Ditopological texture spaces and fuzzy topology, I. Basic concepts

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Received 19 February 2001; received in revised form 11 February 2004; accepted 16 February 2004

Abstract

This is the first of three papers which develop various fundamental aspects of the theory of ditopological texture spaces in a categorical setting and present important links with the theory of \(L\)-topological spaces. The authors begin by defining the notion of q-sets, which together with the p-sets considered earlier, enable the formulation of a powerful concept of duality. This plays an important role in the theory of direlations and difunctions, which is described here in detail. Difunctions are then taken as the morphisms of a category \(dfTex\), whose objects are texture spaces. Several important subcategories are defined and the closely related construct \(fTex\) defined. Some properties of the functors between these categories are obtained.

MSC: 54A40; 54B30; 03E72; 06A99

Keywords: Texture; Hutton algebra; Hutton texture; \(L\)-Fuzzy subset; Product texture; Disjoint union; Direlation; Difunction; Category; Construct; Adjoint functor

1. Introduction

Ditopological texture spaces were introduced by the first author as a natural extension of the work of the second author on the representation of lattice-valued topologies by bitopologies [9–11]. However, in place of the full lattice of subsets of some base set \(S\), attention is now focused on a suitable subfamily of subsets, called a texturing of \(S\), and within this context bitopologies are replaced by dichotomous topologies, or ditopologies for short. Some results on ditopological texture spaces may be found in [3,4,7,8,14,16]. Our aim in this series of papers is to place the investigation...
of textures and ditopological texture spaces on a firm categorical footing, and in particular to use
the representations of fuzzy sets as texture spaces discussed in [5,6] to study the relation between
\(\mathbb{L}\)-topologies and ditopological texture spaces in greater detail.

Let \(S\) be a set. We recall [2–8,14,16] that a texturing on \(S\) is a point separating, complete,
completely distributive lattice \(\mathcal{S}\) of subsets of \(S\) with respect to inclusion, which contains \(S, \emptyset\), and
for which meet \(\wedge\) coincides with intersection \(\cap\) and finite joins \(\vee\) coincide with unions \(\cup\). The pair
\((S, \mathcal{S})\) is then known as a texture.

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only
if \(\mathcal{S}\) is closed under arbitrary unions. In this case \((S, \mathcal{S})\) is said to be plain.

In general, a texturing of \(S\) need not be closed under set complementation, but it may be that there
exists a mapping \(\sigma : \mathcal{S} \rightarrow \mathcal{S}\) satisfying \(\sigma(\sigma(A)) = A, \forall A \in \mathcal{S}\) and \(A \subseteq B \implies \sigma(B) \subseteq \sigma(A), \forall A, B \in \mathcal{S}\).
In this case \(\sigma\) is called a complementation on \((S, \mathcal{S})\), and \((S, \mathcal{S}, \sigma)\) is said to be a complemented
texture.

For \(s \in S\) the set \(P_s\) is defined by \(P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}\), and is therefore the smallest element
of \(\mathcal{S}\) containing \(s\). As may be expected, the sets \(P_s\) play an important role in the study of texture
spaces. Recall that \(M \in \mathcal{S}\) is called a molecule if \(M \neq \emptyset\) and \(M \subseteq A \cup B, A, B \in \mathcal{S}\) implies \(M \subseteq A\) or
\(M \subseteq B\). Molecules are also known as coprimes and as \(\lor\)-irreducible elements, these concepts being
equivalent in the current setting [12]. Clearly, the sets \(P_s, s \in S\) are molecules, and the texture \((S, \mathcal{S})\)
is called simple if these are the only molecules in \(\mathcal{S}\).

It will be convenient to introduce a second element of \(\mathcal{S}\) associated with \(s \in S\), namely the set
\[
Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.
\]
We will call the sets \(P_s, Q_s, s \in S\), respectively, the p-sets and q-sets of \((S, \mathcal{S})\).

Example 1.1. (1) For any set \(X\), \((X, \mathcal{P}(X), \pi_X), \pi_X(Y) = X \setminus Y\) for \(Y \subseteq X\), is the complemented
discrete texture representing the usual set structure of \(X\). Clearly, \(P_x = \{x\}, Q_x = X \setminus \{x\}\) for all
\(x \in X\). Hence, \((X, \mathcal{P}(X))\) is both plain and simple.

(2) (See [5].) Let \((\mathbb{L}', \mathcal{L}')\) be a Hutton algebra, that is a complete, completely distributive lattice \(\mathbb{L}\)
equipped with an order-reversing involution \(\hat{\cdot}\). If we denote by \(M_{\mathbb{L}}\) the set of molecules in \(\mathbb{L}\), set
\(\hat{a} = \{m \in M_{\mathbb{L}} \mid m \leq a\}\) for \(a \in \mathbb{L}\), and \(\mathcal{M}_{\mathbb{L}} = \{\hat{a} \mid a \in \mathbb{L}\}\), then \((M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})\) is a simple texture. Moreover,
\(\mu_{\mathbb{L}} : \mathcal{M}_{\mathbb{L}} \rightarrow \mathcal{M}_{\mathbb{L}}\) defined by \(\mu_{\mathbb{L}}(\hat{a}) = (\hat{d}')\) is a complementation on \((M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})\). We will refer to the
complemented texture \((M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})\) as the Hutton texture of \((\mathbb{L}', \mathcal{L}')\) (called the complemented fuzzy
texture of \((\mathbb{L}', \mathcal{L}')\) in [5]). Conversely, every complemented simple texture \((S, \mathcal{S}, \sigma)\) is the Hutton
texture of \((\mathcal{S}, \sigma)\), regarded as a Hutton algebra.

(3) Let \(L = (0, 1], \mathcal{L} = \{(0, r) \mid r \in [0, 1]\}\) and \(\lambda((0, r)) = (0, 1 - r), r \in [0, 1]\). Clearly \((L, \mathcal{L}, \lambda)\) is
the Hutton texture of \((\mathbb{L}', \mathcal{L}')\), where \(\mathbb{L} = [0, 1]\) with its usual order and \(r' = 1 - r\) for \(r \in \mathbb{L}\). Here
\(P_r = Q_r = (0, r)\) for all \(r \in L\). Although \((L, \mathcal{L})\) is simple it is clearly not plain.

(4) (See [5].) Let \(X\) be a non-empty set and \((\mathcal{L}', \mathcal{L})\) a fixed Hutton algebra. Then the molecules
of the Hutton algebra \(\mathcal{W}_X = \mathbb{L}^X\) of \(\mathbb{L}\)-valued subsets of \(X\) [13] are just the “fuzzy points” \(x_m\) for
\(x \in X\) and \(m \in M_{\mathbb{L}}\). These are in one to one correspondence with the points of \(W_X = X \times M_{\mathbb{L}}\), and for
\(f \in \mathcal{W}_X\) we now have \(\hat{f} = \{(x, m) \in W_X \mid m \leq f(x)\}\). Setting \(\mathcal{W}_X = \{f \mid f \in \mathcal{W}_X\}\) and \(\omega_X(f) = (f')\),
\(f'(x) = f(x)'\), gives us the Hutton texture \((W_X, \mathcal{W}_X, \omega_X)\) of \(W_X\).
(5) For \( \mathbb{I} = [0,1] \) define \( \mathcal{I} = \{ [0,t] \mid t \in [0,1] \} \cup \{ [0,t) \mid t \in [0,1] \} \), \( \nu([0,t]) = [0,1 - t) \) and \( \nu([0,t)) = [0,1 - t], t \in [0,1] \). Again \((\mathbb{I}, \mathcal{I}, \nu)\) is a complemented texture, which we will refer to as the unit interval texture. Here \( P_t = [0,t] \) and \( Q_t = [0,t) \) for all \( t \in \mathbb{I} \). This time \((\mathbb{I}, \mathcal{I})\) is plain but not simple since the sets \( Q_s, 0 < s \leq 1 \), are also molecules.

(6) Let \( B = \{ a, b, c \}, \mathcal{B} = \{ \emptyset, \{ a \}, \{ a, b \}, B \}, \beta(\emptyset) = B, \beta(\{ a \}) = \{ a \}, \beta(\{ a, b \}) = \{ a \} \) and \( \beta(B) = \emptyset \). Then \((B, \mathcal{B}, \beta)\) is a complemented texture. Clearly, \( P_a = \{ a \}, P_b = \{ a, b \}, P_c = B \) and \( Q_a = \emptyset, Q_b = \{ a \}, Q_c = \{ a, b \} \) so \((B, \mathcal{B})\) is both plain and simple.

Examples 1.1(2) and (4) show the very close relationship between complemented simple textures, Hutton algebras and \( L \)-valued subsets of a set \( X \). This, indeed, was the original motivation for the development of the notion of texture, and we will return to this relationship throughout these papers. On the other hand, the unit interval texture \((\mathbb{I}, \mathcal{I}, \nu)\) is not simple and therefore does not arise from a Hutton algebra in the way that, for example, \((L, \mathcal{L}, \lambda)\) does. This shows that the notion of texture is more general than that of Hutton algebra, and since we will be seeing evidence that non-simple textures such as \((\mathbb{I}, \mathcal{I}, \nu)\) play an important role in the theory (see also [16]), we believe that textures will provide significant new results.

For the discrete texture \((X, \mathcal{P}(X))\), \( \mathbb{Q} \) is the largest set in \( \mathcal{P}(X) \) not containing \( x \), but this is certainly not the case in general. From Example 1.1(3) we see that for a texture \((S, \mathcal{S})\) we can have \( s \in \mathbb{Q}_s \), and even \( \mathbb{Q}_s = S \), and it is not difficult to show that \( s \notin \mathbb{Q}_s \) for all \( s \in S \) if and only if \((S, \mathcal{S})\) is plain. Nonetheless, the p-sets and q-sets do set up a form of duality which enables concepts related to set complementation to be encoded in general textures. The importance of this notion of duality will be apparent in this paper and its sequel, and also plays a basic role in [2,4,14,16].

To establish the basic results concerning p-sets and q-sets, and thereby expose the nature of this duality, it will be useful to introduce the auxiliary notion of the core of a set \( A \) in \( \mathcal{I} \). This is the set \( A^p \) defined by

\[
A^p = \bigcap \left\{ \bigcup \{ A_j \mid j \in J \} \mid \{ A_j \mid j \in J \} \subseteq \mathcal{S}, \ A = \bigvee \{ A_j \mid j \in J \} \right\}.
\]

For plain textures we clearly have \( A^p = A \), but for the texture \((L, \mathcal{L})\) of Example 1.1(3), for instance, we have \( (0,r)^p = (0,r) \notin \mathcal{L} \). The relation between these concepts in an arbitrary texture is given below.

**Theorem 1.2.** (1) \( s \notin A \implies A \subseteq \mathbb{Q}_s \implies s \notin A^p \) for all \( s \in S, A \in \mathcal{S} \).

(2) \( A^q = \{ s \mid A \notin \mathbb{Q}_s \} \) for all \( A \in \mathcal{S} \).

