in (2.11) above, for the metric \( g \) such that \( \rho = (0, 0) \). Thus, the \( x \)-coordinates are constant along some transversal to
the orbit $e$ through $p$ ($i = \text{dir}(e)$) and the $y$-coordinates are constant along the orbits of $\delta(\mathcal{U}) = G$. We will keep track of the effect at $p$ of the changes of metric corresponding to $j = 1, \ldots, N$ in coordinate systems derived from $(x_i, j)$.

Observe that for $j = 1, \cdots, N$ the functions $y_j$ depend only on $y$ (and $\delta$) since they are constant on orbits. Moreover, in the coordinate system

$$x_j = x_j \cdot \log \delta, \quad y_j = y_j \cdot \log \delta,$$

the matrix representing the metric $\log \delta \cdot g$ has bounded partial derivatives (of all orders) and the functions $|\partial y_j / \partial x_i|, |\partial y_j / \partial y_j|$ ($j = 1, \cdots, N$) are bounded independent of $\delta$ (as is immediate from (3.7)).

Finally, we need only consider the effect of the changes of metric corresponding to those $j$ for which $p \in U_j$. For such $j$, the orbits of $\delta_j$ are contained in those of $\delta$ on $U_j \cap U_j$.

Let $\beta$ be the first value of $j$ for which $p \in U_j$. By making a linear change of coordinates we can suppose that the orbits of $\delta_j$ are given by $x_i = \text{const} \cdots, y_j = \text{const} \cdots, y_{i+1} = \text{const}, \cdots, y_N = \text{const}$ near $p$. We then introduce new coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_{\beta-1}, y_{\beta}, \cdots, y_N)$ as in (3.3). Since the $y_j$ depend only on $y_1, \cdots, y_{\beta-1}$, they have the same expressions as before. Thus, (4) follows by proceeding by induction.

Remark 3.1. The initial step in Theorems 3.1 in which distances are expanded in all directions by a factor $|\log \delta|$ is not optional, i.e. $\text{Vol}(Y, \delta)$ does not always approach zero as rapidly as possible as $\delta \to 0$. This loss of sharpness is not very serious in the present context since at best one could replace $\text{Vol}(Y, \delta) \sim \delta^3|\log \delta|^6$ by $\text{Vol}(Y, \delta) \sim \delta^3$; compare Example 0.2 (continued). However, in Example 4.2 and in §5 we proceed more carefully.

4. Nonpolarized collapses

(a) Introduction. To be able to collapse when no polarization exists, we must:

(i) Describe a structure (referred to, somewhat informally, as a slice polarization) which replaces that of a polarization and which exists in general.

(ii) Check that there is a collapsing procedure based on this structure.

We begin by illustrating (i) and (ii) in an example which was considered in §1. Then we do the calculation behind (ii). Next we explain how the structure of (i) is constructed in general. Finally, we describe the collapse.

Example 1.7 (continued). Let $Z \subset \mathcal{M}$ denote the union of the zero sections of the flat bundle $\mathcal{D}_z$, i.e. the 2-torus on which the dimensions of the orbits drops from 2 to 1. The $\mathcal{T}$-structure has nilpotent holonomy given by the
matrix (1,10) in the direction of the $\theta$-circle in $Z^2$ (the circle transverse to the orbit). As a consequence, the sub-bundle defined by the isometry subgroups $H_{\theta}$, $p \in Z^2$, has no complementary flat sub-bundle. Equivalently there is no 1-dimensional polarization near $Z^2$.

However, for each fixed $\theta$, the restriction of our structure to a 3-dimensional slice, $d^3 \subset \mathcal{M}$, has no holonomy, and hence admits a 1-dimensional polarization. For example, on each slice we can choose the 1-parameter subgroup of the isometry group of $d^3$ induced by parallel translation of $d^3$ (see (1.6)). This family of "slice-polarizations" varies continuously with $\theta$ (the corresponding family of infinitesimal generators gives rise to a vector field $V$ on $\mathcal{M}^6$, which is tangent to $d^3$ for each fixed $\theta$, and such that $V(d^3)$ is a Killing field).

If the metric is collapsed in the direction of $V$, the curvature does not remain bounded, because $V$ deviates from being a Killing field through its dependence on $\theta$. To obtain a collapse with bounded curvatures, we must simultaneously expand the metric in the $\theta$-direction (at an equal rate). This has the effect of making the above deviation negligible.

(b) Main computation. The essential quantitative features of nonpolarized collapse are captured by the following 3-dimensional situation. To simplify notation, we will only write the computation explicitly in this case.

Let $R^4 = R^2 \times \mathcal{R}$, where the third coordinate is denoted by $z$. Let $g$ be a Riemannian metric on $R^3$ and let $V$ be a nonvanishing vector field such that

1. $V$ is tangent to the slice, $z = \text{const}$.
2. The restriction of $V$ to any slice is a Killing field.
3. There is an abelian Lie algebra, $\mathcal{F}$, of Killing fields such that if $X \in \mathcal{F}$, then $X$ is tangent to every slice, $z = z_0$. Moreover, for each $z_0$, there exists $X_{z_0} \in \mathcal{F}$, with $X_{z_0} = V|_{z_0}$. It suffices to consider, say, $z_0 = 0$. Choose a local coordinate system $(x, y, z)$ with $\partial / \partial x = X_0$.

To begin with we observe that

\[ \left[ \frac{\partial}{\partial x}, V \right] = 0, \]

where $[\ , \ ]$ is the Lie bracket. In fact, for each fixed $z_0$,

\[ \left[ \frac{\partial}{\partial x}, X_{z_0} \right] = 0, \]

since $\partial / \partial x = X_0 \in \mathcal{F}$. But $\partial / \partial x$, $V$, $X_{z_0}$ are all tangent to the slice $z = z_0$. So for $z = z_0$, the brackets in (4.1) and (4.2) can be computed in this slice, and there $V = X_0$.\[ \]
Let
\[ \frac{\partial}{\partial z} = U + N \]
be the decomposition of \( \partial/\partial z \) into components tangent and normal to \( z = z_0 \) respectively. Since \( \partial/\partial z \) is a Killing field tangent to \( z = z_0 \), we have
\[ 0 = \left[ \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right], \quad 0 = \left[ \frac{\partial}{\partial z}, U \right], \quad 0 = \left[ \frac{\partial}{\partial z}, N \right]. \]

Now let \((b_1^y, b_2^y)\) denote the components of \( \partial/\partial x \) with respect to an orthonormal basis adapted to the decomposition \( \{V\}, \{V\} \cap (\partial/\partial x, \partial/\partial y) \), \( \{\partial/\partial x, \partial/\partial y\} \). Let \((b_1^y, b_2^y, b_3^y, b_4^y)\) be the corresponding component functions for \( \partial/\partial y, \partial/\partial z \). Notice, that all seven of these functions do not depend on \( x \). This follows from (4.1), (4.4) and the fact that \( \partial/\partial x \) is a Killing field. For example,
\[ \frac{\partial}{\partial x} (b_1^y) = \left[ \frac{\partial}{\partial x}, (U, V) \right] = \left( \frac{\partial}{\partial x}, U \right) (V, V) = \frac{1}{(V, V)} + \cdots = 0. \]

Finally, observe that, by (3),
\[ b_4^y(y, 0) = 0, \]
since \( \partial/\partial x|_{z=0} = V|_{z=0} \).
Let \( g_{i,j}(y, z) \) denote the metric obtained by the following operations (as above the subscripts \( i, j = 1, 2, 3 \) correspond to the variables \( x, y, z \) respectively).

