Gromov-Hausdorff distance
and applications

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Summer school
Metric Geometry
Felix Hausdorff, 1868–1942

Mikhail Gromov *1943
Sources

- Burago, Burago, Ivanov: A course in metric geometry
- Bridson, Haefliger: Metric spaces of non-positive curvature
- Heinonen: Geometric embeddings of metric spaces
- Gromov: Groups of polynomial growth and expanding maps
- Petersen: Riemannian geometry and several articles
- Hausdorff: Set theory
- Kuratowski: Topology
A pseudometric on a set $X$ is a function $d : X \times X \rightarrow [0; \infty)$ with
$d(x, x) = 0$, $d(x, y) = d(y, x)$, and triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z)$$

A metric is a pseudometric such that $d(x, y) = 0$ only when $x = y$.

Remarks

- For a pseudometric space $(X, d)$,
  
  $$xRy \iff d(x, y) = 0$$
  
  is an equivalence relation. The pseudometric $d$ induces a metric on the quotient $X/R$.
- If $d$ is a pseudometric and $\delta > 0$, then $d_\delta(x, y) = d(x, y) + \delta$ (for $x \neq y$) defines a metric.
- Often useful to admit $d : X \times X \rightarrow [0; \infty]$. Then $d(x, y) < \infty$ is an equivalence relation.
A metric space \((X, d)\) is called
- **complete** if every Cauchy sequence converges
- **compact** if every sequence has a convergent subsequence
- **separable** if there is countable dense subset
- **totally bounded** (= precompact) if \(\forall \varepsilon > 0 \exists \) a finite \(\varepsilon\)-dense subset \(X_\varepsilon \subseteq X\), i.e.

\[
X = \bigcup_{x \in X_\varepsilon} B_\varepsilon(x)
\]

**Implications**
- separable \(\iff\) topology has a countable basis
- totally bounded \(\Rightarrow\) separable
- totally bounded \(\iff\) every sequence has a Cauchy subsequence
- compact \(\iff\) complete and totally bounded
Examples: classical sequence spaces

Examples

- Closed balls in $\ell^2$ are complete, separable, not compact.
- Closed balls in $\ell^\infty$ are complete, not separable.

Recall definitions: For $1 \leq p \leq \infty$, $\ell^p$ is the space of sequences

$$x = (x_k)_{k \in \mathbb{N}} = (x_1, x_2, \ldots)$$

of real numbers such that the $\ell^p$-norm

$$||x||_p = \begin{cases} 
  \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} & \text{for } p < \infty \\
  \sup_{k \in \mathbb{N}} |x_k| & \text{for } p = \infty 
\end{cases}$$

is finite. Banach spaces, for $p=2$ Hilbert. For $p \leq q$,

$$\ell^1 \subset \ell^p \subset \ell^q \subset c_0 \subset c \subset \ell^\infty$$
**Theorem.** Every separable metric space \((X, d)\) admits an isometric embedding into \(\ell^\infty\).

**Proof.** Choose a dense sequence \((x_k)_{k \in \mathbb{N}}\) in \(X\) and define \(\phi : X \to \ell^\infty\) by

\[
\phi(x) = (\phi_k(x))_{k \in \mathbb{N}} = (d(x, x_k) - d(x_k, x_0))_{k \in \mathbb{N}}
\]

Then for \(x, y \in X\),

\[
|\phi_k(x) - \phi_k(y)| = |d(x, x_k) - d(y, x_k)| \leq d(x, y)
\]

with equality obtained when \(x_k\) approaches \(x\) or \(y\). Therefore,

\[
\|\phi(x) - \phi(y)\|_\infty = d(x, y) . \quad \Box
\]

**Exercise.** Find a metric space \((X, d)\) consisting of four points that does not admit an isometric imbedding into Hilbert space \(\ell^2\).
Cauchy completion and precompactness

Theorem. For every metric space \((X, d)\) there is a \textit{complete} metric space \((\hat{X}, \hat{d})\) with an isometric embedding \(\iota: X \to \hat{X}\) such that \(\iota(X)\) is dense in \(\hat{X}\).

- \((\hat{X}, \hat{d})\) unique up to isometry, called the \textit{completion} of \((X, d)\).

- Construction: Generalize Cantor’s definition of the real numbers from the rationals. Define a pseudometric on the set of all Cauchy-sequences in \(X\) by

\[
d(((x_1, x_2, \ldots), (y_1, y_2, \ldots)) := \lim_{k \to \infty} d(x_k, y_k).
\]

Then define \(\hat{X}\) to be the quotient metric space identifying elements with distance zero. Thus points of \(\hat{X}\) are equivalence classes

\[
\xi = [(x_1, x_2, \ldots)]
\]

of Cauchy sequences in \(X\), where equivalence means having distance zero.

Theorem. \((X, d)\) precompact \(\iff\) \((\hat{X}, \hat{d})\) compact.
Hausdorff distance

For a subset $A \subseteq X$ of a metric space $(X, d)$, the $r$-neighbourhood of $A$ is defined as

$$U_r(A) := \{ x \in X \mid \text{dist}(x, A) < r \} = \bigcup_{x \in A} B_r(x)$$

Hausdorff-distance of non-empty subsets $A, B \subseteq X$:

$$d_H(A, B) := \inf \{ r > 0 \mid A \subseteq U_r(B) \text{ and } B \subseteq U_r(A) \}$$

$$= \max \{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \}$$

Properties

- $d_H$ satisfies triangle inequality, is a pseudometric on the set of bounded subsets of $X$.
- $d_H(A, B) = d_H(A, \overline{B})$
- $d_H(A, B) = 0 \iff \overline{A} = \overline{B}$
- $d_H(\{a\}, \{b\}) = d(a, b)$
Hausdorff compactness theorem

Let $\mathcal{C}(X)$ be the set of all non-empty closed bounded subsets of $X$, equipped with the metric $d_H$.