(3) For \( A_j \in \mathcal{S}, j \in J \) we have \( (\bigcup_{j \in J} A_j)^q = \bigcup_{j \in J} A_j^q \).

(4) \( A \) is the smallest element of \( \mathcal{S} \) containing \( A^p \) for all \( A \in \mathcal{S} \).

(5) For \( A, B \in \mathcal{S} \), if \( A \nsubseteq B \) then there exists \( s \in S \) with \( A \nsubseteq \mathbb{Q}_s \) and \( P_s \nsubseteq B \).

(6) \( A = \bigcap \{ \mathbb{Q}_s \mid P_s \nsubseteq A \} \) for all \( A \in \mathcal{S} \).

(7) \( A = \bigvee \{ P_s \mid A \nsubseteq \mathbb{Q}_s \} \) for all \( A \in \mathcal{S} \).

**Proof.** (1) The first implication follows immediately from the definitions. For the second, note that \( A \subseteq \mathbb{Q}_s = \bigvee \{ P_s \mid s \notin P_s \} \) implies \( A = \bigvee \{ A \cap P_s \mid s \notin P_s \} \) by complete distributivity, whence \( s \notin \bigcup \{ A \cap P_s \mid s \notin P_s \} \supseteq A^p \) as required.
Proposition 1.3. Let \((S_i, \mathcal{S}_i), j \in J\) be textures and \((S, \mathcal{S})\) their product. Then for \(s = (s_j) \in S\),
\[
(1) \quad P_s = \bigcap_{j \in J} E(j, P_{s_j}) = \prod_{j \in J} P_{s_j}, \quad \text{and}
\]
\[
(2) \quad Q_s = \bigcup_{j \in J} E(j, Q_{s_j}).
\]

Proof. Equality (1) is immediate from the definitions, so we establish (2).

Firstly, suppose \(Q_s \not\subseteq \bigcup_{j \in J} E(j, Q_{s_j})\). Then by the definition of \(Q_s\) there exists \(t \in S\) satisfying \(P_t \not\subseteq P_s\) and \(P_t \not\subseteq \bigcup_{j \in J} E(j, Q_{s_j})\). On the one hand, there exists \(j \in J\) with \(P_{s_j} \not\subseteq P_t\) by (1) and on the other \(P_{t_j} \not\subseteq Q_{s_j}\). This leads to a contradiction by Theorem 1.2(1), so \(Q_s \subseteq \bigcup_{j \in J} E(j, Q_{s_j})\).
To establish the reverse inclusion let $T = \{ t \in S \mid s \notin P_t \}$. Then

$$Q_t = \bigvee_{i \in T} P_i = \bigvee_{i \in T} \bigcap_{j \in J} E(j, P_{t(j)}) = \bigcap_{\gamma \in J^T} \bigvee_{i \in T} E(\gamma(t), P_{t(\gamma)}),$$

since $\mathcal{J}$ is completely distributive. Also, by [5, Lemma 2.3], we may write

$$\bigvee_{i \in T} E(\gamma(t), P_{t(\gamma)}) = \bigcup_{j \in J} E\left(j, \bigvee\{ P_t \mid \gamma(t) = j \}\right).$$

Hence, if $\bigcup_{j \in J} E(j, Q_s) \nsubseteq Q_s$ then there exist $k \in J$ and $\gamma \in J^T$ so that $E(k, Q_s) \nsubseteq \bigcup_{j \in J} E(j, \bigvee\{ P_t \mid \gamma(t) = j \})$. Clearly, for $j \in J$, $j \neq k$, we may choose $u_j \in S_j$ with $P_{u_j} \nsubseteq \bigvee\{ P_t \mid \gamma(t) = j \}$. Also, since $Q_s \nsubseteq \bigvee\{ P_t \mid \gamma(t) = k \}$ we have $u_k \in S_k$ satisfying $s_k \notin P_{u_k}$ and $P_{u_k} \nsubseteq \bigvee\{ P_t \mid \gamma(t) = k \}$ by the definition of $Q_s$. Consider $u = (u_j) \in S$. Clearly $s \notin P_u$, so $u \in T$. However we now have $\gamma(u) = k$ or $\gamma(u) = j \neq k$, and both possibilities lead to an immediate contradiction. This completes the proof of the proposition. □

**Corollary 1.4.** Let $(S, \mathcal{J})$ be the product of the textures $(S_j, \mathcal{J}_j)$, $j \in J$.

1. $S^0 = \prod_{j \in J} S_j^0$.
2. For $s, t \in S$, $P_s \nsubseteq Q_t \iff P_s \nsubseteq Q_t \forall j \in J$.

**Proof.** Straightforward. □

The sum of disjoint textures was also defined in [5]. If for the textures $(S_j, \mathcal{J}_j)$, $j \in J$, the sets $S_j$ are not necessarily pairwise disjoint, we shall mean by the *disjoint sum* of the textures $(S_j, \mathcal{J}_j)$ the sum of the disjoint textures $(S_j \times \{ j \}, \{ A \times \{ j \} \mid A \in \mathcal{J}_j \})$. Hence the disjoint sum is $(S, \mathcal{J})$ where $S = \bigcup_{j \in J}(S_j \times \{ j \})$ and $\mathcal{J} = \{ A \subseteq S \mid A \cap (S_j \times \{ j \}) = A_j \times \{ j \} \Rightarrow A_j \in \mathcal{J}_j, \forall j \in J \}$. We have:

**Proposition 1.5.** Let $(S_j, \mathcal{J}_j)$, $j \in J$, be textures and $(S, \mathcal{J})$ their disjoint sum. Then for $s = (s_j, j) \in S$ we have

1. $P_s = P_{s_j} \times \{ j \}$, and
2. $Q_s = (Q_{s_j} \times \{ j \}) \cup \bigcup_{k \in J \setminus \{ j \}}(S_k \times \{ k \})$

where $P_{s_j}$, $Q_{s_j}$ denote the $p$-sets and $q$-sets, respectively, for $s_j$ in the texture $(S_j, \mathcal{J}_j)$.

**Proof.** Straightforward. □

The reader is referred to [12] for terms from lattice theory not defined here. Generally we follow the terminology of [1] for general concepts relating to category theory. If $A$ is a category, $\text{Ob } A$ will denote the class of objects and $\text{Mor } A$ the class of morphisms of $A$. We will sometimes use the notation $A(A_1, A_2)$ for the set of morphisms in $A$ from $A_1$ to $A_2$. The categorical results presented in this paper form part of the third author’s continuing Ph.D. studies in the Department of Mathematics of Hacettepe University.
The first two papers in this series, subtitled Basic Concepts and Topological Considerations, were originally submitted as a single paper. Consequently, the planned sequel, Separation Axioms, will now be the third and not the second paper of this series.

2. Relations and functions between textures

In order to define categories whose objects are textures it is necessary to describe the morphisms between textures. One natural approach would be to define a morphism with domain \((S, \mathcal{I})\) and range \((T, \mathcal{F})\) to be a function from \(S\) to \(T\), but we should certainly require some form of comparability with the texturings \(\mathcal{I}, \mathcal{F}\), and it not immediately clear which of the very many possible such conditions would be appropriate. It will transpire that there is indeed at least one such class of functions which are important in this respect, but we will obtain this class indirectly by considering a fundamental notion of morphism which is not in general a function from \(S\) to \(T\) at all. This is the notion of difunction between textures, which in turn is derived from a notion of direlation in much the same way that normal (point) functions are derived from binary relations in the usual sense.

The current theory of direlations and difunctions has been developed by the first author over the past few years and the results collected in a preprint [2], which has circulated mainly within the Hacettepe topology group. This has provided the stimulus for several studies, most importantly [16], where the notion of direlation is used as the foundation of a concept of textural uniformity. A systematic account with mainly new proofs of a significant portion of [2] appears here in print.

2.1. Relations, corelations and direlations

Let \((S, \mathcal{I})\) and \((T, \mathcal{F})\) be textures. What are we to mean by a (binary) relation from \((S, \mathcal{I})\) to \((T, \mathcal{F})\)? If we specialize to \((X, \mathcal{P}(X))\) and \((Y, \mathcal{P}(Y))\) and note that \(\mathcal{P}(X \times Y)\) (the set of binary relations from \(X\) to \(Y\)) is the product \(\mathcal{P}(X) \otimes \mathcal{P}(Y)\), then we might think of looking in \(\mathcal{I} \otimes \mathcal{F}\) for the relations from \((S, \mathcal{I})\) to \((T, \mathcal{F})\). However, it is easy to see that in general this is too small a set to support a useful theory. For example, for the basic texture \((L, \mathcal{I})\) of Example 1.1(3), the only set in \(\mathcal{I} \otimes \mathcal{I}\) containing the diagonal of \(L \times L\) is \(L \times L\) itself. We could, of course, consider arbitrary elements of \(\mathcal{P}(S) \otimes \mathcal{P}(T)\) (i.e., arbitrary subsets of \(S \times T\)), and impose conditions relating to the textures \(\mathcal{I}\) and \(\mathcal{F}\), but we will find it convenient to take a middle course and consider elements of the texturing \(\mathcal{P}(S) \otimes \mathcal{F}\). It remains now only to impose conditions relating to \(\mathcal{I}\), and we do this in two ways producing dual notions of relation and corelation. The clue to obtaining a concept of “relation” for which symmetry, as well as reflexivity and transitivity, can be defined, is now to consider pairs, called direlations, consisting of a relation and a corelation.

Before going into details let us establish some notation. For \(s \in S\) \((t \in T)\) \(P_s, Q_s\) \((P_t, Q_t)\) will denote the p-sets and q-sets for \((S, \mathcal{I})\) (respectively \((T, \mathcal{F})\)), while \(\overline{P}_{(s,t)}, \overline{Q}_{(s,t)}\) will denote the p-sets and q-sets for the product \((S \times T, \mathcal{P}(S) \otimes \mathcal{F})\) of the textures \((S, \mathcal{P}(S))\) and \((T, \mathcal{F})\). This leaves \(P_{(s,t)}, Q_{(s,t)}\) available to denote the p-sets, q-sets of \((S \times T, \mathcal{I} \otimes \mathcal{F})\). Note that, by Proposition 1.3, \(\overline{P}_{(s,t)} = \{s\} \times P_t\) and \(\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t)\), whence \(\overline{P}_{(s,t)} \not\subseteq \overline{Q}_{(s',t')}\iff s = s'\) and \(P_s \not\subseteq Q_{s'}\). Likewise, \(\overline{P}_{(t,s)} \not\subseteq \overline{Q}_{(t',s')}\iff t = t'\) and \(P_s \not\subseteq Q_{s'}\), where \(\overline{P}_{(t,s)}, \overline{Q}_{(t,s)}\) denote the p-sets,
q-sets, respectively, in $(T \times S, \mathcal{P}(T) \otimes \mathcal{I})$. These facts will be used without further comment in the sequel. Now we may give

**Definition 2.1.** Let $(S, \mathcal{I}, (T, \mathcal{F}))$ be textures. Then

1. $r \in \mathcal{P}(S) \otimes \mathcal{F}$ is called a relation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$ if it satisfies
   
   \[ R1 \ r \not\subseteq Q_{(s,t)} \rightarrow r \not\subseteq Q_{(r',t)}. \]
   
   \[ R2 \ r \not\subseteq Q_{(s,t)} \rightarrow \exists s' \in S \text{ such that } P_s \not\subseteq Q_{s'} \text{ and } r \not\subseteq Q_{(s',t)}. \]

2. $R \in \mathcal{P}(S) \otimes \mathcal{F}$ is called a correlation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$ if it satisfies
   
   \[ CR1 \ P_{(s,t)} \not\subseteq R \rightarrow P_s \not\subseteq Q_{s'} \rightarrow P_{(s',t)} \not\subseteq R. \]
   
   \[ CR2 \ P_{(s,t)} \not\subseteq R \rightarrow \exists s' \in S \text{ such that } P_{s'} \not\subseteq Q_s \text{ and } P_{(s',t)} \not\subseteq R. \]

3. A pair $(r, R)$, where $r$ is a relation and $R$ a correlation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$, is called a direlation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$.

