**Problem 2.**

Various simplicial constructions encode topologically the global combinatorial structure of $G$, for example:

- The clique complex $Cl(G)$ is a simplicial complex whose faces are the complete subgraphs of $G$.
- The independence complex $Ind(G)$ is a simplicial complex whose faces are the independent sets of $G$.

**Example.** The clique complex $Cl(G)$ is the union of two disks glued along the boundary, that is $S^2$.

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**Stability**

The first example: cycles

We want to know the homotopy types of $Cl(T_n)$.

With the parameter $k = n - 2r - 1$ consider the complements $T_n - K_r$ known as the circular complete graphs.

So the equivalent question is to identify $\text{Ind}(T_n) - K_r$.

Since $T_n$ are triangle-free we can exhibit their independence complexes $\text{Ind}(T_n)$ as suspensions:

1. Every maximal independent set contains 0 or its neighbour: $\text{Ind}(T_n) = 0(0) \cup \bigcup_{u \neq 0} u(u)$.
2. Since the vertex 0 is not in any triangle, $N(0)$ is a simplex in $\text{Ind}(T_n)$ and both summands are contractible (as a result of Barmak). Therefore $\text{Ind}(T_n) = \Sigma K$, where $K = 0(0) \cup \bigcup_{u \neq 0} u(u)$.
3. The constraints defining $K$ can be encoded in the independence complex of some graph $S_{n,k}$.
4. Steps 1), 2), 3) can be repeated for the new graph $S_{n,k}$. The outcome can be identified with $T_{2k+1}$.

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**Universality**

For every $r \geq 1$ and every finite simplicial complex $K$ there exists a graph $G$ with a homotopy equivalence $Cl(G) \simeq K$.

- A folklore result when $r = 1$, even with a homeomorphism $K = Cl(G)$ for $G = (bdK)^3$, the 1-skeleton of the barycentric subdivision of $K$.
- For $r \geq 2$ we use $G = (bdK)^3$, the 1-skeleton of a large iterated subdivision (roughly $s = (K^3)^3$ isomorphic).
- Cover $Cl(G')$ with subcomplexes $Cl(G_{i,j})$ where $G_i$ consists of the vertices which belong to the open star of $i$ in $K$ (see fig.).
- Dochtermann proved $G_i$ is dismantlable. It follows also for $(G_{i,j})'$, so the covering complexes $Cl(G_{i,j})'$ are contractible. An analysis of the intersections of $G_i$ and the nerve lemma do the rest.

Another representation, which is off by just some 2-cells, comes from an equivalence $Cl((bdK)^3) \simeq Cl(G) \vee \bigcup_{r=1}^{\infty} S^2$.

- $(r)$ - number of triangles in $G$.
- $\pi_2 \text{Ind}(G) - \text{an edge subdivision of } G$ by $r - 1$ new vertices on each edge.

**The answer**

For $\frac{1}{2} < r < \frac{3}{2}$ the clique complexes $Cl(C_r)$ of cycle powers satisfy $Cl(C_r) \simeq \Sigma^{r-\frac{3}{2}} C_r \simeq \Sigma^{r-\frac{3}{2}} + \Sigma^{r+\frac{3}{2}}$.

It means that all $Cl(C_r)$ are generated by the double suspension operator $\Sigma^2$, acting along lines of slope $(2,1)$, as shown by the arrows below.

For any $n \geq 1$ and $0 \leq r \leq \frac{1}{2}$ $Cl(C_r) \simeq \bigcup_{i=1}^{r+1} S^0$ if $\frac{3}{2} < r < \frac{3}{2}$ for some $I \geq 0$.

**Bringing the two together**

The space $Cl(G')$ is the Vietoris-Rips complex of subsets of diameter at most $r$ in $G$.

- Problem 1. What are the interesting features of the spaces $Cl(G')$ for higher $r$?
- Problem 2. What are the interesting features of the induced inclusions $Cl(G) \rightarrow Cl(G') \rightarrow Cl(G'')$?

Warm-up exercise. Prove that the induced map of fundamental groups

$$\pi_1(Cl(G)) \rightarrow \pi_1(Cl(G'))$$

is surjective.