Theorem

- If $X$ is complete, then $\mathcal{C}(X)$ is complete (Hahn 1932).
- If $X$ is totally bounded, then $\mathcal{C}(X)$ is totally bounded.
- If $X$ is compact, then $\mathcal{C}(X)$ is compact (Hausdorff, Blaschke).

Remark

- Same if $\mathcal{C}(X)$ denotes the set of all non-empty compact subsets.

History

Blaschke selection theorem (1916). Every $d_H$-bounded sequence of compact convex sets $A_k \subseteq \mathbb{R}^n$ subconverges to a compact convex set $A \subseteq \mathbb{R}^n$.

- Proof. There is a compact $X \subseteq \mathbb{R}^n$ that contains every $A_k$. Apply previous theorem to obtain a subsequence converging to some $A \in \mathcal{C}(X)$ and check that compact Hausdorff-limits of convex sets are convex.
Hausdorff compactness: proof

- **C(X) is totally bounded:** Given \( \varepsilon > 0 \), choose a finite subset \( X_\varepsilon \subseteq X \) that is \( \varepsilon \)-dense in \( X \). Then the power set \( \mathcal{P}(X_\varepsilon) \) is an \( \varepsilon \)-dense finite subset of \( C(X) \).

- **C(X) is complete:** Let \( A_k \in C(X) \) be a Cauchy sequence. Define

\[
A := \text{Flim sup } A_k = \bigcap_{n=1}^{\infty} A_n \cup A_{n+1} \cup \ldots
\]

**Claim.** \( A \in C(X) \), and \( d_H(A_n, A) \to 0 \) as \( n \to \infty \).

- **Remark:** The set \( \text{Flim sup } A_k \) is called the upper closed limit of the sequence \( (A_k)_{k \in \mathbb{N}} \). An alternative description is

\[
\text{Flim sup } A_k = \{ x \in X \mid \forall \varepsilon > 0 : B_\varepsilon(x) \cap A_k \neq \emptyset \text{ for } \infty \text{ many } k \}
\]

\[
= \{ \text{accumulation points of sequences } a_n \in A_n \}.
\]
Proof of claim

- \( A \subseteq U_\varepsilon(A_n) \) for all large \( n \): \( a \in A \) implies that the \( \varepsilon/2 \)-ball around \( a \) meets infinitely many of the \( A_k \), and so

\[
a \in U_{\varepsilon/2}(A_k)
\]

for these \( k \). Since the sequence is Cauchy, we have \( A_k \subseteq U_{\varepsilon/2}(A_n) \) for all large \( k \) and \( n \). Hence

\[
a \in U_{\varepsilon/2}(U_{\varepsilon/2}(A_n)) \subseteq U_\varepsilon(A_n).
\]

- \( A_n \subseteq U_\varepsilon(A) \) for all large \( n \): If \( x \in A_n \) for sufficiently large \( n \), then there is a subsequence \( n = n_1 < n_2 < \ldots \) and a sequence of points \( a_{n_k} \in A_{n_k} \) starting at \( a_1 = x \) such that \( d(a_{n_k}, a_{n_k+1}) < \varepsilon/2^{k+1} \). The sequence \( (a_{n_k})_{k \in \mathbb{N}} \) is Cauchy, hence converges to some \( a \in X \), and by definition of \( A \) we have \( a \in A \). By the triangle inequality,

\[
d(x, a) \leq \sum_{k=1}^{\infty} d(a_{n_k}, a_{n_k+1}) < \varepsilon. \quad \square
\]
Recall the upper closed limit of the sequence \((A_k)_{k \in \mathbb{N}}\):

\[
F \limsup A_k = \{x \in X \mid \forall \varepsilon > 0 : \mathcal{B}_\varepsilon(x) \cap A_k \neq \emptyset \text{ for } \infty \text{ many } k\}
\]

The lower closed limit is defined as

\[
F \liminf A_k = \{x \in X \mid \forall \varepsilon > 0 : \mathcal{B}_\varepsilon(x) \cap A_k \neq \emptyset \text{ for nearly all } k\}
\]

The closed limit is said to exist if both are equal:

\[
F \lim A_k := F \liminf A_k = F \limsup A_k.
\]

**Theorem**

Consider \(A_k, A \in \mathcal{C}(X)\).

- If \(d_H(A_k, A) \to 0\), then \(F \lim A_k\) exists and is equal to \(A\).
- If \(X\) is compact and \(F \lim A_k\) exists, then \(d_H(A_k, A) \to 0\).
Examples

- \((X, d)\) euclidean plane, \(F\lim A_k\) exists, but sequence not Hausdorff convergent:

- \((X, d)\) unit ball in Hilbert space \(\ell^2\), \(F\lim A_k\) exists, but sequence not Hausdorff convergent:

- Menger sponge
**Gromov-Hausdorff distance**

**Definition.** The Gromov-Hausdorff distance between metric spaces $X$ and $Y$ is defined as

$$d_{GH}(X, Y) = \inf_Z \inf_{X', Y'} d_H^Z(X', Y')$$

where the infimum $\in [0, \infty]$ is taken over all metric spaces $Z$ and all subspaces $X', Y'$ of $Z$ that are isometric to $X, Y$.