Normally relations will be denoted by lower case and correlations by upper case letters, as in the above definition. If $(r_1, R_1), (r_2, R_2)$ are both direlations from $(S, \mathcal{I})$ to $(T, \mathcal{F})$ we write $(r_1, R_1) \subseteq (r_2, R_2)$ if $r_1 \subseteq r_2$ and $R_2 \subseteq R_1$.

For a general texture $(S, \mathcal{I})$ we define

\[ i_{(S, \mathcal{I})} = \bigvee \{ P_{(s,t)} \mid s \in S \} \quad \text{and} \quad I_{(S, \mathcal{I})} = \bigcap \{ Q_{(s,t)} \mid s \in S \}. \]

It is trivial to verify that $i_{(S, \mathcal{I})} \not\subseteq Q_{(s,t)} \iff P_s \not\subseteq Q_{s'} \iff i_{(S, \mathcal{I})} \not\subseteq Q_{s'}$, and $I_{(S, \mathcal{I})} \not\subseteq Q_{(s,t)} \iff P_{(s,t)} \not\subseteq R \iff I_{(S, \mathcal{I})} \not\subseteq R$. Moreover, it will be clear from the context whether it is applied to a relation, correlation or direlation.

**Definition 2.2.** The direlation $(i_{(S, \mathcal{I})}, I_{(S, \mathcal{I})})$ is called the identity direlation on $(S, \mathcal{I})$.

Where there can be no confusion we will denote the identity direlation on $(S, \mathcal{I})$ by $(i, I)$, or even by $(i, i)$. A direlation $(r, R)$ on $(S, \mathcal{I})$ is called reflexive if $(i, I) \subseteq (r, R)$. In particular the identity direlation is reflexive.

**Definition 2.3.** Let $(S, \mathcal{I}, (T, \mathcal{F}))$ be textures and $(r, R)$ a direlation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$. Then the direlation $(r, R)^\rightarrow = (R^\rightarrow, r^\rightarrow)$ from $(T, \mathcal{F})$ to $(S, \mathcal{I})$ defined by

\[ r^\rightarrow = \bigcap \{ Q_{(s,t)} \mid r \not\subseteq O_{(s,t)} \}, \quad R^\rightarrow = \bigvee \{ P_{(t,s)} \mid P_{(s,t)} \not\subseteq R \} \]

is called the inverse of $(r, R)$. Likewise, $r^\rightarrow$ is called the inverse of $r$ and $R^\rightarrow$ the inverse of $R$.

It is trivial to verify that $R^\rightarrow$ is indeed a relation and that $r^\rightarrow$ is a correlation from $(T, \mathcal{F})$ to $(S, \mathcal{I})$. No confusion will arise from the use of the single symbol $\rightarrow$ to denote these different inverses, since it will be clear from the context whether it is applied to a relation, correlation or direlation.

A direlation $(r, R)$ on $(S, \mathcal{I})$ is called symmetric if $(r, R)^\rightarrow = (r, R)$. For any texture $(S, \mathcal{I})$, the identity direlation is clearly symmetric.

**Lemma 2.4.** Let $(r, R)$ be a direlation from $(S, \mathcal{I})$ to $(T, \mathcal{F})$. Then

1. $r \not\subseteq Q_{(s,t)} \iff P_{(t,s)} \not\subseteq r^\rightarrow$ and $P_{(s,t)} \not\subseteq R \iff R^\rightarrow \not\subseteq Q_{(t,s)}$ for all $s \in S, t \in T$. 

De/Definition 2.5. For a second direlation \((m,M)\) from \((S,\mathcal{S})\) to \((T,\mathcal{F})\), \((r,R)\subseteq(m,M)\iff(r,R)^\leftarrow\subseteq(m,M)^\leftarrow\).

Proof. (1) Suppose \(r \not\subseteq \overline{O}_{(s,t)}\). Then by condition R2 we have \(s' \in S\) with \(P_s \not\subseteq Q_{s'}\) and \(r \not\subseteq \overline{O}_{(s',t)}\). By Definition 2.3 we have \(r^\leftarrow \subseteq \overline{O}_{(s',t)}\), whence \(P_{(t,s)} \not\subseteq r^\leftarrow\). Conversely, if \(P_{(t,s)} \not\subseteq r^\leftarrow\) then \(P_{(t,s)} \not\subseteq \overline{O}_{(s',t)}\) for some \(s' \in S\) with \(r \not\subseteq \overline{O}_{(s',t)}\), and then \(r \not\subseteq \overline{O}_{(s,t)}\) by RI. The proof of the second equivalence is dual to this.

(2) Bearing in mind that \(r^\leftarrow\) is a corelation we have
\[
(r^\leftarrow)^\leftarrow \not\subseteq \overline{O}_{(s,s')} \iff \overline{P}_{(s',s)} \not\subseteq r^\leftarrow \iff r \not\subseteq \overline{O}_{(s,s')}
\]
for all \(s,s' \in S\) by (1). This verifies that \((r^\leftarrow)^\leftarrow = r\), and the proof of the second equality is similar.

(3) Immediate from the definitions. \(\Box\)

It is interesting to ask what our definitions give when applied to discrete textures. We will call an element \(\varphi\) of \(\mathcal{P}(S \times T) = \mathcal{P}(S) \otimes \mathcal{P}(T)\), that is any binary relation from \(S\) to \(T\) in the ordinary sense, a point relation from \(S\) to \(T\). Clearly, \(\varphi\) is both a relation and a corelation from the texture \((S,\mathcal{P}(S))\) to the texture \((T,\mathcal{P}(T))\) in the sense of Definition 2.1. Under both interpretations the inverse is given by
\[
\varphi^\leftarrow = (\varphi^{-1})^c = (\varphi^c)^{-1},
\]
where \(\varphi^{-1} = \{(t,s) \mid (s,t) \in \varphi\}\) is the usual point inverse, and \(^c\) denotes set complementation. Set complementation is also involved in the identity direlation on \((S,\mathcal{P}(S))\), since clearly \(I_{(S,\mathcal{P}(S))} = A_S\), the diagonal of \(S\), and \(I_{(S,\mathcal{P}(S))} = A_S^c\).

Naturally this relationship with set complementation disappears in the general case since textures are not necessarily closed under set complementation. For example, for the texture \((L,\mathcal{L},\lambda)\) of Example 1.1(3) we have \(I\) and its inverse \(I\) both equal to the set \(\{(s,t) \mid 0 < t \equiv s \leq 1\}\). Nonetheless, the potential for using direlations to encode set-complement-related information in general textures still remains.

We now wish to generalize the formation of the set \(\varphi[A] = \{ t \mid \exists s \in A, (s,t) \in \varphi \} = \bigcap\{T \setminus \{t\} \mid \forall s, \varphi \not\subseteq (S \times T) \setminus \{(s,t)\} \implies A \subseteq S \setminus \{s\}\}\), where \(\varphi\) is a point relation from \(S\) to \(T\) and \(A \subseteq S\). The following gives an appropriate formulation for general textures. Note that an essentially dual definition has been adopted for corelations.

Definition 2.5. Let \((S,\mathcal{S})\), \((T,\mathcal{F})\) be textures, \((r,R)\) a direlation from \((S,\mathcal{S})\) to \((T,\mathcal{F})\) and \(A \in \mathcal{S}\).

(1) The \(A\)-section of \(r\) is the element \(r^\rightarrow A\) of \(\mathcal{F}\) defined by
\[
r^\rightarrow A = \bigcap \{Q_s \mid \forall s, r \not\subseteq \overline{O}_{(s,t)} \implies A \subseteq Q_s\}.
\]

(2) The \(A\)-section of \(R\) is the element \(R^\rightarrow A\) of \(\mathcal{S}\) defined by
\[
R^\rightarrow A = \bigvee \{P_s \mid \forall s, \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\}.
\]

Clearly, the sections preserve inclusion, that is \(A_1 \subseteq A_2 \implies r^\rightarrow A_1 \subseteq r^\rightarrow A_2\) and \(R^\rightarrow A_1 \subseteq R^\rightarrow A_2\).
Note that in [2,14,16], the $A$-sections are denoted by $r(A)$, $R(A)$, respectively. The following technical results will be useful.

**Lemma 2.6.** For a direlation $(r,R)$ from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ we have

1. $r \not\subseteq \overline{O}_{(s,t)} \iff r^\rightarrow P_s \not\subseteq Q_t$, and
2. $\overline{P}_{(s,t)} \not\subseteq R \iff P_t \not\subseteq R^\rightarrow \overline{Q}_s$.

**Proof.** We prove (1), leaving the dual proof of (2) to the reader.

It is easy to verify $r^\rightarrow P_s \not\subseteq Q_t \implies r \not\subseteq \overline{O}_{(s,t)}$, so assume $r \not\subseteq \overline{O}_{(s,t)}$. By $R2$ we have $s' \in S$ with $r \not\subseteq \overline{O}_{(s',t')}$, $P_s \not\subseteq Q_{s'}$, and we may choose $t' \in T$ with $P_{t'} \not\subseteq Q_t$ and $r \not\subseteq \overline{O}_{(s',t')}$. If we have $r^\rightarrow P_s \subseteq Q_t$, then $P_{t'} \not\subseteq r^\rightarrow P_s$, from which we may obtain the contradiction $P_s \subseteq Q_{t'}$. Hence $r^\rightarrow P_s \not\subseteq Q_t$, as required. □

**Lemma 2.7.** For relations $r_1, r_2$ and corelations $R_1, R_2$ from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ we have

1. $(r_1 \subseteq r_2) \iff (r_1^\rightarrow A \subseteq r_2^\rightarrow A \forall A \in \mathcal{S})$,
2. $(R_1 \subseteq R_2) \iff (R_1^\rightarrow A \subseteq R_2^\rightarrow A \forall A \in \mathcal{S})$.

**Proof.** We give the proof for (1), that for (2) being dual.

$\implies$. Straightforward from Definition 2.5(1).