(*) Multiply the metric \( g_{i,j}(y, z) \) by the factor \( \delta^2 \) in the direction \( \partial/\partial x, \partial/\partial y \), while leaving it unchanged on \( \{\partial/\partial x, \partial/\partial y\} \).

(**) Multiply the metric obtained in (*) by a factor \( \delta^2 \) in the direction of \( \{V\} \), while leaving it unchanged on \( \{V\} \).
We have
\begin{align*}
(4.7) & \quad g_{11}(y, z, \delta) = \delta^2 (b_1^y)^2 + (b_2^y)^2, \\
(4.8) & \quad g_{12}(y, z, \delta) = \delta^2 b_1^y b_2^y + (b_3^y)^2, \\
(4.9) & \quad g_{13}(y, z, \delta) = \delta^2 (b_3^y)^2 + (b_4^y)^2, \\
(4.10) & \quad g_{14}(y, z, \delta) = \delta^2 b_1^y b_3^y + b_2^y b_4^y, \\
(4.11) & \quad g_{22}(y, z, \delta) = \delta^2 (b_2^y)^2 + (b_3^y)^2, \\
(4.12) & \quad g_{23}(y, z, \delta) = \delta^2 (b_3^y)^2 + (b_4^y)^2 + \frac{1}{\delta^2} (b_1^y)^2. 
\end{align*}
Make the change of variables \( u = \delta x, w = z/\delta \). In terms of these new coordinates

\[
\begin{align*}
(4.13) & \quad g_{11}(y, \delta w, \delta) = (\delta y)^2 + \left( \frac{1}{\delta} \delta y \right)^2, \\
(4.14) & \quad g_{12}(y, \delta w, \delta) = \delta \delta y \partial_y \delta y + \delta y \partial_y \delta y, \\
(4.15) & \quad g_{22}(y, \delta w, \delta) = \delta^2 \left( \delta y \right)^2 + \left( \delta y \right)^2, \\
(4.16) & \quad g_{11}(y, \delta w, \delta) = \delta^2 \delta y \partial_y \delta y + \delta y \partial_y \delta y, \\
(4.17) & \quad g_{22}(y, \delta w, \delta) = \delta^2 \delta y \partial_y \delta y + \delta y \partial_y \delta y, \\
(4.18) & \quad g_{33}(y, \delta w, \delta) = \delta^2 \left( \delta y \right)^2 + \delta^2 \left( \delta y \right)^2 + \left( \delta y \right)^2.
\end{align*}
\]

In view of (4.6),

\[
(4.19) \quad \lim_{\delta \to 0} \frac{1}{\delta} \delta f(y, \delta w) = \frac{3}{\delta^2} \delta^2 f(y, 0).w.
\]

Moreover

\[
(4.20) \quad \lim_{\delta \to 0} \left( g_{ij}(y, 0, \delta) \right) = \begin{pmatrix}
(\delta y)^2 & 0 & 0 \\
0 & (\delta y)^2 & 0 \\
0 & 0 & (\delta y)^2
\end{pmatrix}
\]

which is positive definite. Thus, it is clear that the curvature stays bounded as \( \delta \to 0 \).

The calculation just given can be generalized. Since the details are straightforward, we will merely state the results.

(A) First of all, instead of the coordinates \( x, y \), we can as well have several coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_m \), where \( x_1, \ldots, x_n \) correspond to \( V_1, \ldots, V_n \). Moreover, we can collapse only, say, \( x_1, \ldots, x_n \) and make the changes of coordinates \( u_1 = \delta x_1 \ldots u_n = \delta x_n, u_m = x_m/\delta \) \( \ldots u_m = x_m/\delta \). Finally, we can artificially treat a subset of \( x_1 \ldots x_m \) as \( \epsilon \)-coordinates, even though this is not required in order to keep the curvature bounded.

(B) As in (3.1)–(3.4), \( \delta \) can be replaced by a function \( \rho(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_m, \delta) \). The curvature of the collapsed metric depends on \( \rho/\rho^2 \). Compare (3.4).

(c) Construction of slice polarizations. We now explain how the "slice polarizations" which are described in the continuation of Example 1.7 (at the beginning of this section) are obtained in general, starting with the case of a pure structure, \( F'. \) For this we must consider the orbit stratification associated
to $\mathcal{F}$ and construct an invariant metric on $Y_p$ and inner products in the stalks, $G_u$, which are suitably compatible with this stratification.

Let $\mathcal{F}$ be a pure $p$-structure of positive rank on $Y_p$ and let $g$ be an invariant metric. There is a natural stratification of $Y_p$ into maximal strata, $\Sigma_k$, such that rank $\mathcal{F} = i$ for $p \in \Sigma_i$. Since the groups $G_u$ are abelian, it follows that the identity components, $H^0_u$, of isotropy groups, $H_u$, are invariant under parallel translation along any curve in $\Sigma_i$. (Recall that the structure sheaf of $\mathcal{F}$ can be regarded as a flat bundle.) Each $\Sigma_i$ is totally geodesic for the metric $g$, since locally it can be viewed as the set of common zeros of a collection of Killing fields.

Let $q \in \Sigma_i$ and let $(\Sigma_k)_q$ denote the tangent plane to $\Sigma_k$ at $q$. Let $p \in \Sigma_k$ and consider the collection of subspaces of $Y_q$ of the form $\lim_{k \rightarrow \infty} (\Sigma_k)_q = T_q$ ($k > i$). Since all groups $G_u$ are abelian, it follows that the $G_u$ are coordinate hyperplanes relative to some fixed orthogonal basis of $Y_q$. Moreover, $(Q, \Sigma_k)$ is invariant under parallel translation in the normal bundle $\nu(\Sigma_k)$.

Let $\Sigma_k$ denote the set of points of $\Sigma_k$ at distance $\geq \varepsilon$ from $\Sigma_k$. Let $\exp$ be the exponential map of the normal bundle $\nu(\Sigma_k)$. For $\varepsilon$ sufficiently small, $\exp$ restricted to the subset $\Sigma_{\varepsilon} = \{ v \in \nu(\Sigma_k) \mid ||v|| < \varepsilon \}$ is a diffeomorphism onto a set $\Sigma_{\varepsilon}$. Let $\pi : \Sigma_{\varepsilon} \rightarrow \Sigma_k$ denote the corresponding projection map.