**Comments**

- $d_H^Z$ denotes the Hausdorff distance in the metric space $(Z, d^Z)$.
- $X', Y'$ carry the metrics obtained by restriction of $d^Z$.
- Distance depends only on the isometry classes of $X$ and $Y$.
- Reformulate:

$$d_{GH}(X, Y) = \inf_Z \inf_{\phi, \psi} d_H^Z(\phi(X), \psi(Y))$$

where the infimum is over all isometric embeddings $\phi : X \to Z$ and $\psi : Y \to Z$. 
Examples

- If $X_\varepsilon$ is an $\varepsilon$-dense subset of $X$ with the induced metric, then $d_{GH}(X, X_\varepsilon) < \varepsilon$. So totally bounded metric spaces admit approximation by finite metric spaces.

- If $\{p\}$ is a one-point space, then $d_{GH}(X, \{p\}) = \frac{1}{2} \text{diam}(X)$, where

$$
\text{diam}(X) = \sup_{x, y \in X} d(x, y)
$$

Proof: Take $Z = X \sqcup \{p\}$ and extend the given metric from $X$ to $Z$ by setting $d(x, p) := \frac{1}{2} \text{diam}(X)$.

- If diameters are finite, then

$$
\frac{1}{2} |\text{diam}(X) - \text{diam}(Y)| \leq d_{GH}(X, Y) \leq \frac{1}{2} \max\{\text{diam}(X), \text{diam}(Y)\}.
$$
Lipschitz-close implies GH-close

A map $F : X \rightarrow Y$ between metric spaces is called $L$-bi-lipschitz if

$$\frac{1}{L} d(x, x') \leq d(Fx, Fx') \leq L d(x, x').$$

Claim. If $F : X \rightarrow Y$ is $(1+\varepsilon)$-bi-lipschitz and bijective, then

$$d_{GH}(X, Y) \leq \varepsilon \max\{\text{diam}(X), \text{diam}(Y)\}.$$

Proof. Can assume diameters are finite. Take $Z = X \sqcup Y$ and extend the metrics $d^X$ and $d^Y$ to a metric(!) on $Z$ by setting

$$d(x, y) := \inf_{a \in X} (d^X(x, a) + d^Y(y, Fa)) + \varepsilon C$$

where $C = \max\{\text{diam}(X), \text{diam}(Y)\}$. Given $y \in Y$, we show that $x = F^{-1}y \in X$ has distance at most $\varepsilon C$ from $y$: For every $a \in X$

$$d(x, y) \leq d^X(F^{-1}y, a) + d^Y(y, Fa) + \varepsilon C.$$

Choose $a = F^{-1}y$ to obtain $d(x, y) \leq \varepsilon C$. □
Alternative definition 1

Proposition

\[ d_{GH}(X, Y) = \inf_d d^X \sqcup Y \|H \| (X, Y) \]

where \(X \sqcup Y\) is the disjoint union and the infimum is taken over all admissible metrics \(d\) on \(X \sqcup Y\), i.e. metrics that extend \(d^X\) and \(d^Y\).

Proof

- If \(\widehat{d}_{GH}(X, Y)\) denotes the right hand side, then \(d_{GH} \leq \widehat{d}_{GH}\) because the infimum for \(d_{GH}\) is extended over a larger set.

- Conversely given \(\varepsilon > 0\), choose \(Z, X'\) and \(Y'\) such that

\[ d^Z_H(X', Y') \leq d_{GH}(X, Y) + \varepsilon. \]  

\((\ast)\)

- If \(X', Y'\) disjoint, restrict the metric of \(Z\) to the union \(X' \cup Y'\) to get

\[ \widehat{d}_{GH}(X, Y) = \widehat{d}_{GH}(X', Y') \leq d_{GH}(X, Y) + \varepsilon. \]  

\((\ast\ast)\)

- If \(X', Y'\) are not disjoint, replace \(Z, X', Y'\) by \(Z \times [0, 1]\), \(X' \times \{0\}\), \(Y' \times \{\varepsilon\}\). Obtain equations \((\ast)\) and \((\ast\ast)\) with \(\varepsilon\) replaced by \(2\varepsilon\). \(\square\)
Claim

\[ d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z) \]

Proof. Take admissible metrics \( d^{X \sqcup Y} \) and \( d^{Y \sqcup Z} \) and, for \( \delta > 0 \), define an admissible metric \( d^{X \sqcup Z} \) on the disjoint union \( X \sqcup Z \) by

\[ d^{X \sqcup Z}(x, z) = \inf_{y \in Y} (d^{X \sqcup Y}(x, y) + d^{Y \sqcup Z}(y, z)) + \delta \]

for \( x \in X \) and \( z \in Z \). Then

\[ d^{X \sqcup Z}_H(X, Z) \leq d^{X \sqcup Y}_H(X, Y) + d^{Y \sqcup Z}_H(Y, Z) + \delta. \]

Now take the infimum over all admissible metrics \( d^{X \sqcup Y} \) and \( d^{Y \sqcup Z} \), and finally let \( \delta \to 0 \). \( \square \)

Proposition. \( X, Y \) compact with \( d_{GH}(X, Y) = 0 \), then \( X, Y \) are isometric.