$\iff$. If $r_1 \not\subseteq r_2$ then we have $s \in S$, $t \in T$ with $r_1 \not\subseteq \overline{O}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq r_2$. By Lemma 2.6(1), $r_1^\rightarrow P_s \not\subseteq Q_t$. Since $P_s \in \mathcal{S}$, by hypothesis $r_1^\rightarrow P_s \subseteq r_2^\rightarrow P_s$ and so $r_2^\rightarrow P_s \not\subseteq Q_t$, whence $r_2 \not\subseteq \overline{O}_{(s,t)}$ by Lemma 2.6(1). This gives the contradiction $\overline{P}_{(s,t)} \not\subseteq r_2$. □

It follows in particular that the sections characterize relations and corelations.

**Definition 2.8.** Let $(r,R)$ be a direlation from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ and $B \in \mathcal{T}$. Then:

1. The $B$-presection of the relation $r$ is the $B$ section of the corelation $r^\rightarrow$ from $(T,\mathcal{T})$ to $(S,\mathcal{S})$. This set will be denoted by $r^\rightarrow B$, so $r^\rightarrow B = (r^\rightarrow)^\rightarrow B = \bigvee \{P_s \mid \forall t, \ r \not\subseteq \overline{O}_{(t,s)} \implies P_t \subseteq B\} \in \mathcal{S}$.

2. The $B$-presection of the corelation $R$ is the $B$ section of the relation $R^\rightarrow$ from $(T,\mathcal{T})$ to $(S,\mathcal{S})$. This set will be denoted by $R^\rightarrow B$, so $R^\rightarrow B = (R^\rightarrow)^\rightarrow B = \bigwedge \{Q_t \mid \forall s, \ \overline{P}_{(s,t)} \not\subseteq R \implies B \subseteq Q_s\} \in \mathcal{S}$.

Since presections are just sections of the inverse, they again preserve inclusion. Note that if $\phi$ is a point relation between discrete textures, its $A$-section as a relation is indeed $\phi[A]$, but the corelational section, and the presections, all involve set complementation in one way or another, as the interested reader may easily verify.

**Lemma 2.9.** (1) For any relation $r$ we have $r^\rightarrow \emptyset = \emptyset$, $A \subseteq r^\rightarrow (r^\rightarrow A)$ for $A \in \mathcal{S}$ and $r^\rightarrow (r^\rightarrow B) \subseteq B$ for $B \in \mathcal{T}$.

(2) For any corelation $R$ we have $R^\rightarrow S = T$, $R^\rightarrow (R^\rightarrow A) \subseteq A$ for $A \in \mathcal{S}$ and $B \subseteq R^\rightarrow (R^\rightarrow B)$ for $B \in \mathcal{T}$.

(3) For the identity direlation $(i,I)$ on $(S,\mathcal{S})$ and $A \in \mathcal{S}$ we have $i^\rightarrow A = I^\rightarrow A = A$ and hence $i^\rightarrow A = I^\rightarrow A = A$. 


(4) If a relation \( r \) (corelation \( R \)) on \( (S,T) \) is reflexive then for all \( A \in T \) we have \( A \subseteq r^{-1}A \) (\( R^{-1}A \subseteq A \)).

**Proof.** (1) Since the implication \( \forall s, \ r \notin \overline{Q}_{(s,t)} \implies \emptyset \subseteq Q_s \) holds for all \( t \in T \) we have \( r^{-1}\emptyset = \bigcap\{Q_t \mid t \in T\} = \bigcap\{Q_t \mid P_t \notin \emptyset\} = \emptyset \) by Theorem 1.2(6).

Now suppose that \( A \notin r^{-1}(r^{-1}A) \) for some \( A \in T \). Then \( A \notin Q_s, \ P_t \notin r^{-1}(r^{-1}A) \) for some \( s \in S \), so by Definition 2.8(1) we have \( t \in T \) with \( r \notin \overline{Q}_{(s,t)} \) and \( P_t \notin r^{-1}A \). Now by Definition 2.5(1) we have \( t' \in T \) satisfying \( P_{t'} \notin Q_{t'} \) and \( \forall s', \ r \notin \overline{O}_{(s',t')} \implies A \subseteq Q_{s'} \). But \( Q_{t'} \subseteq Q_t \), whence \( r \notin \overline{O}_{(s,t')} \) and we have the contradiction \( A \subseteq Q_t \) by using the above implication with \( s' = s \). This establishes \( A \subseteq r^{-1}(r^{-1}A) \) for all \( A \in T \).

If \( r^{-1}(r^{-1}B) \not\subseteq B \) for some \( B \in T \), then \( r^{-1}(r^{-1}B) \not\subseteq Q_s \), \( P_t \not\subseteq B \) for some \( t \in T \). Now \( r \notin \overline{Q}_{(s,t)} \), \( r^{-1}B \not\subseteq Q_s \) for some \( s \in S \) whence \( \exists s' \in S \) with \( P_{t'} \not\subseteq Q_{s'} \) and \( \forall t', \ r \notin \overline{O}_{(s',t')} \implies P_{t'} \not\subseteq B \). By the condition \( R \) we have \( r \notin \overline{O}_{(s',t')} \), which gives the contradiction \( P_{t'} \not\subseteq B \). This verifies \( r^{-1}(r^{-1}B) \subseteq B \forall B \in T \).

(2) Dual to (1).
(3) Straightforward.
(4) By the above, \( A = r^{-1}A \subseteq r^{-1}A \) and \( R^{-1}A \subseteq T^{-1}A = A \) for all \( A \in T \). \( \square \)

The inclusions in Lemma 2.9(1),2(2) suggest the presence of adjoint situations. Let us verify that this is so. Given a texture \( (S,T) \), consider the category \( S \) whose objects are the sets \( A \in T \) and for which there is just one morphism \( [A_1,A_2]:A_1 \to A_2 \) if and only if \( A_1 \subseteq A_2 \), composition being defined by \([A_2,A_3] \circ [A_1,A_2] = [A_1,A_3] \iff A_1 \subseteq A_2 \subseteq A_3 \). Then if \( T \) is the category associated with \( (T,S) \) in the same way, and \( R \) is a corelation from \( (S,T) \) to \( (T,S) \), we may set \( R^{-1}([A_1,A_2]) = [R^{-1}A_1,R^{-1}A_2] \) since sections preserve inclusion, and it is trivial to verify that \( R^{-1} \) is now a functor from \( S \) to \( T \).

**Lemma 2.10.** \( R^{-1}:S \to T \) is an adjoint functor.

**Proof.** We must show that for every object \( B \in \text{Ob} T = T \) there is a \( R^{-1} \)-universal arrow with domain \( B \). Let us define \( A = R^{-1}B \in \text{Ob} S \subseteq T \). Since \( B \subseteq R^{-1}(R^{-1}B) \) by Lemma 2.9(2), \( ([B,R^{-1}A],A) \) is a \( R^{-1} \)-structured array with domain \( B \). It remains to show the universal property, so let \( ([B,R^{-1}A'],A') \) be a \( R^{-1} \)-structured arrow with domain \( B \). Then \( B \subseteq R^{-1}A' \) and so \( A = R^{-1}B \subseteq R^{-1}(R^{-1}A') \subseteq A' \) by Lemma 2.9(2). Hence \( [A,A'] \) is a morphism from \( A \) to \( A' \) and \( R^{-1}[A,A'] \circ [B,R^{-1}A] = [R^{-1}A,R^{-1}A'] \) \( \circ [B,R^{-1}A] = [B,R^{-1}A'] \). Moreover this morphism is unique and we have established that \( ([B,R^{-1}A],A) \) is a \( R^{-1} \)-universal arrow with domain \( B \). \( \square \)

It is clear from the proof of Lemma 2.10 and [1, Theorem 19.1] that a co-adjoint of \( R^{-1} \) is \( (R^{-1})^{\circ}:T \to S \), which is made into a functor by setting \( (R^{-1})^{\circ}[B_1,B_2] = [R^{-1}B_1,R^{-1}B_2] \). Hence the functors \( R^{-1}, (R^{-1})^{\circ} \) give rise to an adjoint situation. Likewise, if \( r \) is a relation from \( (S,T) \) to \( (T,T) \) then \( r^{-1} \) is a corelation from \( (T,T) \) to \( (S,S) \) and the above argument (with \( (S,T), (T,T) \) interchanged) shows that \( (r^{-1})^{\circ} \) is an adjoint functor and that \( r^{-1} \) is a co-adjoint of \( (r^{-1})^{\circ} \). We summarize this below:

**Theorem 2.11.** For a direct relation \( (r,R) \) from \( (S,T) \) to \( (T,T) \):

1. \( (r^{-1})^{\circ} \) is an adjoint of \( r^{-1} \), \( r^{-1} \) a co-adjoint of \( (r^{-1})^{\circ} \).
(2) \( R \to \) is an adjoint of \( (R^\to)^\to \), \( (R^\to)^\to \) a co-adjoint of \( R \to \).

Note that an adjoint is sometimes known as a right adjoint, and a co-adjoint as a left adjoint [15].

**Corollary 2.12.** Let \((r,R)\) be a direlation from \((S,\mathcal{S})\) to \((T,\mathcal{T})\), \(J\) an index set, \(A_j \in \mathcal{S} \; \forall j \in J\) and \(B_j \in \mathcal{T} \; \forall j \in J\). Then:

1. \( \left( \bigcap_{j \in J} B_j \right)^{(r)} = \bigcap_{j \in J} B_j^{(r)} \) and \( R^{(r)}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} R^{(r)} A_j \).
2. \( R^{(r)}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} R^{(r)} B_j \).

**Proof.** (1) Adjoint functors preserve limits [1], and hence products. But products in \( \mathcal{S} \) and \( \mathcal{T} \) are meets and hence intersections. The given results now follow from the fact that \( (r^\to)^\to \) and \( R^\to \) are adjoints.

(2) Dual to (1). \( \square \)

**Definition 2.13.** Let \((S,\mathcal{S}), (T,\mathcal{T}), (U,\mathcal{U})\) be textures.

1. If \( c \) is a relation from \((S,\mathcal{S})\) to \((T,\mathcal{T})\) and \( d \) a relation from \((T,\mathcal{T})\) to \((U,\mathcal{U})\) then their **composition** is the relation \( d \circ c \) from \((S,\mathcal{S})\) to \((U,\mathcal{U})\) defined by

\[
d \circ c = \bigvee \{ P_{(s,t)} \mid \exists t \in T \text{ with } c \not\subseteq Q_{(s,t)} \text{ and } d \not\subseteq \bar{Q}_{(t,u)} \}.
\]

2. If \( C \) is a corelation from \((S,\mathcal{S})\) to \((T,\mathcal{T})\) and \( D \) a corelation from \((T,\mathcal{T})\) to \((U,\mathcal{U})\) then their **composition** is the corelation \( D \circ C \) from \((S,\mathcal{S})\) to \((U,\mathcal{U})\) defined by

\[
D \circ C = \bigcap \{ \bar{Q}_{(s,u)} \mid \exists t \in T \text{ with } P_{(s,t)} \not\subseteq C \text{ and } P_{(t,u)} \not\subseteq D \}.
\]

3. With \( c, d; C, D \) as above, the **composition** of the direlations \((c,C)\), \((d,D)\) is the direlation \( (d,D) \circ (c,C) = (d \circ c, D \circ C) \).