Lemma 4.1. The invariant metric $g$ and numbers $e_i, r_j$ can be chosen such that

\begin{enumerate}
\item $\Sigma_{\varepsilon} = Y_{\varepsilon}$.
\item If $i_j < i_k$, then $\pi_j = \pi_k \circ \pi_{i_k}$ on $\Sigma_{\varepsilon_{i_k}} \cap \Sigma_{\varepsilon_{i_j}}$.
\end{enumerate}

Proof. Start with any invariant metric $g_0$. Choose $\varepsilon_1 = 0$ and $r_j$ so small that $\exp|_{\Sigma_{\varepsilon_{i_j}}}$ is a diffeomorphism onto $\Sigma_{\varepsilon_{i_j}}$. There is a natural metric on $\Sigma_{\varepsilon_{i_j}}$ which is flat on the fibers, for which the subspace orthogonal to the fibers is given by the connection on $\nu(\Sigma_k)$ and for which projection onto the zero section is a riemannian submersion. Push this metric down to a metric $g_j$ on $\Sigma_{\varepsilon_{i_j}} \cap \Sigma_{\varepsilon_{i_k}}$ via exp. Note that $g_1$ is compatible with $\mathcal{F}|_{\Sigma_{\varepsilon_{i_1}}}$ and hence that $\Sigma_i \cap \Sigma_{\varepsilon_{i_j}}$ is totally geodesic for $g_j$.

It follows easily from the construction that (2) is satisfied on $\Sigma_{\varepsilon_{i_j}}$. Moreover (using (2)) it follows that near $\Sigma_i / \Sigma_{\varepsilon_{i_j}}$, the pullback of $g_j$ via the exponential map of $(\Sigma_i, \Sigma_{\varepsilon_{i_j}})$ actually coincides with the natural metric of $\mathcal{F}|_{\Sigma_{\varepsilon_{i_j}}}$.

Now we can proceed by induction. Extend $g_i$ to an invariant metric for $\mathcal{F}$ on all of $Y_p$, choose $\varepsilon_i \ll \varepsilon_j$, and replace the metric $g_j|_{\Sigma_{\varepsilon_{i_j}}}$ with the push down of the natural metric on $\Sigma_{\varepsilon_{i_j}}$. Let $g_j$ be the metric on $\Sigma_{\varepsilon_{i_j}} \cup \Sigma_{\varepsilon_{i_j}}$ so obtained. The $\pi_i$ that was noted above, $g_i$ coincides with $g_j$ on $\Sigma_{\varepsilon_{i_j}}$. By proceeding in this way, we obtain the required metric.

Put $U_i = \Sigma_{\varepsilon_{i_j}}$. Let $q \in U_i$ and let $\gamma$ be the unique minimal geodesic from $q$ to $\pi_i(q)$. Parallel translation along $\gamma$ induces an isomorphism $G_u \rightarrow G_{\varepsilon_{i_j}}$, which we will also denote by $\gamma$.
Note that at each point \( p \), the metric \( g \) of Lemma 4.1 induces a natural inner product on the Lie algebra, \( g_p \), of \( G_p \). For this, we identify a Killing field \( X \) with \((X(p),\nabla X(p))\) (we assume \( G_p \) acts effectively; see Remark 1.2). The resisting inner product is invariant under the local action of \( G_p \) but not under the maps \( \sigma, G_p \rightarrow G_{\sigma(p)} \).

**Lemma 4.2.** There exists an inner product \( \langle \cdot, \cdot \rangle_q \) on \( g_q \) which is invariant under the local action of \( G_q \) and under the projection \( \pi_q \) (for \( q \in U_i \)).

**Proof.** On \( \Sigma_i \), define \( \langle \cdot, \cdot \rangle_q \) to be the inner product above. Extend \( \langle \cdot, \cdot \rangle_q \) to \( U_i \) by making it invariant under \( \pi_q \). In view of 2) of Lemma 4.1, \( \langle \cdot, \cdot \rangle_q \) is invariant under the local action of \( G_q \) and under \( \pi_q \) on \( U_i \cup U_j \). Clearly, we can extend \( \langle \cdot, \cdot \rangle_q \) to \( U_i \cap \Sigma_{\alpha} \) to an inner product on \( g_q \), for all \( q \in \Sigma_{\alpha} \), which is invariant under the local action \( G_q \). Then extend to \( U_j \) by composing with \( \pi_q \). This is consistent with \( \langle \cdot, \cdot \rangle_q \) as defined previously on \( U_i \cap U_j \). By proceeding in this way, we construct \( \langle \cdot, \cdot \rangle_q \) on all of \( Y \) with the desired properties.

For \( p \in S_{\alpha,\beta} \), we let \( K_{\alpha} \) denote the connected (but not necessarily closed) subgroup whose Lie algebra is the orthogonal complement of that of the isotropy group, \( H_{\alpha} \). For \( q \in U_i \), we put

\[
K_{\alpha}^q = \pi_q^{-1}(K_{\alpha(q)}).
\]

It follows from Lemmas 4.1 and 4.2 that the assignment \( q \rightarrow K_{\alpha}^q \) is invariant under the local action of \( G_q \). Moreover, if \( q \in U_i \cap U_j \), \( i \neq j \), then

\[
K_{\alpha}^q \subset K_{\alpha}^r \quad \text{for} \quad q \in U_i \cap U_j \cap U_r.
\]

**Finally, if** \( q_i, q_j \in U_i \cap U_j \), \( i \neq j \), and \( \pi(q_i) = \pi(q_j) \), then \( K_{\alpha}^{q_i} = K_{\alpha}^{q_j} \).

**d. Collapse.** We can now collapse \( Y_{\alpha} \) by a straightforward variant of the procedure of \( \S 3 \). Choose functions \( f, \rho \) on \( U_i \) as in (3.5) and (3.6). Fix \( q \) and let \( U_i, \ldots, U_j, U_k, \ldots, U_l \) denote the \( U_i \) with \( q \in U_i \). Let \( Z_0 \subset \cdots \subset Z_l \) denote the subspaces of \( Y_{\alpha} \) tangent to the orbits of \( K_{\alpha}^1 \cdots K_{\alpha}^l \) and \( W_0 \subset \cdots \subset W_l \) denote the tangent spaces of \( E_{\alpha}^1 \cdots E_{\alpha}^l \). Then

\[
Z_0 \subset \cdots \subset Z_l \subset W_0 \subset \cdots \subset W_0.
\]

Let \( g \) be as in Lemma 4.1 and put

\[
\log_0^\alpha g = g_i + h_i + k_i,
\]

where the decomposition (4.24) corresponds to \( Z_0 \cup Z_1 \cup \cdots \cup Z_l \). Set

\[
g_1 = \left\{ \begin{array}{ll} p_i^1 g_i + h_i + p_i^1 k_i, & q_i, \\ \log_0^\alpha g_i, & \text{otherwise} \end{array} \right.
\]

(4.25)
Define \( g_{j+1} \) by induction:

\[
(4.26) \quad g_{j+1} = \left[ \phi_j^2 g_{j+1}^1 + \phi_j^2 \gamma_j \phi_j, \quad U_{j+1}, \quad Y \setminus U_{j+1}, \right.
\]

where the decomposition corresponds to \( \mathbb{Z}_{q+1}, Z_{q+1} \cap W_{q+1}, W_{j+1}, \)

We claim that \( g_j \) (n = \( \dim Y \)) collapses with bounded curvature as \( \delta \to 0 \)

where \( \phi_j \) depends on \( \delta \) as in (3.6).