Notation. Let \( \mathcal{M} \) denote the set of isometry classes of compact metric spaces \( \neq \emptyset \). Then \( (\mathcal{M}, d_{GH}) \) is a metric space.
Proof of proposition. Take a sequence of admissible metrics $d_k$ on $X \sqcup Y$ such that the Hausdorff distance between $X$ and $Y$ with respect to $d_k$ is $\leq 1/k$. Then there are (discontinuous) maps $I_k : X \to Y$ and $J_k : Y \to X$ with

$$d_k(x, I_k(x)) \leq \frac{1}{k} \quad \text{and} \quad d_k(y, J_k(y)) \leq \frac{1}{k}.$$ 

The triangle inequality for $d_k$ then implies

$$d(I_k(x_1), I_k(x_2)) \leq \frac{2}{k} + d(x_1, x_2)$$
$$d(J_k(y_1), J_k(y_2)) \leq \frac{2}{k} + d(y_1, y_2)$$
$$d(x, J_k \circ I_k(x)) \leq \frac{2}{k}$$
$$d(y, I_k \circ J_k(y)) \leq \frac{2}{k}$$

An Arzela-Ascoli argument yields limits $I : X \to Y$ and $J : Y \to X$ for $k \to \infty$. (Obtain $I : X \to Y$ first on a countable dense subset $A \subseteq X$ using a diagonal argument and the compactness of $Y$, then extend from $A$ to $X$. Similarly for $J$.) Then $I$ and $J$ are the required isometries. \qed
Example. Two proper metric spaces with $d_{GH}(X, Y) = 0$ that are not isometric.

- Both $X$ and $Y$ are metric graphs obtained from the real line by attaching segments of suitable length at all integer points.

- For $X$ attach a segment of length $|\sin(m)|$ to the point $m \in \mathbb{Z}$.

- For $Y$ attach a segment of length $|\sin(m + \frac{1}{2})|$ to the point $m \in \mathbb{Z}$.

- To see that $d_{GH}(X, Y) \leq \varepsilon$ for every $\varepsilon > 0$, observe that $X$ and $Y$ are isometrically embedded into the grid

$$Z = \{(x, y) \in \mathbb{R}^2 \mid x \text{ or } y \in \mathbb{Z}\}$$

equipped with its path metric. A suitable integer translation in the $x$-direction will move $X$ into an $\varepsilon$-neighborhood of $Y$. 
Proposition. For separable metric spaces $X, Y$,

$$d_{GH}(X, Y) = \inf_{\phi, \psi} d^\infty_H(\phi(X), \psi(Y))$$

where the infimum is taken over all isometric embeddings $\phi : X \to \ell^\infty$ and $\psi : Y \to \ell^\infty$, and $d^\infty_H$ is the Hausdorff distance in $\ell^\infty$.

Proof. The inequality $\leq$ is clear. Conversely given $\varepsilon > 0$, choose an admissible metric $d$ on $Z = X \sqcup Y$ such that

$$d^Z_H(X, Y) \leq d_{GH}(X, Y) + \varepsilon.$$ 

Since $(Z, d)$ is also separable, there is an isometric embedding $\iota : Z \to \ell^\infty$, and we obtain isometric embeddings

$$\phi : X \to X \sqcup Y \to \ell^\infty \quad \psi : Y \to X \sqcup Y \to \ell^\infty.$$ 

Then

$$d^\infty_H(\phi(X), \psi(Y)) = d^Z_H(X, Y) \leq d_{GH}(X, Y) + \varepsilon. \quad \square$$
Correspondences

**Definition.** Consider metric spaces $X$ and $Y$.

- A **correspondence** (or *surjective relation*) between $X$ and $Y$ is a subset 
  \[ R \subseteq X \times Y \]
  such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ remain surjective when restricted to $R$.

  - Example: If $f : X \to Y$ is a surjective map, then the graph
    \[ R = \{ (x, f(x)) \mid x \in X \} \]
    is a correspondence.

- The **distortion** of a correspondence is defined as
  \[ \text{dis}(R) = \sup_{(x,y),(x',y') \in R} |d^Y(y,y') - d^X(x,x')| \]

  - Remark: If $\text{dis}(R) = 0$, then $R$ is the graph of an isometry.
Theorem

\[ d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}) \]

where the infimum is taken over all correspondences \( \mathcal{R} \subseteq X \times Y \).

Proof

- \[ d_{GH}(X, Y) \geq \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}) : \]

  If \( r > d_{GH}(X, Y) \), then there is a metric space \((Z, d)\) containing \(X\) and \(Y\) such that the Hausdorff distance in \(Z\) satisfies \(d_H(X, Y) < r\). Then

  \( \mathcal{R} := \{(x, y) \mid d(x, y) < r\} \)

  is a correspondence, and

  \[ \frac{1}{2} \text{dis}(\mathcal{R}) < r \]

  because for \((x, y), (x^\prime, y^\prime) \in \mathcal{R}\)

  \[ |d(y, y^\prime) - d(x, x^\prime)| \leq d(x, y) + d(x^\prime, y^\prime) < 2r . \]
Alternative definition 3

\[ d_{GH}(X, Y) \leq \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}) : \]

Let \( \mathcal{R} \) be a correspondence and \( r := \frac{1}{2} \text{dis}(\mathcal{R}) \). We may assume \( r > 0 \).

Define an admissible metric(!) on \( Z = X \sqcup Y \) by

\[
d(x, y) = \inf_{(x', y') \in \mathcal{R}} \left( d(x, x') + r + d(y', y) \right)
\]

Then the Hausdorff distance of \( X, Y \subseteq Z \) is

\[
d_H(X, Y) \leq r = \frac{1}{2} \text{dis}(\mathcal{R}) : \]

Given \( x \in X \), choose \( y \in Y \) such that \( (x, y) \in \mathcal{R} \). Then

\[
d(x, y) \leq d(x, x) + r + d(y, y) = r,
\]

and so the distance from \( x \) to \( Y \) is \( \leq r \). \( \square \)
Definition. A map $f : X \to Y$ is called an $\varepsilon$-isometry if its distortion

$$\text{dis}(f) := \sup_{x,x' \in X} |d^Y(fx, fx') - d^X(x, x')| \leq \varepsilon$$

and if $f(X)$ is $\varepsilon$-dense in $Y$.