No confusion will be caused by the use of the single symbol “\( \circ \)” to denote these various compositions. But we must of course verify that the composition of two relations (corelations) is indeed a relation (corelation).

**Proposition 2.14.** With the notation as in Definition 2.13, \( d \circ c \) is a relation and \( D \circ C \) a corelation from \((S,\mathcal{S})\) to \((U,\mathcal{U})\).

**Proof.** We give the proof for \( d \circ c \), leaving the dual proof for \( D \circ C \) to the reader. Take \( s \in S, u \in U \) with \( d \circ c \not\subseteq \bar{Q}_{(s,u)} \). Then we have \( w \in U, t \in T \) with \( P_{(s,w)} \not\subseteq \bar{Q}_{(s,u)} \), \( c \not\subseteq Q_{(s,t)} \) and \( d \not\subseteq \bar{Q}_{(t,u)} \). To establish \( R1 \) take \( s' \in S \) with \( P_{s'} \not\subseteq Q_{s} \). Since \( c \) satisfies \( R1 \), \( c \not\subseteq \bar{Q}_{(s',t)} \), whence \( P_{(s',w)} \subseteq d \circ c \) and \( d \circ c \not\subseteq \bar{Q}_{(s',u)} \) as required. On the other hand to prove \( R2 \), note that since \( c \not\subseteq \bar{Q}_{(s,t)} \) and \( c \) satisfies \( R2 \) we have \( s' \in S \) with \( c \not\subseteq \bar{Q}_{(s',t)} \) and then again we have \( d \circ c \not\subseteq \bar{Q}_{(s',u)} \) as before. \( \square \)

It will be noted that the above proof does not actually require that \( d \) be a relation, and likewise \( D \) need not be a corelation either.
The reader may easily verify that for point relations between discrete textures composition reduces to the ordinary composition of binary relations if the point relations are regarded as relations, whereas it gives the complement of the composition of the complements if they are regarded as corelations.

**Definition 2.15.** A direlation \((r,R)\) on \((S,\mathcal{S})\) is called transitive if \((r,R) \circ (r,R) \subseteq (r,R)\). In this case \(r\) and \(R\) are also said to be transitive.

Composition combines with sections and presections as one would expect.

**Lemma 2.16.** With the notation as in Definition 2.13:

1. \((d \circ c)^\rightarrow A = d^\rightarrow (c^\rightarrow A)\) and \((D \circ C)^\rightarrow A = D^\rightarrow (C^\rightarrow A)\) \(\forall A \in \mathcal{S}\).
2. \((d \circ c)^\leftarrow B = c^\leftarrow (d^\leftarrow B)\) and \((D \circ C)^\leftarrow B = C^\leftarrow (D^\leftarrow B)\) \(\forall B \in \mathcal{U}\).

**Proof.** We establish the first equality in (1), leaving the other cases to the interested reader.

Suppose first that \((d \circ c)^\rightarrow A \not\subseteq d^\rightarrow (c^\rightarrow A)\) for some \(A \in \mathcal{S}\). Then \((d \circ c)^\leftarrow A \not\subseteq Q_u\) and \(P_u \not\subseteq d^\rightarrow (c^\rightarrow A)\) for some \(u \in U\). By Definition 2.5(1) we have \(s \in S\) with \(d \circ c \subseteq \overline{Q}_{(s,u)}\) and \(A \subseteq Q_u\). Now we have \(u' \in U\) with \(P_{(s,u')} \subseteq \overline{Q}_{(s,u)}\) and \(t \in T\) with \(c \subseteq \overline{Q}_{(s,t)}\) and \(d \not\subseteq \overline{Q}_{(t,u')}\). Also, we may choose \(t' \in T\) with \(c \subseteq \overline{Q}_{(s,t')}\) and \(P_{(s,t')} \subseteq \overline{Q}_{(s,t)}\), On the other hand, from \(P_u \not\subseteq d^\rightarrow (c^\rightarrow A)\) we have \(u'' \in U\) with \(P_u \not\subseteq Q_{w''}\) and \(t'' \in T\) with \(d \subseteq \overline{Q}_{(w',u'')}\) such that \(c^\rightarrow A \subseteq Q_t\) with taking \(v = t\) in the above implication. Since \(P_{t'} \not\subseteq Q_t\) we have \(P_{t'} \not\subseteq c^\rightarrow A\), whence for some \(t'' \in T\) with \(P_{t'} \not\subseteq Q_{u'}\) we have \(\forall w, c \subseteq \overline{Q}_{(w,u'')} \implies A \subseteq Q_w\). But from \(c \subseteq \overline{Q}_{(s,t')}\) and \(Q_{w''} \subseteq Q_u\) we have \(c \subseteq \overline{Q}_{(s,t'')},\) from which we obtain the contradiction \(A \subseteq Q_s\) for taking \(w = s\) in this implication. This verifies that \((d \circ c)^\rightarrow A \subseteq d^\rightarrow (c^\rightarrow A)\) for all \(A \in \mathcal{S}\).

Now suppose that \(d^\rightarrow (c^\rightarrow A) \not\subseteq (d \circ c)^\rightarrow A\) for some \(A \in \mathcal{S}\). Then for some \(u \in U\) we have \(d^\rightarrow (c^\rightarrow A) \not\subseteq Q_u\) and \(P_u \not\subseteq (d \circ c)^\rightarrow A\). Firstly, we have \(t \in T\) with \(d \not\subseteq \overline{Q}_{(t,u)}\) and \(c^\rightarrow A \subseteq Q_t\), then \(s \in S\) with \(c \subseteq \overline{Q}_{(s,t)}\) and \(A \subseteq Q_s\). Hence \(P_{(s,u)} \subseteq d \circ c\). On the other hand, from \(P_u \not\subseteq (d \circ c)^\rightarrow A\) we have \(u' \in U\) with \(P_u \not\subseteq Q_{w''}\) so that \(\forall w, d \circ c \subseteq \overline{Q}_{(w,u'')} \implies A \subseteq Q_w\). Since \(Q_{w''} \subseteq Q_u\) we have \(d \circ c \subseteq \overline{Q}_{(w,u'')}\), whence we obtain the contradiction \(A \subseteq Q_s\) by taking \(w = s\) in this implication. Hence \(d^\rightarrow (c^\rightarrow A) \subseteq (d \circ c)^\rightarrow A\) for all \(A \in \mathcal{S}\), which completes the proof. \(\Box\)

The following proposition gives some important properties of the operations of composition. They generalize in a natural way corresponding properties of point relations.

**Proposition 2.17.** (1) If \((r,R)\) is a direlation from \((S,\mathcal{S})\) to \((T,\mathcal{F})\), \((i_S, I_S)\) the identity on \((S,\mathcal{S})\) and \((i_T, I_T)\) that on \((T,\mathcal{F})\) then

\[(r,R) \circ (i_S, I_S) = (r,R) \text{ and } (i_T, I_T) \circ (r,R) = (r,R).
\]

(2) If \((c,C)\) is a direlation from \((S,\mathcal{S})\) to \((T,\mathcal{F})\), \((d,D)\) a direlation from \((T,\mathcal{F})\) to \((U,\mathcal{U})\) then

\[
[(d,D) \circ (c,C)]^\leftarrow = (c,C)^\leftarrow \circ (d,D)^\leftarrow.
\]
(3) Composition is associative. That is, if \((c, C), (d, D)\) and \((r, R)\) are direlations from \((S, \mathcal{S})\) to \((T, \mathcal{F})\), \((T, \mathcal{F})\) to \((U, \mathcal{H})\) and \((U, \mathcal{H})\) to \((V, \mathcal{Y})\), respectively, then

\[
[(r, R) \circ (d, D)] \circ (c, C) = (r, R) \circ [(d, D) \circ (c, C)].
\]

(4) Let \((c_1, C_1), (c_2, C_2)\) be direlations from \((S, \mathcal{S})\) to \((T, \mathcal{F})\) and \((d_1, D_1), (d_2, D_2)\) direlations from \((T, \mathcal{F})\) to \((U, \mathcal{H})\) satisfying \((c_1, C_1) \sqsubseteq (c_2, C_2)\) and \((d_1, D_1) \sqsubseteq (d_2, D_2)\). Then \((d_1, D_1) \circ (c_1, C_1) \sqsubseteq (d_2, D_2) \circ (c_2, C_2)\).

**Proof.** All these results may be easily established using Lemma 2.16 and Lemma 2.7. We illustrate this by looking at (2). We have to show \((D \circ C)^\sim = C^\sim \circ D^\sim\) and \((d \circ c)^\sim = c^\sim \circ d^\sim\) for relations, corelations and direlations.

Now let us consider complemented textures. The following definition gives notions of complement for relations, corelations and direlations.

**Definition 2.18.** Let \((S, \mathcal{S}, \sigma)\) and \((T, \mathcal{F}, \theta)\) be complemented textures and \((r, R)\) a direlation from \((S, \mathcal{S})\) to \((T, \mathcal{F})\). Then

1. The **complement** \(r'\) of the relation \(r\) is the corelation \(r' = \bigcap\{\overrightarrow{D}(s,t) \mid \exists u, v \text{ with } r \nsubseteq \overrightarrow{D}(u,v), \sigma(Q_s) \nsubseteq Q_u \text{ and } P_t \nsubseteq \theta(P_t)\}\).
2. The **complement** \(R'\) of the corelation \(R\) is the relation \(R' = \bigvee\{\overrightarrow{P}(s,t) \mid \exists u, v \text{ with } \overrightarrow{P}(u,v) \nsubseteq R, P_u \nsubseteq \sigma(P_s) \text{ and } \theta(Q_t) \nsubseteq Q_v\}\).
3. The **complement** \((r, R)'\) of the direlation \((r, R)\) is the direlation \((r, R)' = (R', r')\).

The direlation \((r, R)\) is said to be **complemented** if \((r, R)' = (r, R)\).

It is, of course, necessary to prove that \(r'\), \((R')\) is indeed a corelation, (relation). The following lemma is useful when dealing with complementations.

**Lemma 2.19.** Let \((S, \mathcal{S}, \sigma)\) be a complemented texture. Then for all \(A, B \in \mathcal{S}\),

1. \(A = \bigcap\{\sigma(Q_s) \mid A \nsubseteq \sigma(P_s)\} = \bigcap\{\sigma(P_s) \mid A \nsubseteq \sigma(Q_s)\}\), and
2. \(A \nsubseteq B \implies \exists s \in S \text{ with } A \nsubseteq \sigma(P_s) \text{ and } \sigma(Q_s) \nsubseteq B\).

**Proof.** (1) Using Theorem 1.2(6) for \(\sigma(A)\) gives \(\sigma(A) = \bigcap\{Q_s \mid P_s \nsubseteq \sigma(A)\}\), and \(A = \bigcap\{\sigma(Q_s) \mid A \nsubseteq \sigma(P_s)\}\) follows by applying \(\sigma\) to both sides. The second equality is proved likewise using Theorem 1.2(7).