To see that the curvature remains bounded, let \( U_0, \ldots, U_q \) be those \( U_q \) with \( q \in U_q \), and choose local coordinates near \( q \) as follows. Let

\[
(4.27) \quad m_j = \dim \Sigma_j - i.
\]

Choose local coordinates functions \( s_1, \ldots, s_{m_j} \) on \( \Sigma_j \), which are constant on

the orbits. Extend these to \( U_0 \cap \cdots \cap U_i \) by composing with \( \pi_i \). Next choose \( s_{m_j+1}, \ldots, s_{m_{j+1}} \) on \( \Sigma_{j+1} \) so that \( s_1, \ldots, s_{m_j} \) are coordinates transverse to the orbits on \( \Sigma_j \). Extend these to \( U_0 \cap \cdots \cap U_i \) by composing with \( \pi_i \) (tread \( \pi_i = \pi_j \)). By proceeding in this way, we obtain \( s_1, \ldots, s_{m_{j+1}} \).

Extend \( s_1, \ldots, s_{m_{j+1}} \) to a complete system of local coordinates transverse to the orbit of \( K_Y \) through \( g_j \),

by choosing additional functions, \( t_1, \ldots, t_{m_{j+1}} \), which are constant on the orbits of \( K_Y \) (for \( p \) near \( q \)). Finally, choose \( s_{m_{j+1}+1}, \ldots, s_{m_{j+2}} \), \( i = 1, \ldots, m \), such that for fixed \( t_1, \ldots, t_{m_{j+1}} \), the fields \( \partial/\partial x_i, \ldots, \partial/\partial x_{m_{j+1}} \) are Killing fields generated by the action of \( K_Y \).

\[
(4.28) \quad s_{m_j} = s_{m_j}.
\]

\[
(4.29) \quad s_{m_j+1} = s_{m_j+1}.
\]

\[
(4.30) \quad s_{m_{j+1}+n} = s_{m_{j+1}+n}.
\]

The effect of the change of metric corresponding to \( U_q \) in this coordinate system is to collapse only \( x_1, \ldots, x_i \) while expanding all directions orthogonal to \( z_1 = \text{const}, \ldots, z_m = \text{const} \). The change corresponding to \( U_{j+1} \) collapses all \( x_1, \ldots, x_i \) directions while expanding directions normal to \( y_1 = \text{const}, \ldots, y_{m-m_j} = \text{const} \), as well as \( z_1 = \text{const}, \ldots, z_{m-m_j} = \text{const} \).

The change corresponding to \( U_q, q < k < j \), has an effect intermediate between the two above. Thus, by successively
changing coordinates as in (B) above and observing, as in §3, that \( |p_1^\mathcal{A}/r_1| \), \( |p_2^\mathcal{A}/r_2| \) remain bounded in the new coordinate systems, we see that the curvature of \( g_\delta \) is bounded independent of \( \delta \).

To see that \( g_\delta \) collapses as \( \delta \to 0 \), for each \( p \in \mathcal{Y} \), choose \( r = r(\delta) \) such that the exponential map of the normal bundle to \( \mathcal{E}_p \) is a diffeomorphism when restricted to vectors of length \( r \). Take a finite covering of \( \mathcal{Y}^* \) by tubular open neighborhoods \( T_{\delta}(\mathcal{E}_p) \). For \( q \in T_{\delta}(\mathcal{E}_p) \), it follows that \( \text{dist}(q, \partial T_{\delta}(\mathcal{E}_p)) > c(q) > 0 \) for some \( c(q) \) independent of \( \delta \). But through every such \( q \) passes a curve of length \( c_3(q) \delta \), which is not contractible in \( T_{\delta}(\mathcal{E}_p) \). For \( c_3(q) \delta < c(q)/2 \), this implies that there is a closed not-contractible geodesic loop on \( q \) of length \( < c_3(q) \delta \). Hence \( i(d) < c_3(q) \delta \).

Finally, we note that if \( m_1 \) as in (4.27) and we put

\[
\kappa = \min_{\xi(\cdots, \eta) \ge 0} \{ (l_1 - m_1) + \cdots + (l_j - m_j) \},
\]

then

\[
\text{Vol}(\mathcal{Y}^*, g_\delta(\delta)) \le C(\log \delta)^\kappa.
\]

In particular, for this method of collapsing (which we indicate how to sharpen in §5) the volume goes either to infinity or to zero. In fact,

\[
\lim_{\delta \to 0} \text{Vol}(\mathcal{Y}^*, g_\delta(\delta)) = 0
\]

if and only if for all \( i \),

\[
i - m_i > 0,
\]

or equivalently,

\[
i > \frac{1}{i} \dim \Sigma_i.
\]

The procedure just described has a straightforward generalization to structures which are not pure. For this, we choose a regular atlas, \( \{ U_i \} \), for a substructure of positive rank. Over each \( U_i \) we have a pure substructure, \( G_i \) (see Remark 1.2), and as in Lemma 1.3 there is a natural partial ordering among the \( U_i \). Note that if \( G_i \subset G_j \), then the orbit stratification for the local action of \( G_i \) refines that for \( G_j \). From this, we easily obtain the existence of a metric \( g \) and inner product \( \langle \cdot, \cdot \rangle \) on \( g_\delta \), with properties which generalize in the obvious way those of Lemmas 4.1 and 4.2.

Now on each \( U_i \) we collapse as above, except that we modify the cut off functions \( \eta_i \) in such a way that \( \eta_i^\mathcal{A} = 1 \) in a small neighborhood of \( \partial U_i \). By performing these collapses successively, we collapse \( \mathcal{Y}^* \) with bounded curvature. Thus, we have the following result.
Let $Y^*$ admit an $F$-structure of positive rank. Let $\{U_\alpha\}$ be a covering as above. Let $m_F^* = \dim \Sigma_2^*$, where $\Sigma_2^*$ is defined by the action of $G_2$ and put

$$k = \inf \kappa_\alpha,$$

where $\kappa_\alpha$ is defined as in (4.31).