Proposition

- If $d_{GH}(X, Y) < \varepsilon$, then there is a $2\varepsilon$-isometry $f : X \to Y$.
- If there is an $\varepsilon$-isometry $f : X \to Y$, then $d_{GH}(X, Y) \leq \frac{3}{2} \varepsilon$.

Proof. Use the previous theorem. If $d_{GH}(X, Y) < \varepsilon$, take a correspondence with $\text{dis}(\mathcal{R}) < 2\varepsilon$. For each $x$ choose $y$ such that $(x, y) \in \mathcal{R}$ and define $f(x) = y$. Then $f$ is a $2\varepsilon$-isometry.

Given an $\varepsilon$-isometry $f : X \to Y$, define $\mathcal{R} := \{(x, y) \mid d(fx, y) < \varepsilon\}$. This is a correspondence with $\text{dis}(\mathcal{R}) \leq 3\varepsilon$. □
GH-limits

**Definition.** A sequence of metric spaces $X_k$ converges to $X$ in the Gromov-Hausdorff sense (short: GH-converges to $X$) if $d_{GH}(X_k, X) \to 0$ as $k \to \infty$. Notation:

$$X_k \xrightarrow{GH} X \quad (k \to \infty)$$

**Remarks**

- If $X$ is compact, then $X$ is unique up to isometry.
- Example: Every compact $X$ is a GH-limit of a sequence of finite metric spaces.
- Hausdorff convergent implies GH-convergent.
- Assume $X_k$, $X$ compact, and $X_k \xrightarrow{GH} X$. Then there are $X'_k, X' \subseteq \ell^\infty$ isometric to $X_k, X$ such that $X'_k \xrightarrow{d_H^\infty} X'$.

Proof later.
In this section we first present Gromov-Hausdorff convergence, then metric measure convergence, then the intrinsic flat convergence and finally, weakest of all, the notion of an ultralimit. We include a few key examples, applications and further resources for each notion.


The Gromov-Hausdorff distance is defined between any pair of compact metric spaces,

\[ d_{GH}(M_1, M_2) = \inf \left\{ d_Z(H_1(M_1), H_2(M_2)) : \text{isom}'_i : M_i \to Z \right\} \]

where the infimum is taken over all metric spaces \( Z \) and all isometric embeddings \( \text{isom}'_i : M_i \to Z \).

An isometric embedding, \( \text{isom}' : X \to Z \) satisfies

\[ d_Z(\text{isom}'(x_1), \text{isom}'(x_2)) = d_X(x_1, x_2) \quad \forall x_1, x_2 \in X. \]

We write \( M_i \xrightarrow{GH} X \) if \( d_{GH}(M_j, X) \to 0 \). See Figure 4.

Figure 4. Gromov-Hausdorff Convergence

The sequences of Riemannian manifolds depicted in Figure 4 reveal a variety of properties that are not conserved under Gromov-Hausdorff convergence. The first sequence \( A_j \) are the flat manifolds.

Source: Christina Sormani, How Riemannian manifolds converge
Examples: bounded curvature collaps

- Circles; flat tori; $M \times S^1$

- The Hopf fibration 
  \[ S^3 \xrightarrow{h} \mathbb{C}P^1 = S^2 \]
  is the quotient map of the free isometric $S^1$-action
  \[ e^{i\theta} (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) \]
  on the standard sphere $S^3 \subseteq \mathbb{C}^2$. This is a Riemannian submersion for a metric of constant curvature $=4$ on $\mathbb{C}P^1$.

  - Take cyclic groups $C_k \subseteq S^1$ of order $k$. Then
  \[ S^3 / C_k \xrightarrow{GH} \mathbb{C}P^1 \text{ as } k \to \infty \]

  - Berger spheres: Define $S^3_\varepsilon = (S^3, g_\varepsilon)$, where the Riemannian metric $g_\varepsilon$ is obtained by multiplying the standard Riemannian metric of $S^3$ with a factor $\varepsilon > 0$ in the fiber direction. Then
  \[ S^3_\varepsilon \xrightarrow{GH} \mathbb{C}P^1 \text{ as } \varepsilon \to 0. \]
Examples: Heisenberg group

The 3-dimensional Heisenberg group $\mathbb{H}$ is the set of all

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } x, y, z \in \mathbb{R}.$$  

The subset $\Gamma \subseteq \mathbb{H}$ of integral matrices is a discrete subgroup. Consider the compact manifold $M = \Gamma \backslash \mathbb{H}$.

1. For $\varepsilon > 0$, take basis for left invariant 1-forms

$$\omega^1 = \varepsilon \, dx \quad \omega^2 = \varepsilon \, dy \quad \omega^3 = \varepsilon^3(dz - xdy)$$

Define Riemannian metric so that this is an ON-basis:

$$g_\varepsilon = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

This is left invariant, and $g_\varepsilon \to 0$ as $\varepsilon \to 0$. Conclusion:

$$(M, g_\varepsilon) \xrightarrow{GH} \text{point} \quad \text{as } \varepsilon \to 0.$$
Examples: Heisenberg (continued)

- The curvature in this example remains bounded: The Maurer-Cartan equations \( d\omega^k = c_{ij}^k \omega^i \wedge \omega^j \) are
  \[
  d\omega^1 = d\omega^2 = 0 \quad d\omega^3 = -\varepsilon \omega^1 \wedge \omega^2
  \]
  For the curvature tensor \( R \) one then calculates \( ||R|| \leq 6 ||d\omega|| = 6\varepsilon \).
  So curvature \( \to 0 \) as \( \varepsilon \to 0 \).

- This works for general nilpotent Lie groups \( G \): choose basis for \( g^* \) such that \( c_{ij}^k = 0 \) unless \( i, j < k \). These metrics descend to nilmanifold quotients \( \Gamma \backslash G \); and to compact infranil-quotients \( \Lambda \backslash G \) after averaging over \( \Lambda / \Gamma \).