(2) By (1), \(A \nsubseteq B = \bigcap\{\sigma(P_s) \mid \sigma(Q_s) \nsubseteq B\}\) so we have \(s \in S \text{ with } A \nsubseteq \sigma(P_s) \text{ and } \sigma(Q_s) \nsubseteq B\), as required. □
\(\sigma(Q_s) \not\subseteq Q_u\) and Lemma 2.19(2) note that we have \(s' \in S\) satisfying \(\sigma(Q_s) \not\subseteq \sigma(P_{r'})\) and \(\sigma(Q_{r'}) \not\subseteq Q_u\). It is clear that \(s'\) satisfies \(P_{r'} \not\subseteq Q_s\) and, as above, \(\overline{P_{(s',t)}} \not\subseteq r'\), so CR2 is satisfied and \(r'\) is seen to be a correlation. The proof that \(R'\) is a relation is dual to this, and is omitted.

It is interesting to note that if \(\phi\) is a point relation from \(X\) to \(Y\), regarded as a relation or a correlation from \((X, \mathcal{P}(X), \pi_X)\) to \((Y, \mathcal{P}(Y), \pi_Y)\), then \(\phi' = \phi^\circ\), the set theoretic complement of \(\phi\) in \(X \times Y\). The proof is straightforward and is left to the interested reader.

Lemma 2.20. With the notation above:

1. \((r')^\rightarrow A = \theta(r^\rightarrow \sigma(A))\) and \((R')^\rightarrow A = \theta(R^\rightarrow \sigma(A))\) for all \(A \in \mathcal{F}\).
2. \((r')^\leftarrow B = \sigma(r^\leftarrow \theta(B))\) and \((R')^\leftarrow B = \sigma(R^\leftarrow \theta(B))\) for all \(B \in \mathcal{F}\).

Proof. We prove the first equality in (1), leaving the remaining cases to the interested reader.

Suppose first that \((r')^\rightarrow A \not\subseteq \theta(r^\rightarrow \sigma(A))\). Then we have \(t \in T\) with \(P_t \not\subseteq \theta(r^\rightarrow \sigma(A))\) so that

\[\forall z \in S, \quad \overline{P_{(z,t)}} \not\subseteq r' \implies P_z \subseteq A.\]

Take \(t' \in T\) with \(P_t \not\subseteq Q_{r'}, P_{r'} \not\subseteq \theta(r^\rightarrow \sigma(A))\). Now \(r^\rightarrow \sigma(A) \not\subseteq \theta(P_{r'})\) so we have \(v \in T\) with \(r^\rightarrow \sigma(A) \not\subseteq Q_v\) and \(P_v \not\subseteq \theta(P_{r'})\). Hence we have \(u \in S\) satisfying \(r \not\subseteq Q_{(u,v)}\) and \(\sigma(A) \not\subseteq Q_u\). By Lemma 2.19(2) we have \(s \in S\) with \(\sigma(A) \not\subseteq \sigma(P_s)\), i.e. \(P_s \not\subseteq A\), and \(\sigma(Q_s) \not\subseteq Q_u\). By the definition of \(r'\) we have \(r' \subseteq Q_{(s',t')}\), whence \(\overline{P_{(s,t)}} \not\subseteq r'\) and the implication above with \(z = s\) gives the contradiction \(P_s \subseteq A\).

Conversely, suppose that \(\theta(r^\rightarrow \sigma(A)) \not\subseteq (r')^\rightarrow A\). First, we have \(t \in T\) with \(\theta(r^\rightarrow \sigma(A)) \not\subseteq Q_t\), \(P_t \not\subseteq (r')^\rightarrow A\). Now \(\theta(Q_{r'}) \not\subseteq Q_t\), for some \(t' \in T\) for which

\[\forall z \in S, \quad r \not\subseteq Q_{(z,t')} \implies \sigma(A) \subseteq Q_z.\]

Secondly, for some \(s \in S\), \(\overline{P_{(s,t)}} \not\subseteq r', P_s \subseteq A\). Hence \(\overline{P_{(s,t)}} \not\subseteq Q_{(s,t'')}\) for some \(t'' \in T\) for which we have \(u \in S\), \(v \in T\) satisfying \(r \not\subseteq Q_{(u,v)}\), \(\sigma(Q_u) \not\subseteq Q_u\) and \(P_v \not\subseteq \theta(P_{r'})\). From \(\theta(Q_{r'}) \not\subseteq Q_t\), \(P_t \not\subseteq Q_{r'\prime}\) and \(P_{r'\prime} \not\subseteq \theta(P_s)\) we obtain \(P_s \not\subseteq Q_{r'\prime}\), so from \(r \not\subseteq Q_{(u,v)}\) we obtain \(r \not\subseteq Q_{(u,v')}\), whence \(\sigma(A) \subseteq Q_u\) by the above implication with \(z = u\). On the other hand from \(P_s \not\subseteq A\) and \(\sigma(Q_s) \not\subseteq Q_u\) we obtain \(\sigma(A) \not\subseteq Q_u\), which is a contradiction. \(\square\)

The following proposition gives some basic properties of the complementation operators.

Proposition 2.21. Let \((S, \mathcal{F}, \sigma), (T, \mathcal{F}, \theta)\) be complemented textures, \((r, R)\) a darelation from \((S, \mathcal{F})\) to \((T, \mathcal{F})\) and \((r, R)' = (R', r')\) the complement of \((r, R)\). Then

1. \((r')' = r\) and \((R')' = R\). That is, the operations of taking the complement are idempotent.
2. \((r')^\rightarrow = (r^\rightarrow)'\) and \((R')^\rightarrow = (R^\rightarrow)'\). That is, the inverse operations commute with complementation.
3. If \((U, \mathcal{U}, v)\) is a complemented texture and \((d, D)\) a darelation on \((T, \mathcal{F})\) to \((U, \mathcal{U})\) then
\[d \circ r' = d' \circ r\] and \[D \circ R' = D' \circ R\].
4. The identity darelation \((i_s, i_s)\) on \((S, \mathcal{F}, \sigma)\) is complemented.

Proof. In each case we may make use of Lemmas 2.20 and 2.7. For example, to show the first equality in (2), note that each side is a relation from \((T, \mathcal{F})\) to \((S, \mathcal{F})\) and take \(B \in \mathcal{F}\). Then \(((r')^\rightarrow)'B = (r')^\rightarrow B = \sigma(r^\rightarrow \theta(B))\) by the first equality in Lemma 2.20(2), while \(((r^\rightarrow)'\)
The equality of these sets for arbitrary \( B \) shows \( (r')^\leftarrow = (r^\leftarrow)' \) by Lemma 2.7. \( \square \)

### 2.2. Difunctions

The notion of difunction is based on that of direlation.

**Definition 2.22.** Let \((f, F)\) be a direlation from \((S, \mathcal{S})\) to \((T, \mathcal{T})\). Then \((f, F)\) is called a difunction from \((S, \mathcal{S})\) to \((T, \mathcal{T})\) if it satisfies the following two conditions:

DF1 For \( s, s' \in S, P_s \not\subseteq Q_{s'} \implies \exists t \in T \text{ with } f \not\subseteq Q_{(s, t)} \text{ and } P_{(s', t)} \not\subseteq F. \)

DF2 For \( t, t' \in T \text{ and } s \in S, f \not\subseteq Q_{(s, t)} \text{ and } P_{(s, t')} \not\subseteq F \implies P_r \not\subseteq Q_t. \)

It is clear that \((i_S, I_S)\) is a difunction on \((S, \mathcal{S})\). In this context it is called the identity difunction.

In the particular case of discrete textures \((X, \mathcal{P}(X)), (Y, \mathcal{P}(Y))\), the pair \((\phi, \psi)\) of point relations from \( X \) to \( Y \) is a difunction if and only if \( \phi \) is a point function \( \phi : X \rightarrow Y \) and \( \psi = \phi^c \). As noted earlier \( \phi^c = \phi' \), where \( \phi' \) is the complement of \( \phi \) regarded as a relation from \((X, \mathcal{P}(X), \pi_X)\) to \((Y, \mathcal{P}(Y), \pi_Y)\).

**Lemma 2.23.** Let \((r, R)\) be a direlation from \((S, \mathcal{S})\) to \((T, \mathcal{T})\).

1. The following are equivalent:
   1. \((r, R)\) satisfies DF1.
   2. \( r^\leftarrow \circ R \subseteq I_S. \)
   3. \( r^\leftarrow (R^\rightarrow A) \subseteq A \forall A \in \mathcal{S}. \)
   4. \( r^\leftarrow B \subseteq R^\leftarrow B \forall B \in \mathcal{T}. \)
   5. \( \forall A_1, A_2 \in \mathcal{S}, r^\rightarrow A_1 \subseteq R^\rightarrow A_2 \implies A_1 \subseteq A_2. \)

2. The following are equivalent:
   1. \((r, R)\) satisfies DF2.
   2. \( r \circ R^\leftarrow \subseteq I_T. \)
   3. \( r^\rightarrow (R^\leftarrow B) \subseteq B \forall B \in \mathcal{T}. \)

**Proof.** (1) (i) \( \implies \) (ii) Suppose that DF1 holds but that \( r^\leftarrow \circ R \not\subseteq I_S \). Then for some \( s, u \in S \) we have \( r^\leftarrow \circ R \not\subseteq Q_{(s, u)} \) and \( P_{(s, u)} \not\subseteq I_S \). Then \( P_u \not\subseteq Q_s \) and so by DF1 there exists \( t \in T \) with \( r \not\subseteq Q_{(u, t)} \) and \( P_{(s, t)} \not\subseteq R \). But now \( P_{(t, u)} \not\subseteq r^\leftarrow \) by Lemma 2.4(1), and we obtain the contradiction \( r^\leftarrow \circ R \subseteq Q_{(s, u)} \) from Definition 2.13(2).
   (ii) \( \implies \) (iii) Immediate from the second equality in Lemma 2.16(1).
   (iii) \( \implies \) (iv) Let \( B \in \mathcal{T} \). Then noting that \( B \subseteq R^\rightarrow (R^\leftarrow B) \) by Lemma 2.9(2) and applying (iii) to \( A = R^\leftarrow B \) gives
   \[
   r^\rightarrow B \subseteq r^\rightarrow (R^\rightarrow (R^\leftarrow B)) \subseteq r^\leftarrow B
   \]
   as required.
   (iv) \( \implies \) (v) Suppose that \( r^\rightarrow A_1 \subseteq R^\rightarrow A_2 \). Then
   \[
   A_1 \subseteq r^\leftarrow (r^\rightarrow A_1) \subseteq r^\leftarrow (R^\rightarrow A_1) \subseteq R^\leftarrow (R^\rightarrow A_1) \subseteq R^\leftarrow (R^\rightarrow A_2) \subseteq A_2
   \]
   by Lemma 2.9(1), (iv) and Lemma 2.9(2).
(v) \implies (i) Take \( s, s' \in S \) satisfying \( P_s \subseteq Q_{s'} \). By (v) we have \( r \rightarrow P_s \subseteq R^{-1}Q_{s'} \) and so there exists \( t \in T \) with \( r \rightarrow P_s \subseteq Q_t \) and \( P_t \subseteq R^{-1}Q_{s'} \). By Lemma 2.6 we deduce \( r \not\subseteq \overline{Q}_{(s,t)} \) and \( \overline{P}_{(s',t)} \not\subseteq R \), which verifies DF1.