**Theorem 4.1.** There exists a family of invariant metrics, $g_\alpha$, on $Y^*$ such that for $\delta < 1/2$,

1. $(Y^*, g_\delta)$ is $c^\delta$-collapsed.
2. $\text{diam}(Y^*, g_\delta) < c\log \delta$.
3. $\text{Vol}(Y^*, g_\delta) < c\delta^{-\delta \log \delta} \text{Vol}^n$. 

**Example 4.1 (Nonpolarized volume collapse).** Let $R^3 \to \mathcal{M} \to S^1 \times S_2^1$ be the space of flat bundles considered in Example 1.7, where $Y^1$ denotes the zero section of $\mathcal{M}$ and $S_2^1$ the circle which parameterizes the $\eta^1(\theta)$. If $\mathcal{M}_1, \mathcal{M}_2$ are 2-covers of $\mathcal{M}$, we can form

$$R^3 \to \mathcal{M}_1 \times \mathcal{M}_2 \to S^1 \times S_2^1 \times S^1 \times S_2^1,$$

Let

$$T^3 = \{(x_1, \theta, x_2, \theta)\} \subset S^1 \times S_2^1 \times S^1 \times S_2^1,$$

and let $\mathcal{M}^* = (\mathcal{M}_1 \times \mathcal{M}_2)^- (T^3)$. The $T$-structure on $\mathcal{M}$ gives rise to an obvious nonpolarizable $T$-structure on $\mathcal{M}^*$ with orbits of dimension 2, 3, 4. The corresponding strata satisfy $\dim \Sigma_2 = 3, \dim \Sigma_3 = 3, \dim \Sigma_4 = 7$. Thus, (4.35) holds. By regarding $R^3 \to \mathcal{M} \to T^3$ and letting $Y^1$ denote the double of the corresponding disc bundle, we obtain a specific example of a compact manifold which can be volume collapsed by means of a nonpolarizable $T$-structure.

**Remark 4.1.** The above $Y^1$ actually does admit a polarized $T$-structure. But probably there exist manifolds which can be volume collapsed although they admit no such structure.

The following example indicates how the construction of Theorem 4.1 can be sharpened.

**Example 4.2 (Collapsing $M_2^3$ with bounded volume).** The manifolds $M_2^3$ of Example 3.9 have $F$-structures which are of rank 2, except on $\Sigma_1$, which is the union of three connected codimension 2 submanifolds. These are either tori or Klein bottles depending on the particular choice of $F$. It will suffice to collapse tubular neighborhoods of the components of $\Sigma_2$, such that the volume stays finite and near the boundary the collapse agrees with the standard collapse of a pure rank 2 structure.
Let $M^2$ be a component of $\Sigma_t$. We start with a metric on the normal bundle, $\nu(M^2)$, which is cylindrical on the fibers. That is, on each fiber, $\mathbb{R}^1$, it is of the form $dr^2 + f(r)^2 dt^2$, where $f(r) = 1$ for $r > 1$.

Given $\delta > 0$, we can construct a $\delta$-collapsed metric on the disk-bundle, $\nu \leq r \leq 1$, by means of a slice polarization. Thus, we multiply the metric by $\delta^3$ on the subspace, $X$, tangent to the $1$-dimensional orbits of the slice polarization and multiply the metric by $\delta^{-3}$ on the subspace $W'$, orthogonal to the slices.

Let $V'$ be the orthogonal complement of $X$ in the tangent space to the orbit. We extend the collapse to the annular region $1 < r < \log \delta$ by multiplying the metric in the direction of $V'$ by a factor $p^3(r)$, where $|p'/p|, |p''/p|$ are bounded, $p = 1$ near $r = 1$, and $p = \delta$ near $r = \log \delta$. Observe that the volume of this region is bounded independent of $\delta$. Moreover, near $r = \log \delta$, we have the standard collapse of a pure structure. However, the metric is still expanded by a factor $p^3(r - [\log \delta] + 1)$ on the subspace $W'$ and for different components $M_i^2$, $M_j^2$ the subspaces $W_i, W_j$ do not correspond. Thus, we extend the collapse to the region $[\log \delta] < r < 2[\log \delta] - 1$ by multiplying the metric by a factor $\delta^3(r - [\log \delta] + 1)$ on the subspace $W'$. It is easy to see that the curvature remains bounded independent of $\delta$ as does the volume.

By gluing the metrics just constructed onto the standard $\delta$-collapsed metric for the rank 2 polarization on the remaining piece of $M_i^2$, we obtain the required $\delta$-collapsed metric on $M_i^2$, with curvature and volume bounded independent of $\delta$.

5. $F$-structures and complete metrics on open manifolds

(a) Introduction. In this section, we consider an open manifold, $Y$, which carries an $F$-structure, $\mathcal{F}$, or polarization, $\mathcal{P}$, on the complement of some compact subset. We treat in detail the case of a polarization, showing that $Y^\ast$ admits a complete metric, $g_{\ast}$, such that $|K_{g_{\ast}}| \leq 1$, Vol($Y^\ast, g_{\ast}$) $< \infty$. The analogous result for $F$-structures is the existence of a complete metric, $g_{\ast}$, such that $|K_{g_{\ast}}| \leq 1$ and the injectivity radius goes uniformly to zero as $p \to \infty$, i.e., the family $Y^\ast \setminus B_3(p)$ collapses as $R \to \infty$. The proof of this latter result will be omitted since the ingredients which are required (beyond those of §4) will be presented in proving the existence of metrics of finite volume.

It is necessary to refine the constructions of the previous sections at two points.
Point 1. The invariant metric constructed in Lemma 1.3 is not guaranteed to have additional nice properties such as completeness, bounded curvature, etc. in case the manifold is open. Thus, we must begin by showing the existence of such an invariant metric, $g_0$, for a subpolarization $\mathcal{P}' \subset \mathcal{P}$. Moreover, this $g_0$ can be chosen such that $\mathcal{P}'$, when measured with respect to $g_0$, has essentially the same kind of uniform local behavior as in the compact case. This is achieved by making the metric grow sufficiently fast at infinity, and there is no attempt to control the volume at this stage.

Point 2. Rather than passing from $\mathcal{E}_0$ to $\log ^{\Delta} - \mathcal{E}_0$ as in (3.27) and (4.24) we will make a sequence of changes which expand $\mathcal{E}_0$ by a factor $\log ^{\Delta} - \mathcal{E}_0$ in a single (radial) direction near the boundary of such $U'_x$. The construction is such that the numbers $\log \delta_i$ can be selected independently. If $c_\alpha = \text{Vol}(U'_x, g_0)$, we choose $(\log \delta_i)$ so small that

$$\sum_{\alpha = 1} c_\alpha A_{\alpha} |\log \delta_i| < \infty.$$  

Then, by proceeding as in §3, we obtain

$$\text{Vol}(Y^\infty, g_0) < c \sum_{\alpha = 1} c_\alpha A_{\alpha} |\log \delta_i|,$$

where $g_0$ is the required metric.