2. Now consider the Riemannian metrics \( g'_\varepsilon \) given by the ON-basis
  \[
  \omega^1 = dx \quad \omega^2 = dy \quad \omega^3 = \frac{1}{\varepsilon} (dz - xdy).
  \]
  For \( \varepsilon \to 0 \), \( (M, g'_\varepsilon) \) converges to a metric space \( X \) which is \( M \) equipped with the subriemannian metric defined by \( \omega^1, \omega^2 \) on the plane field \( \ker \omega^3 \). Curvatures go to \( \pm \infty \).
Proposition. Suppose $X_k \xrightarrow{\text{GH}} Y$. If each $X_k$ is/has ... , then $Y$ is/has ...

- separable
- totally bounded
- a proper space – if $Y$ is complete
- a length space – if $Y$ is complete
- a proper geodesic space – if $Y$ is complete
- diameter $\leq D$ (in fact diam $X_k \to$ diam $Y$)
- properties of the form $F(d_{11}, d_{12}, \ldots, d_{k-1,k}) \geq 0$ or $= 0$, where $d_{ij} = d(x_i, x_j)$, and where $F$ is continuous, e.g.
  - $\delta$-hyperbolic
  - $\text{CBB}^\kappa$ ($\iff (1+3)^\kappa$-condition)
  - $\text{CAT}^\kappa$ ($\iff (2+2)^\kappa$-condition)
- complete geodesic with curv $\geq \kappa$
- **NOT:** complete geodesic with curv $\leq \kappa$ (counterexample: hyperboloids $\to$ double-cone)
Proofs: totally bounded

- For CBB$^κ$, CAT$^κ$ and curv see the lectures of Stephanie Alexander at this summer school.

- **Totally bounded.** Pick a finite $\varepsilon$-dense subset in some $X_k$ GH-close to $Y$, then move it to $Y$ via a correspondence. Details:

  - Given $\varepsilon > 0$, fix $k$ so large that $d_{GH}(X_k, Y) < \varepsilon/4$. Then there is a correspondence $\mathcal{R} \subseteq X_k \times Y$ with distortion $\text{dis}(\mathcal{R}) < \varepsilon/2$. Take a finite $\varepsilon/2$-dense subset $X'_k \subset X_k$. For each $x' \in X'_k$ choose a $y' \in Y$ such that $(x', y') \in \mathcal{R}$, and let $Y'$ be the set of all such $y'$.

  - We claim that $Y'$ is $\varepsilon$-dense in $Y$:

  - Given $y \in Y$, find $x \in X_k$ such that $(x, y) \in \mathcal{R}$, and then $x' \in X'_k$ at distance $< \varepsilon/2$ from $x$. For the $y' \in Y'$ that corresponds to this $x'$ we obtain

    \[
    d(y, y') \leq |d(y, y') - d(x, x')| + d(x, x') \\
    \leq \text{dis}(\mathcal{R}) + d(x, x') \\
    < \varepsilon.
    \]

  □
Proper. A metric space $X$ is called proper if all closed balls

$$\bar{B}_r(x) := \{d(\cdot, x) \leq r\}$$

are compact.

- Given a ball $\bar{B}_r(y) \subseteq Y$, there are $x_k \in X_k$ corresponding to $y$, and then for radii $r_k \downarrow r$ the balls $\bar{B}_{r_k}(x_k)$ Hausdorff-converge to $\bar{B}_r(y)$.

- Since all $\bar{B}_{r_k}(x_k)$ are totally bounded, so is the limit $\bar{B}_r(y)$. Since $Y$ is complete, $\bar{B}_r(y)$ is complete, hence compact. □
Proofs: length space

- **Length space.** A length space is a metric space $X$ such that $d(x, x')$ is the infimal length of curves joining $x$ and $x'$. Recall the approximate mid point condition:

For all $x, x' \in X$ and $\varepsilon > 0$, there is an $\varepsilon$-midpoint $m \in X$, i.e.

$$
\max\{d(x, m), d(m, x')\} \leq \frac{1}{2} d(x, x') + \varepsilon.
$$

Then
- length space $\implies$ approximate mid point condition
- $\Leftarrow$ is true for *complete* metric spaces

- Verify this condition for $Y$: Given $y, y' \in Y$ and $\varepsilon > 0$, fix $k$ so large that there is a correspondence with small distortion between $X_k$ and $Y$. Take points $x_k, x'_k \in X_k$ corresponding to $y, y'$ and find an $\varepsilon$-midpoint $m_k \in X_k$. Finally, let $m \in Y$ be a point corresponding to $m_k$. Then $m$ is a $3\varepsilon$-midpoint for $y, y'$. Since $Y$ is assumed complete, it is a length space. $\square$
Proper and geodesic. By definition, a geodesic space is a length space such all pairs \(x, x'\) can be joined by a curve of length \(= d(x, x')\). So every \(X_k\) is a proper length space, and so is the limit \(Y\). By the Hopf-Rinow-theorem, every proper length space is geodesic. □

Example. A complete limit \(Y\) of geodesic spaces \(X_k\) that is not geodesic: \(Y\) is the metric graph constructed by joining two vertices with a sequence of edges \(e_n\) of length \(1 + \frac{1}{n}\) for \(n = 1, 2, \ldots\) \(X_k\) is obtained from \(Y\) by replacing the edge \(e_k\) by an edge of length 1.
Packing and covering