(2) Left to the interested reader. \( \square \)

**Theorem 2.24.** For a direlation \((f, F)\) from \((S, \mathcal{I})\) to \((T, \mathcal{F})\) the following are equivalent:

1. \((f, F)\) is a difunction.
2. The following inclusions hold:
   a. \( f^{-\leftarrow}(F^{-\rightarrow}A) \subseteq A \subseteq F^{-\rightarrow}(f^{-\leftarrow}A) \ \forall A \in \mathcal{I} \), and
   b. \( f^{-\leftarrow}(F^{-\rightarrow}B) \subseteq B \subseteq F^{-\rightarrow}(f^{-\leftarrow}B) \ \forall B \in \mathcal{F} \).
3. \( f^{-\leftarrow}B = F^{-\rightarrow}B \ \forall B \in \mathcal{F} \).

**Proof.** (1) \implies (2) Let \((f, F)\) be a difunction. Then \( f^{-\leftarrow}(F^{-\rightarrow}A) \subseteq A \) holds for all \( A \in \mathcal{I} \) by Lemma 2.23(1 iii). On the other hand, \( f^{-\leftarrow}F \subseteq I_S \) by Lemma 2.23(1 ii), so \( I_S \subseteq F^{-\leftarrow} \circ f \), and taking the \( A \)-sections gives \( A \subseteq F^{-\leftarrow}(f^{-\leftarrow}A) \ \forall A \in \mathcal{I} \). This proves (a), and (b) is proved likewise using Lemma 2.23(2).

(2) \implies (3) Let the inclusions (2) hold. Then the equivalent conditions in Lemma 2.23(1) all hold and in particular \( f^{-\leftarrow}B \subseteq F^{-\leftarrow}B \ \forall B \in \mathcal{F} \) by (1(iv)). On the other hand, \( f^{-\leftarrow}(F^{-\rightarrow}B) \subseteq F^{-\rightarrow}(f^{-\leftarrow}B) \ \forall B \in \mathcal{F} \) by (2)(b), and applying (v) now gives \( F^{-\leftarrow}B \subseteq f^{-\leftarrow}B \ \forall B \in \mathcal{F} \).

(3) \implies (1) If the equalities (3) hold then \((f, F)\) satisfies DF1 by the equivalence of (i) and (iv) in Lemma 2.23(1). On the other hand, by Lemma 2.9(1) we have \( f^{-\leftarrow}(f^{-\leftarrow}B) \subseteq B \ \forall B \in \mathcal{F} \), so \( f^{-\leftarrow}(F^{-\leftarrow}B) \subseteq B \ \forall B \in \mathcal{F} \) by (3) and \((f, F)\) satisfies DF2 by the equivalence of (i) and (iii) in Lemma 2.23(2). Hence \((f, F)\) is a difunction. \( \square \)

If \((f, F)\) is a difunction we will refer to \( f^{-\leftarrow}A \) as the **image** and to \( F^{-\rightarrow}A \) as the **co-image** of \( A \) under \((f, F)\). Generally the image and co-image will be unequal. Indeed, if \((\varphi, \varphi')\) is a difunction from \((X, \mathcal{P}(X))\) to \((Y, \mathcal{P}(Y))\), and \( A \subseteq X \), it is easy to see that \( \varphi^{-\leftarrow}A = \varphi[A] \), \( (\varphi')^{-\rightarrow}A = Y \setminus \varphi[X \setminus A] \), these sets being unequal in general. On the other hand the above theorem says that the *inverse image* \( f^{-\leftarrow}B \) and *inverse co-image* \( F^{-\rightarrow}B \) of \( B \) are equal, and indeed that this is characteristic of difunctions. For a difunction \((\varphi, \varphi')\) between discrete textures this turns out to be the usual inverse image \( \varphi^{-1}[B] \) of \( B \) under the point function \( \varphi \).

Theorem 2.11 and its corollary specialize as follows to difunctions:

**Theorem 2.25.** Let \((f, F)\) be a difunction from \((S, \mathcal{I})\) to \((T, \mathcal{F})\). Then

1. The functor \( (f^{-\leftarrow})^{-\rightarrow} = (F^{-\rightarrow})^{-\leftarrow} \) is an adjoint of the functor \( f^{-\leftarrow} \).
2. The functor \( F^{-\rightarrow} \) is an adjoint of the functor \( (f^{-\leftarrow})^{-\rightarrow} = (F^{-\leftarrow})^{-\rightarrow} \).

**Corollary 2.26.** Inverse (co) images of difunctions preserve intersections and joins. Images of difunctions preserve joins and co-images preserve intersections.

Another important consequence of the equality of inverse images and inverse co-images is the following:
Proposition 2.27. If \((f, F), (g, G)\) are difunctions from \((S, \mathcal{F})\) to \((T, \mathcal{F})\) then
\[
f \subseteq g \iff F \subseteq G.
\]
Hence \((f, F) \subseteq (g, G) \iff F = G \iff f = g \iff (f, F) = (g, G)\).

**Proof.** \(f \subseteq g \iff g^{-} \subseteq f^{-} \iff g^{-} B \subseteq f^{-} B \quad \forall B \in \mathcal{F} \iff G^{-} B \subseteq F^{-} B \quad \forall B \in \mathcal{F} \iff G^{-} \subseteq F^{-} \iff F \subseteq G\).
The remaining equivalences are now clear. \(\square\)

This result shows that difunctions are characterized by their relational part and by their corelational part. It also shows that, just as each point function from \(X\) to \(Y\) is a maximal element in the set of all such functions ordered by inclusion, each difunction from \((S, \mathcal{F})\) to \((T, \mathcal{F})\) is a maximal element in the set of all such difunctions ordered by \(\subseteq\).

Proposition 2.28.

1. Let \((f, F)\) be a difunction from \((S, \mathcal{F})\) to \((T, \mathcal{F})\). Then
   a. For \(A \in \mathcal{F}\), \(A = \emptyset \iff f^{-} A = \emptyset\).
   b. For \(A \in \mathcal{F}\), \(A = S \iff F^{-} A = T\).
   c. \(f^{-} \emptyset = F^{-} \emptyset = \emptyset\) and \(f^{-} T = F^{-} T = S\).
2. Let \((S, \mathcal{F})\), \((T, \mathcal{F})\) and \((U, \mathcal{U})\) be textures, \((f, F) : (S, \mathcal{F}) \to (T, \mathcal{F})\), \((g, G) : (T, \mathcal{F}) \to (U, \mathcal{U})\) difunctions. Then \((g, G) \circ (f, F) : (S, \mathcal{F}) \to (U, \mathcal{U})\) is a difunction.

**Proof.** (1) \(f^{-} \emptyset = \emptyset\) follows from Lemma 2.9(1). On the other hand, suppose that \(A \neq \emptyset\). By Theorem 1.2(4) we have \(A' \neq \emptyset\), so by Theorem 1.2(2) there exists \(s \in S\) with \(A \not\subseteq Q_s\). Now we have \(s' \in S\) with \(A \not\subseteq Q_{s'}\) and \(P_{s'} \not\subseteq Q_s\). By DF1 we have \(t \in T\) satisfying \(f \not\subseteq \overline{Q}_{(s', t)}\), and \(\overline{P}_{(s, t)} \not\subseteq F\). From Lemma 2.6(1) we obtain \(f^{-} P_{s'} \not\subseteq Q_t\), whence \(P_t \subseteq f^{-} A \subseteq f^{-} a\) since \(P_{s'} \subseteq A\). Thus \(f^{-} A \neq \emptyset\). This proves (a), and the proof of (b) is essentially dual. Finally, \(F^{-} \emptyset = \emptyset\) is obtained by applying Lemma 2.9(1) to the relation \(F^{-}\) from \((T, \mathcal{F})\) to \((S, \mathcal{F})\), and \(f^{-} T = S\) is proved in the same way using Lemma 2.9(2). The proof of (c) is now completed using Theorem 2.24(3).

(2) Take \(C \in \mathcal{U}\). Then by the first equality in Lemma 2.16(2), \((g \circ f)^{-} C = f^{-} (g^{-} C)\). Now \(g^{-} C = G^{-} C\) by Theorem 2.24(3), and since this set belongs to \(\mathcal{F}\), \((g \circ f)^{-} C = (G \circ F)^{-} C\) by the same theorem. By the second equality in Lemma 2.16(2) we obtain \((g \circ f)^{-} C = (G \circ F)^{-} C\), and so \((g \circ f, G \circ G) = (g, G) \circ (f, F)\) is a difunction, again from Theorem 2.24(3). \(\square\)

Proposition 2.29. If \((f, F) : (S, \mathcal{F}, \sigma) \to (T, \mathcal{F}, \theta)\) is a difunction then \((f, F) : (S, \mathcal{F}, \sigma) \to (T, \mathcal{F}, \theta)\) is also a difunction.

**Proof.** For \(B \in \mathcal{F}\) we have \((F')^{-} B = \sigma(F^{-} \theta(B)) = \sigma(f^{-} \theta(B)) = (f')^{-} B\) by Lemma 2.20(2) and Theorem 2.24(3). The result now follows from the equivalence of (1) and (3) in Theorem 2.24. \(\square\)

As for direlations, a difunction \((f, F)\) will be called complemented if \((f, F)' = (f, F)\). All point functions are automatically complemented. For if \(\varphi : X \to Y\) is a point function, regarded as a relation from \((X, \mathcal{P}(X), \pi_X)\) to \((Y, \mathcal{P}(Y), \pi_Y)\), and \(\varphi'\) its complement regarded as a corelation, then \((\varphi, \varphi')' = ((\varphi')', \varphi') = (\varphi, \varphi')\), so the difunction \((\varphi, \varphi')\) corresponding to \(\varphi\) is indeed complemented.
However, as we will see, the requirement that a difunction be complemented can be quite restrictive in general (see Example 3.11(3)). Conditions DF1 and DF2 may be modified in a natural way to give us suitable notions of surjectivity and injectivity of difunctions.