(b) Construction of a complete metric $g_0$. We can assume that the polarization $\mathcal{P}$ is regular and that the boundaries $\{\partial U'_x\}$ are smooth and interest transversally. Moreover, after modifying the invariant metric, we can assume that $g$ is such that:

1. The exponential map on the normal bundle, $\nu(\partial U'_x)$, is a diffeomorphism when restricted to vectors of length $\leq 2\varepsilon_0$.

2. Let $\tau_x$ denote the distance function from $\partial U'_x$. Then on $T_{\tau_x}(\partial U'_x) \cap T_{\tau_x}(\partial \mathcal{P})$

$$\langle \text{grad} \tau_x, \text{grad} \tau_x \rangle = 0.$$

3. The sets $(U'_x \setminus T_{\tau_x}(\partial U'_x))$ cover a neighborhood of infinity.

Let $\mathcal{P}' \subset \mathcal{P}$ denote the polarization defined by $\{U'_x\}$, where $U'_x = U'_x \setminus T_{\tau_x}(\partial U'_x)$.

The construction of the metric $g_0$ is based on the following lemma which is essentially a restatement of Lemma 5.4 and Theorem 5.5 of [3]. Unfortunately, the presentation of these results in [3] was somewhat garbled due to a confusion between the functions $k$ and $1/k$ below. For this reason, we will repeat some of the details here.
Lemma 5.1. Let \( k(p) \) be a locally bounded nonnegative function on a riemannian manifold, \( Y^* \), with possibly incomplete metric \( g \). Then there exists a smooth function, \( k^* \), such that

1. \( k \leq 3k^* \).

2. If \( g_0 = (k^*)^*g = e^{2kn^*}g \), then \( g_0 \) is complete with curvature \( |K(g_0)| \leq 1 \) and injectivity radius \( i_d(g_0) \geq \frac{1}{2} \) for all \( p \). Moreover,

\[
\|\text{grad}_{g_0} \log k^*\|_{g_0} \leq c(n),
\]

\[
\|Hess_{g_0} \log k^*\|_{g_0} \leq c(\varepsilon).
\]

3. If \( Y^* \) carries regular polarisations \( \mathcal{P} \subset \mathcal{P} \) and the metric \( g \) is above, then \( g_0 \) can be chosen invariant for \( \mathcal{P}^* \).

Proof. By increasing the function \( \tilde{k} \) if necessary, we can assume

\[
k(p) = \sup_{r \in \mathbb{R}(\mathcal{P})} \|K_g(r)\|^{1/2},
\]

\[
k(p) > 1/i_d(g),
\]

\[
k(p) > 1/\lambda, \text{ for } \lambda > 0,
\]

\[
k(p) \neq 0,
\]

where \( \mathbb{R}(\mathcal{P}) \) denotes the supremum of the radii of open metric balls at \( p \) whose closure is compact.

Put

\[
\bar{k}(p) = \inf \left\{ \frac{1}{R} \left| \sup_{r \in \mathbb{R}(\mathcal{P})} k(q) \leq 1/R \right| \right\}.
\]

It follows directly from (5.10) that if \( \lambda > 0 \) and \( \mu \in \mathcal{P} \), then

\[
\frac{1}{1 + \lambda} k(p) \leq \bar{k}(p).
\]

Moreover, \( \bar{k} \leq 1 \).

\[
\bar{k}(q) \leq \frac{1}{1 + \lambda} \bar{k}(p).
\]

(5.11) and (5.12) replace Lemma 5.4 of [3]. The construction of \( k^* \) now proceeds as in the proof of Theorem 5.5 of [3], but with the following proviso: \( \bar{k} \) is to be replaced by \( 1/\lambda \), except in the expression \( \lambda \bar{k} \). Thus gives (1) and (2).

3. If \( \mathcal{P} \subset \mathcal{P} \) is above, the function \( \bar{k}(p) \) need not be invariant for \( \mathcal{P} \) at points \( p \in U \) with

\[
\bar{k}(p) < 1/\text{dist}(p, U).
\]
However, if we put

\[ \eta (x) = \sup_{\tau \in \mathcal{U}_t} \varepsilon_\tau, \]

(5.14)

where \( \mathcal{U}_t = \mathcal{U}_t \setminus T(\partial \mathcal{U}_t) \) and require in addition to (5.9)–(5.12) that, say,

\[ k(p) > 10 \eta (p), \]

(5.15)

then \( \tilde{F}(p) \) is invariant for \( \mathcal{G}^p \). If we now combine the argument of [3] with a standard averaging argument, (3) follows.

Let \( p \in \mathcal{Y}^n \) and let \( (z_1, \cdots, z_n) \) be a local coordinate system with \( p \) at the origin. Suppose that on the 2-coordinate ball, \( B(p) \), the matrix \( (\tilde{g}_{ij}(z)) \) for the metric \( \tilde{g} \) satisfies, say

\[ 1 < \det \tilde{g}_{ij}(z) \leq 1, \]

(5.16)

\[ |g'_{ij}(z)| < \Omega, \]

(5.17)

\[ |g''_{ij}(z)| < \Omega^2. \]

(5.18)

We choose \( k^* > \max(1/\varepsilon, \Omega) \) and make the change of variables

\[ z_i = k^*(0) z_i, \]

(5.19)

Then the metric \( g_0 = (k^*)^2 \tilde{g} \) satisfies

\[ \frac{1}{2} < \det (g_0)_{ij}(z) \leq 1, \]

(5.20)

\[ |g_0'_{ij}(z)| < \varepsilon(n), \]

(5.21)

\[ |g_0''_{ij}(z)| < \varepsilon(n). \]

(5.22)

on the 2-coordinate ball \( B(0) \) (see (5.4), (5.5), (5.11), (5.12) and (1) above).

Remark 5.1. By making the function \( k \) grow sufficiently rapidly we can find at each point a coordinate system satisfying (5.20)–(5.32) in which the basic computation, (2.11)–(2.14), will apply.