**Definition.** For a metric space $X$ and $\varepsilon > 0$ define the covering and packing numbers by

$$\text{cov}(X, \varepsilon) = \min \{ n \mid X \text{ can be covered by } n \text{ closed } \varepsilon\text{-balls} \}$$

$$\text{pack}(X, \varepsilon) = \sup \{ n \mid X \text{ contains } n \text{ disjoint } \frac{\varepsilon}{2}\text{-balls} \}.$$

**Lemma 1.** $\text{cov}(X, \varepsilon) \leq \text{pack}(X, \varepsilon)$.

Proof. If $B_{\varepsilon/2}(x_1), \ldots, B_{\varepsilon/2}(x_n)$ is a maximal disjoint set of $\varepsilon/2$-balls, then the balls $\bar{B}_\varepsilon(x_1), \ldots, \bar{B}_\varepsilon(x_n)$ cover $X$. □

**Lemma 2.** If $d_{GH}(X, Y) \leq \delta$, then

$$\text{cov}(X, \varepsilon) \geq \text{cov}(Y, \varepsilon + 2\delta)$$

$$\text{pack}(X, \varepsilon) \geq \text{pack}(Y, \varepsilon + 2\delta)$$

Proof. Use a correspondence with distortion $2\delta'$, $\delta' > \delta$. □
Totally bounded sets in \( M \)

Theorem. For a subset \( C \subseteq M \), the following are equivalent:

1. There is a constant \( D > 0 \) and a function \( N : (0, \infty) \to \mathbb{N} \) such that \( \text{diam}(X) \leq D \) and \( \text{pack}(X, \varepsilon) \leq N(\varepsilon) \) for all \( X \in C \).
2. Same as (1), but replace \( \text{pack}(X, \varepsilon) \) by \( \text{cov}(X, \varepsilon) \).
3. \( C \) is totally bounded with respect to \( d_{GH} \).

Proof

(3) \( \Rightarrow \) (1) Recall that (3) means \( \forall \delta > 0 \exists \) finite \( \delta \)-dense subset in \( C \).
Consider such a subset \( C' \subseteq C \) and let \( D' \) and \( N'(\varepsilon) \) be upper bounds for \( \text{diam}(\cdot) \) and \( \text{pack}(\cdot, \varepsilon) \) on \( C' \).

Given \( X \in C \), take \( C \in C' \) such that \( d_{GH}(X, C) < \delta \). Then

\[
\text{diam}(X) \leq \text{diam}(C) + 2\delta \leq D' + 2\delta \\
\text{pack}(X, \varepsilon) \leq \text{pack}(C, \varepsilon - 2\delta) \leq N'(\varepsilon - 2\delta).
\]

(1) \( \Rightarrow \) (2) by Lemma 1.
Totally bounded sets in $\mathcal{M}$

(2)$\implies$(3) Fix $\varepsilon > 0$.

- The set $\mathcal{F}$ of finite metric spaces with at most $N(\varepsilon)$ elements and diameters $\leq D$ is totally bounded with respect to $d_{GH}$.

Proof. With each $F \in \mathcal{F}$ that has $N \leq N(\varepsilon)$ elements associate “the” $N \times N$ matrix $\Delta(F) = (d_{ij})$ of pairwise distances of all the points in $F$. These matrices have entries bounded by $D$, so they form a totally bounded set in $\mathbb{R}^{N \times N}$. If $\Delta(F)$ and $\Delta(F')$ are $\delta$-close, then there is a correspondence (in fact a bijection) between $F$ and $F'$ with distortion $< \delta$, and so $d_{GH}(F, F') \leq \delta$.

- This set $\mathcal{F}$ is $\varepsilon$-dense for $\mathcal{C}$.

Proof. Given $X \in \mathcal{C}$, cover it by $\leq N(\varepsilon)$ balls of radius $\varepsilon$. Let $F$ be the set of centers of these balls. Then $F \in \mathcal{F}$, and $d_{GH}(X, F) \leq \varepsilon$.

- This works for every $\varepsilon > 0$. Conclude that every sequence in $\mathcal{C}$ contains a Cauchy subsequence (diagonal argument). $\square$
Completeness of $\mathcal{M}$

**Lemma** (Gromov). For every totally bounded subset $\mathcal{C} \subseteq \mathcal{M}$ there is a compact subset $K \subseteq \ell^\infty$ such that every $X \in \mathcal{C}$ admits an isometric embedding into $K$.

As a corollary we obtain:

**Theorem.** The metric space $(\mathcal{M}, d_{GH})$ is complete.

**Proof**

Apply the Lemma to the set of terms $\{X_k \mid k \in \mathbb{N}\} \subseteq \mathcal{M}$ of a given Cauchy sequence. The lemma says that the $X_k$ have isometric copies $X'_k$ contained in some compact $K \subseteq \ell^\infty$. The Hausdorff compactness theorem applied to $K$ provides a subsequence $X'_{k_j}$ that $d_H^\infty$-converges to a compact $X \subseteq K$. This implies that $X_{k_j} \xrightarrow{GH} X$. Since the sequence was Cauchy, $X_k \xrightarrow{GH} X$. □
The space \( \ell^\infty(A) \)

It remains to prove Gromov's lemma. Instead of embeddings into \( \ell^\infty = \ell^\infty(\mathbb{N}) \), we construct embeddings into \( \ell^\infty(A) \) for some other countably infinite set \( A \). This is the Banach space of all bounded functions \( f : A \to \mathbb{R} \) with the sup-norm. It is isometric to \( \ell^\infty(\mathbb{N}) \).