(c) Expansion of \( g_0 \). Let \( \mathcal{G}^p \) be as above and put \( \mathcal{I}_p = \mathcal{U}_t \cap T(\partial \mathcal{U}_t) \), where as in (5.3), the tubular neighborhood is with respect to the metric \( g \). If \( p \in \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_n \) we can introduce a local coordinate system, \( (x, y, r) \), near \( p \), as follows. As usual, the fields \( \partial / \partial x_1, \cdots, \partial / \partial x_n \) are Killing fields spanning the orbit \( \mathcal{G}_p \) (with respect to \( \mathcal{G}^p \)). The functions \( r, \cdots, t_n \) are as in (5.3). In view of (5.3) we can find additional coordinates \( z_1, \cdots, z_n \) such that the matrix of \( g \) for these coordinates is of the form

\[ \begin{pmatrix} A(y, r) & 0 \\ 0 & I \end{pmatrix}. \]

(5.23)
where \( I \) is the identity matrix and \( A(x, y, z) \) represents the inner product on the subspace spanned by \( \{ \partial/\partial \xi_1, \ldots, \partial/\partial \xi_r \} \). As in Remark 5.1, we can assume that for the coordinate system

\[
x_i = k^*(p)x_i,
\]

\[
y_i = k^*(p)y_i,
\]

\[
z_i = k^*(p)z_i,
\]

the matrix

\[
\left( g_0(x, y, r) \right) = \begin{pmatrix}
    (k^*(y_i))^2 & A & 0 \\
    (k^*(p))^2 & 0 & I
\end{pmatrix}
\]

satisfies (5.20)-(5.22). Here

\[
A = A(y/k^*(p), r/k^*(p))
\]

and the \( r_i \) take values which include the interval \((0, 1)\).

In order to expand \( 3_n \) we choose functions \( h_d : [0, 1] \to [1, \infty) \), each \( d \in R^* \), such that

\[
(i) \quad h_d = 1 \text{ on fixed intervals } [0, \epsilon), [1 - \epsilon, 1].
\]

\[
(ii) \quad \int_0^1 h_d = d.
\]

(iii) The derivatives of the function \( 1/h_d \) are uniformly bounded independent of \( d \).

Now on each subset \( I_n \), multiply the metric in the direction of \( \text{grad } r_i \) by the function \( h_d^2(r_i) \), while leaving \( x \) unchanged in the orthogonal directions. Call the new metric \( g_{nr} \). The constant \( d_n \) will be specified below.

To see the effect of this change of metric on \( I_n \cap \cdots \cap I_n \), we make the change of variables

\[
\int_0^1 h_{x_n}(x) \, dx_n = s_n, \quad h_{y_n}(r) \, dr_n = ds_n.
\]

\[
\frac{1}{h_{x_n}(x)} \frac{\partial}{\partial x_n} = \frac{\partial}{\partial s_n}.
\]

Then using (iii) and (5.28) it follows that

\[
(\phi_0(x, y, s)) = \begin{pmatrix}
    (k^*(y_i))^2 & A & 0 \\
    (k^*(p))^2 & 0 & I
\end{pmatrix}
\]

satisfies (5.20)-(5.22) (for some \( c(s) \) independent of \( d_1, \ldots, d_r \)). Moreover, the functions \( s_n \) take values which include the interval \((0, d)\). (d) Collapse of the expanded metric. Now choose nonincreasing functions \( \rho_d : [0, d] \to [0, 1] \) such that \( \rho_d = 1 \) on \([0, \epsilon_1]\) and \( \rho_d = c^{-1} \) near \( d \). These can be chosen such that \( |\psi'/\psi|, |\phi'/\phi| \) are bounded independent of \( d \) (compare
(3.4). Finally, by choosing $\epsilon < \epsilon$ (where $\epsilon$ is as in (i)) we can arrange that

$$|h_0(f(x)) - f(x)| < 1,$$

where $x$ and $z$ are related as in (5.27).

The function $h_0(x)$ defined near $U_0^*$ has an obvious extension to all of $Y^*$. Start with the metric $g_0$ and successively multiply the inner product on the subspace tangent to the orbits of each $g_0^i$ by the function $h_0^{i_0}$. Call the resulting metric $g_0$. In view of (5.30), the volume form of $g_0$ is pointwise smaller than that of $g_0$. Now choose $d_0 = \log g_0$, where $g_0$ is as in (5.1). Then it follows from (3) of the subsection above that $\text{Vol}(Y^*, g_0) < \infty$.

Finally, since $F$ is regular, each point is contained in at most $n$ different sets $U_0^*$. So by Remark 5.1 and the bounds for (5.29) it is clear that $\{K_n\}$ is uniformly bounded. Thus we have

**Theorem 5.2.** (1) If $Y^*$ admits a polarization $F$ on the complement of a compact subset, $C$, then $Y^*$ admits a complete metric, $g_{\text{we}}$, invariant for some $F' \subset F$, with $|K_n| < 1$ and $\lim_{t \to 0} \text{Vol}(Y^*, g_{\text{we}}) = 0$.

(2) If $C$ is empty, then $Y^*$ admits a family, $g_{\text{we},n}$, of such complete metrics, with $|K_{n+1}| < 1$ and $\lim_{t \to 0} \text{Vol}(Y^*, g_{\text{we},n}) = 0$.

In the same way we have

**Theorem 5.3.** (1) If $Y^*$ admits an F-structure, $F$, of positive rank of the complement of a compact subset, $C$, then $Y^*$ admits a complete metric $g_{\text{we}}$, invariant for some $F' \subset F$, such that $|K_n| < 1$ and $i_p \to 0$ uniformly as $p \to \infty$.

(2) If $C$ is empty, then $Y^*$ admits a family, $g_{\text{we},n}$, of such metrics, such that $|K_{n+1}| < 1$ and $\{Y^*, g_{\text{we},n}\}$ collapses.

**Remark 5.2.** Clearly, a sharper statement of Theorem 5.2 is possible; see Theorem 4.1 and Example 4.2.

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**Appendix:** Pure polarized structures on essential manifolds

Let $X^*$ be a closed oriented manifold and let $f: X \to (\pi, 1)$ be the classifying map where $\pi = \pi_1(X^*)$. We call $X^*$ essential if the fundamental class, $[X^*] \in H_*(X^*, R)$, satisfies $\langle [X^*], [X^*] \rangle = 0$ (compare [6], [7]).

**Theorem A.1.** Let $F$ be a pure F-structure on an essential manifold $X^*$, such that the group which acts locally is isomorphic to a $k$-torus, $Y^*$. Then $\operatorname{dim} F_p = k$ for all $p \in X^*$. Moreover, there exists a free normal abelian subgroup, $A^j \subset \pi_1(X^*)$, of rank $k$, whose action on the higher homotopy groups $\pi_j(X^*)$ $(j > 2)$ is trivial.
Corollary A.2. The connected sum \(X \# M^s\), where \(M^s\) is an arbitrary \(n\)-dimensional manifold which is not a homology sphere, admits no pure \(F\)-structure of rank \(\geq 1\).