**Definition.** Fix a sequence \( \mathbf{N} = (N_1, N_2, \ldots) \) of positive integers and consider the sets

\[
A_1 = \{ (n_1) \mid n_1 = 1, \ldots, N_1 \}
\]

\[
A_2 = \{ (n_1, n_2) \mid n_1 = 1, \ldots, N_1; \ n_2 = 1, \ldots, N_2 \}
\]

\[
A_3 = \{ (n_1, n_2, n_3) \mid n_1 = 1, \ldots, N_1; \ n_2 = 1, \ldots, N_2; \ n_3 = 1, \ldots, N_3 \}
\]

etc., and then

\[
A = \bigcup_{j=1}^\infty A_j
\]

The elements \( f \in \ell^\infty(A) \) are bounded families of numbers

\[
(f(a))_{a \in A} = (f_a)_{a \in A}
\]

where the indices \( a \) are of the form \( a = (n_1, \ldots, n_k) \). We write \( f(n_1, \ldots, n_k) \) instead of \( f((n_1, \ldots, n_k)) \).
Compact sets in $\ell^\infty(A)$

**Sublemma.** Let $D > 0$, and let $e = (\varepsilon_1, \varepsilon_2, \ldots)$ be a sequence of positive numbers such that $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. Consider the subset $F = F_{D,e} \subseteq \ell^\infty(A)$ defined by the following conditions:

1. $0 \leq f(n_1) \leq D$ for $n_1 = 1, \ldots, N_1$
2. $|f(n_1, \ldots, n_k, n_{k+1}) - f(n_1, \ldots, n_k)| \leq \varepsilon_k$

for all $k$ and all $(n_1, \ldots, n_{k+1}) \in A$. Then $F$ is compact.

**Proof**

$F$ is closed in $\ell^\infty(A)$, hence complete. Therefore it suffices to show that $F$ is totally bounded. Note that we have finite dimensional subspaces

$$
\ell^\infty(A_1 \cup \cdots \cup A_k) \hookrightarrow \ell^\infty(A).
$$

- $F \cap \ell^\infty(A_1 \cup \cdots \cup A_k)$ is compact.
- By condition (2), $F$ is contained in the $\hat{\varepsilon}_k$-neighbourhood of $F \cap \ell^\infty(A_1 \cup \cdots \cup A_k)$, where $\hat{\varepsilon}_k = \varepsilon_k + \varepsilon_{k+1} + \cdots \to 0$ as $k \to \infty$.
- Using this, every sequence in $F$ has a Cauchy subsequence (diagonal sequence argument). So $F$ is totally bounded. □
Recall the statement: For every totally bounded \( C \subseteq M \) there is a compact \( K \subseteq \ell^\infty(\mathbb{N}) \) such that every \( X \in C \) admits an isometric embedding into \( K \).

Proof

▶ Choose \( D > 0 \) and a function \( N : (0, \infty) \rightarrow \mathbb{N} \) such that \( \text{diam}(X) \leq D \) and \( \text{cov}(X, \varepsilon) \leq N(\varepsilon) \) for all \( X \in C \).

▶ Take a decreasing sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \) of positive numbers such that \( \sum_{j=1}^{\infty} \varepsilon_j < \infty \), and let \( N_j := N(\varepsilon_j) \).

▶ Using this sequence \( N_1, N_2, \ldots \), define \( A \) as before, and let

\[
K := F_{D,2\varepsilon} \subseteq \ell^\infty(A) \cong \ell^\infty(\mathbb{N})
\]

be the compact set described in the sublemma. We show that every \( X \in C \) embeds isometrically into this \( K \).
Proof of Gromov lemma (end)

Cover $X$ with $N_1$ balls of radius $\varepsilon_1$, say $B(x_{n_1}, \varepsilon_1)$ where $n_1 = 1, \ldots, N_1$.

Next cover each of the balls $B(x_{n_1}, \varepsilon_1)$ with $N_2$ balls of radius $\varepsilon_2$, say $B(x_{n_1n_2}, \varepsilon_2)$ where $n_2 = 1, \ldots, N_2$.

Then cover each of these balls $B(x_{n_1n_2}, \varepsilon_2)$ with $N_3$ balls of radius $\varepsilon_3$, say $B(x_{n_1n_2n_3}, \varepsilon_3)$ where $n_3 = 1, \ldots, N_3$. Continue like this.

The centers $x_a, a \in A$ of all these balls form a dense set in $X$. Therefore the Fréchet-embedding $\phi : X \to \ell^\infty(A)$ defined by

$$\phi(x) = (\phi_a(x))_{a \in A} = (d(x, x_a))_{a \in A}$$

is isometric.

Verify that $\phi(X) \subseteq F_{D, 2\varepsilon}$: Condition (1) holds since $d(x, x_{n_1}) \leq D$, and condition (2) because of

$$|d(x, x_{n_1\ldots n_kn_{k+1}}) - d(x, x_{n_1\ldots n_k})| \leq d(x_{n_1\ldots n_kn_{k+1}}, x_{n_1\ldots n_k}) \leq 2\varepsilon_k \square$$
Topics

- For non-compact spaces: pointed GH-convergence
- What Gromov does with it: groups of polynomial growth
- Precompact sets of Riemannian manifolds: the Bishop-Gromov relative volume comparison
- If suitable \( X \) and \( Y \) are GH-close, then \( X \) and \( Y \) are diffeomorphic, homeomorphic, homotopy equivalent; corresponding finiteness results; Cheeger, Grove, Petersen, Anderson, Perelman et.al.
- Continuity of quantities under GH-limit; Anderson’s estimate on the harmonic radius of a Riemannian manifold
- Collapsing and fibration theorems: \( Y \) fixed, \( X \) close to \( Y \), then \( X \) fibers over \( Y \) with infranil fiber; Gromov, Fukaya, Yamaguchi
- Structure of limit spaces of Riemannian manifolds under curvature bounds; Fukaya, Cheeger, Colding et.al.