Example A.1. If \(n = 2l + 1\) is odd, and \(M^s \) admits a (possibly nonpure) \(T\)-structure which is of rank \(l\) on some open set, then \(T^s \# M^s\) also admits a (nonpure) \(T\)-structure. In fact, let \(p \in T^s\), \(q \in M^s\) lie on principal orbits \(E_{p}, E_{q}\) of rank \(l\). Let \(T_{p}(E_{p}), T_{q}(E_{q})\) denote the small (saturated) tubular neighborhoods of \(p, q\). If we form \(T^s \# M^s\) by removing balls of radius \(\varepsilon/2\) about \(p\) and \(q\), we can regard \(T^s \setminus T_{p}(E_{p})\) and \(M^s \setminus T_{q}(E_{q})\) as contained in \(T^s \# M^s\). On these sets, the \(T\)-structure on \(T^s \# M^s\) can be taken to coincide with the restrictions of the given structures on \(T^s\) and \(M^s\) (compare [12]).

Proof of Theorem A.1. By Proposition 1.1, \(\pi_1(X^{s}) \cong T^{s}\) acts on \(X^{s}\) the holonomy covering. Let \(s \in X^{s}\) and consider the orbit map \(T^{s} \times s \rightarrow E_{s}\). We claim that it suffices to show that the induced homomorphism

\[
\pi_{1}(T^{s}) \rightarrow \pi_{1}(\tilde{X}^{s}, s) \cong \pi_{1}(X, x)
\]

is injective. To see this, note that if \(\text{dim} E_{s} < k\) for some \(x\), then \(\ker i_{*}\) contains the image in \(\pi_{1}(T^{s})\) of \(\pi_{1}(H_{s})\), where \(H_{s}\) denotes the isotropy group of \(s\). For the second assertion, we observe that it is well known and easy to see that \(\pi_{1}(\pi_{1}(T^{s})) \cong \pi_{1}(X, x)\) is a ceram and acts trivially on \(\pi_{1}(X, x)\).

Let \(T^{s}_{s}\subset T^{s}\) define the unique sub-torsus commensurable with \(\ker i_{*}\). Then \(T^{s}_{s}\) defines a substructure, \(F^{s} \), with the following properties. For each orbit, \(E_{s}\), of \(F^{s}\), there is a finite covering \(\tilde{E}_{s} \rightarrow E_{s}\) such that the induced map \(\pi_{1}(\tilde{E}_{s}) \rightarrow \pi_{1}(E_{s})\) is the zero map (see below for further details)

Suppose first that \(E_{s} \neq F_{s}\) for all \(x\), and so \(\pi_{1}(E_{s}) \rightarrow \pi_{1}(E_{s})\) is the zero map. Then if \(X/F^{s}\) denotes the orbit space, it follows that the induced map \(p_{*}: \pi_{1}(X) \rightarrow \pi_{1}(X/F^{s})\) is an isomorphism. In fact, since the inverse image, \(\omega_{s}^{*}\), of each point in \(X/F^{s}\), is a connected, \(p_{*}\) is surjective. Moreover, \(\ker p_{*}\) is spanned by the normal subgroup generated by \(\bigcup_{s} \omega_{s}(E_{s}) \subset \pi_{1}(X)\). Since homotopy classes of maps \(f: X \rightarrow K(\pi_{1})\) are in 1-1 correspondence with (conjugacy classes of) homomorphisms, \(\omega_{s}(X) \rightarrow \pi_{1}(K(\pi_{1}))\), it follows that \(f\) is homotopic to \(\pi_{1}(X/F^{s}) \rightarrow \pi_{1}(X^{s})\) and \(\pi_{1}(X^{s}) = 0\). Therefore \(H_{s}(X^{s}/F^{s}) = 0\), which is possible only if \(\text{dim} T_{s} = 0\). Thus \(\ker i_{*} = 0\) and the theorem follows in this case.

If \(E_{s} \neq F_{s}\), for some \(x\), the idea is similar but requires some further technical elaboration. Let \(U_{s}\) be a small equivariant tubular neighborhood of \(x\). By passing to a finite covering, \(U_{\prime} \rightarrow U_{s}\), we can assume that the lifted orbit, \(\tilde{O}_{\prime}\), is induced by the action of \(T_{s}\) (see Definition 1.2). Note that \(T_{s}\) is
only commensurable with $\ker_1$ and that $\delta^*$ might be a multiple orbit in its stratum. Thus, $\pi_1(\delta^*) \to \pi_1(\delta^*_*)$ need not be the zero map. But after passing to a finite covering, $\delta^* \to \delta^*$, we can assume that this is the case for $\pi_1(\delta^*) :\pi_1(\delta^*_*)$. Moreover, each $\tilde{U}_j \to U_j$ denotes the corresponding covering of $U_j$, then the same holds for any $\delta^* \subset U_j$, since the inclusion, $\delta^* \subset U_j$, is a homotopy equivalence.

In order to make use of the covering spaces, $\tilde{U}_j \to U_j$, we need the following lemma.

**Lemma 4.3.** Let $Y^*$ be a closed manifold which is the union of open submanifolds $U_j^* \cdots U_2^* \cdots U_1^*$, whose (smooth) boundaries $(\partial U_j^*)$ intersect transversally. Let $\pi : \tilde{U}_j \to U_j$ be finite coverings. Then there exists an $n$-dimensional polyhedron $Y^*$ and a continuous map $g : Y^* \to Y^*$, such that

1. $g : H_i(Y^*; Q) \to H_i(Y^*; Q)$ is surjective,
2. If $C_j \subset U^*_j$ is closed, the map $g^*([C_j]) \subset Y^*$ factors through a map $h_i : 

**Proof.** Let $Z$ be an arbitrary topological space, $U \subset Z$ an open subset and $\pi : \tilde{U} \to U$ a finite covering map. Denote by $\tilde{Z} = \tilde{Y} \times \tilde{U}$ the set $Z \times \tilde{U}$, and by $\tilde{Z} \to Z$ the obvious map. Define the topology in $\tilde{Z}$ by the condition that $A \subset \tilde{Z} \subset Z$ is closed if and only if $\pi(A) \subset Z$ is closed. It is easy to see that $\pi$ is surjective on rational homology and that the covering $\pi$ factors through a unique map $\tilde{U} \to Z$.

To complete the proof of Theorem 4.1, we consider some sufficiently fine open covering, $(U_i)$ of $Y$ by saturated open subsets with smooth boundary such that $(\partial U_i)$ intersect transversally; compare Lemma 4.1. The local actions on $\tilde{U}_j$ induce corresponding actions on $\tilde{Y}$. As in the special case considered above, we want to show that if $f : (X \to K_{\partial, 1})$, then $f \circ g : \tilde{Y} \to \tilde{K}_{\partial, 1}$ lifts to $\tilde{f} \circ \tilde{g}$. But it follows as above, that if $f \circ g : \pi_1(\tilde{X}) \to \pi_1(\tilde{Y})$ and $f \circ g : \pi_1(\tilde{X}) \to \pi_1(\tilde{Y})$, then $\ker(f \circ g)$ contains $\ker(g \circ h)$. This implies that the desired lift exists and suffices to complete the proof.

**References**


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