**Definition 2.30.** Let \((f, F)\) be a difunction from \((S, \mathcal{S})\) to \((T, \mathcal{T})\). Then \((f, F)\) is called **surjective** if it satisfies the condition

\[
\text{SUR. For } t, t' \in T, P_t \nsubseteq Q_{t'} \implies \exists s \in S \text{ with } f \nsubseteq Q_{(s,t')} \text{ and } P_{(s,t)} \nsubseteq F.
\]

Likewise, \((f, F)\) is called **injective** if it satisfies the condition

\[
\text{INJ. For } s, s' \in S \text{ and } t \in T, f \nsubseteq Q_{(s,t)} \text{ and } P_{(s',t)} \nsubseteq F \implies P_s \nsubseteq Q_{s'}.
\]

The conditions SUR and INJ are related to properties of the direlation \((F, F)\leftarrow\) as follows:

**Theorem 2.31.** Let \((f, F)\) be a difunction from \((S, \mathcal{S})\) to \((T, \mathcal{T})\).

1. \((f, F)\) is surjective if and only if \((f, F)\leftarrow\) satisfies DF1.
2. \((f, F)\) is injective if and only if \((f, F)\leftarrow\) satisfies DF2.
3. \((f, F)\) is bijective if and only if \((f, F)\leftarrow\) is a difunction from \((T, \mathcal{T})\) to \((S, \mathcal{S})\) and in this case \((F, F)\leftarrow\) is also bijective.

**Proof.** Immediate from Lemma 2.4(1) and the definitions. \(\square\)

The following theorem and its corollary give some basic properties of surjective and injective difunctions.

**Theorem 2.32.** Let \((f, F)\) be a difunction from \((S, \mathcal{S})\) to \((T, \mathcal{T})\). Then

1. \((f, F)\) is surjective \(\iff\) \(F \circ f^\leftarrow \subseteq I_T \iff i_T \subseteq f \circ F^\leftarrow\).
2. \((f, F)\) is injective \(\iff\) \(F^\leftarrow \circ f \subseteq i_S \iff I_S \subseteq f^\leftarrow \circ F\).

**Proof.** (1) By Theorem 2.31(1) surjectivity is equivalent to \((F^\leftarrow, f^\leftarrow) : (T, \mathcal{T}) \to (S, \mathcal{S})\) satisfying DF1, and by Lemma 2.23(1 ii) this is equivalent to \(F \circ f^\leftarrow = (F^\leftarrow)\circ f^\leftarrow \subseteq I_T\). The second equivalence is obtained by taking the inverse of both sides of this inclusion.

(2) Dual to (1). \(\square\)

**Corollary 2.33.** Let \((f, F)\) be a difunction from \((S, \mathcal{S})\) to \((T, \mathcal{T})\).

1. If \((f, F)\) is surjective then \(F^\rightarrow (f^\leftarrow B) = B = f^\leftarrow (F^\leftarrow B)\) for all \(B \in \mathcal{T}\). In particular
   (i) \(F^\rightarrow A \subseteq f^\rightarrow A, \forall A \in \mathcal{S}, \text{ and}
   (ii) \forall B_1, B_2 \in \mathcal{T}, f^\leftarrow B_1 \subseteq f^\leftarrow B_2 \implies B_1 \subseteq B_2.
2. If \((f, F)\) is injective then \(f^\leftarrow (F^\rightarrow A) = A = f^\leftarrow (F^\rightarrow A)\) for all \(A \in \mathcal{S}\). In particular
   (i) \(f^\rightarrow A \subseteq F^\rightarrow A, \forall A \in \mathcal{S}, \text{ and}
   (ii) \forall A_1, A_2 \in \mathcal{S}, F^\rightarrow A_1 \subseteq F^\rightarrow A_2 \implies A_1 \subseteq A_2.
Proof. (1) Let \((f,F)\) be surjective. From Theorem 2.32(1) we have \(F\to(f\leftarrow B) \subseteq B \subseteq f\to(F\leftarrow B)\) for all \(B \in \mathcal{S}\), whence the result follows from Theorem 2.24(2 b).

(i) Setting \(B = F\to A\) gives \(F\to A = f\to(F\leftarrow (F\to A))\), and the result follows on noting that \(F\leftarrow (F\to A) \subseteq A\) by Proposition 2.9(2).

(ii) If \(f\leftarrow B_1 \subseteq f\leftarrow B_2\) then \(B_1 = F\to(f\leftarrow B_1) \subseteq F\to(f\leftarrow B_2) = B_2\).

(2) Similar.  

It will be noted that for a bijective difunction we will have \(f\to A = F\to A\) for all \(A \in \mathcal{S}\).

Clearly, a difunction \((\phi,\phi')\) from \((X,\mathcal{P}(X))\) to \((Y,\mathcal{P}(Y))\) is surjective (respectively, injective) if and only if the same is true, in the usual sense, of the point function \(\phi\). Hence in this case Corollary 2.33(1 i) (respectively, Corollary 2.33(2 i)) expresses the well known fact that for \(A \subseteq X\) we have \(Y \setminus \phi[X\setminus A] \subseteq \phi[A]\) (respectively, \(\phi[A] \subseteq Y \setminus \phi[X\setminus A]\)) if the point function \(\phi\) is surjective (respectively, injective).

**Proposition 2.34.** Let \((f,F):(S,\mathcal{S},\sigma) \to (T,\mathcal{T},\theta)\) be a difunction.

(1) \((f,F)\) is surjective if and only if \((f,F)'\) is surjective.

(2) \((f,F)\) is injective if and only if \((f,F)'\) is injective.

Proof. (1) \((f,F)\) is surjective \iff \(F \circ f\leftarrow \subseteq I_T \iff I_T' \subseteq F' \circ (f'\leftarrow) \iff i_T \subseteq F' \circ (f'\leftarrow) \iff (F',f')\) is surjective by Theorem 2.32(1) and Proposition 2.21(2).

(2) Similar to (1).  

It is clear that difunctions have many desirable function-like properties, but it should be emphasized that in general difunctions are not functions in the usual sense, that is they do not necessarily set up a mapping from the points of \(S\) to those of \(T\). Indeed, in the next paper in this series, we will present a characterization of difunctions which is point-free in the sense that it depends only on the texturings \(\mathcal{S}, \mathcal{T}\) and not on the sets \(S\) and \(T\) at all. Nonetheless, point functions satisfying a suitable condition do give rise to difunctions, and this can characterize the difunctions in certain special cases, as we will see in the next section.

3. The category dfTex

We may now introduce the category which, together with its various specializations, will mainly concern us in this series of papers.

**Theorem 3.1.** Textures and difunctions form a category.

Proof. Difunctions are closed under composition of direlations by Proposition 2.28. By Proposition 2.17(1) the identity difunctions are identities for composition. Also, composition is associative by Proposition 2.17(3).  

**Corollary 3.2.** Complemented textures and complemented difunctions form a category.
Proof. Since the identity difunctions are complemented by Proposition 2.21(4), it remains only to show that the composition of two complemented difunctions is complemented. Let \((f, F), (g, G)\) be complemented difunctions for which \((f, F) \circ (g, G)\) is defined. Then \((f, F) \circ (g, G)' = (f \circ g, F \circ G)' = ((F \circ G)', (f \circ g)') = (F' \circ G', f' \circ g')\) by Definition 2.18(3) and Proposition 2.21(3). Since \((f, F), (g, G)\) are complemented we have \(F' = f, G' = g, f' = F\) and \(g' = G\) so \((f, F) \circ (g, G) = (f \circ g, F \circ G) = (f, F) \circ (g, G)\), as required. □

Definition 3.3. The category whose objects are textures and whose morphisms are difunctions will be denoted by \(\text{dfTex}\).

If the objects are restricted to be simple textures we obtain the full subcategory \(\text{dfSTex}\) and inclusion functor \(\mathcal{S} : \text{dfSTex} \to \text{dfTex}\).

If the objects are restricted to be plain textures we obtain the full subcategory \(\text{dfPTex}\) and inclusion functor \(\mathcal{P} : \text{dfPTex} \to \text{dfTex}\).

Finally, the category \(\text{cdfTex}\) is obtained by taking the objects to be complemented textures and the morphisms to be complemented difunctions. In this case \(\mathcal{C} : \text{cdfTex} \to \text{dfTex}\) is the forgetful functor \(\mathcal{C}(S, \mathcal{P}, \sigma) = (S, \mathcal{P}), \mathcal{C}(f, F) = (f, F)\).

Here the first part of the name denotes the type of morphism (df = difunction, or cdf = complemented difunction), and the letters \(S\) and \(P\) stand for simple and plain, respectively. The same convention has been followed for the functors, with \(\mathcal{C}\) being named for complemented.

The obvious notation is used where more than one restriction is made. Hence, for example, \(\text{dfPSTex}\) will denote the category whose objects are plain simple textures, and whose morphisms are difunctions. The same name will also be used for the corresponding functors between these new categories, possibly with explanatory subscripts. For example, we may write \(\mathcal{P}_s : \text{dfPSTex} \to \text{dfSTex}\) for the inclusion functor from \(\text{dfPSTex}\) to \(\text{dfSTex}\).

Given a set \(X\), consider the plain simple complemented texture \((X, \mathcal{P}(X), \pi_X)\) of Example 1.1(1). Now, \(\Sigma_c(X) = (X, \mathcal{P}(X), \pi_X)\) defines a mapping from the objects of the category \(\text{Set}\) to those of \(\text{cdfPSTex}\). If \(X, Y\) are sets and \(f : X \to Y\) a point function, then we have already noted that \((f, f')\) is a complemented difunction from \((X, \mathcal{P}(X), \pi_X)\) to \((Y, \mathcal{P}(Y), \pi_Y)\). Hence \(\Sigma_c(f) = (f, f')\) gives a mapping from the morphisms of \(\text{Set}\) to those of \(\text{cdfPSTex}\). Clearly \(\Sigma_c\) maps the identity function on \(X\) to the identity difunction on \((X, \mathcal{P}(X))\), while composition of morphisms in \(\text{Set}\) corresponds to composition of relations in \(\text{dfTex}\), so \(\Sigma_c(f \circ g) = \Sigma_c(f) \circ \Sigma_c(g)\) since \(f' \circ g' = (f \circ g)'\) by Proposition 2.21(3). This establishes that \(\Sigma_c : \text{Set} \to \text{cdfPSTex}\) is a functor, and it is clearly an embedding.

By the comment following Definition 2.22 we know that if \((f, F)\) is any difunction from \((X, \mathcal{P}(X))\) to \((Y, \mathcal{P}(Y))\), then \(f : X \to Y\) is a point function and \(F = f'\). Hence the image of \(\text{Set}\) under \(\Sigma_c\) is a full subcategory of \(\text{cdfPSTex}\).

We may also embed \(\text{Set}\) directly in \(\text{dfPSTex}\) in the same way, and the functor \(\Sigma : \text{Set} \to \text{dfPSTex}\) is again an embedding. Hence, we will be particularly interested in the relations below.
As mentioned earlier, for general textures \((S,\mathcal{S}), (T,\mathcal{T})\) it is not possible to associate a point function from \(S\) to \(T\) with a difunction \((f,F):(S,\mathcal{S}) \rightarrow (T,\mathcal{T})\). However, we will now show that this is possible for both simple textures and for plain textures. This will enable us to represent \textbf{dfSTex} and \textbf{dfPTex}, and their associated categories, as constructs, that is as concrete categories over \textbf{Set}, and to obtain more information about the functors \(\Sigma_c\) and \(\Sigma\) defined above. First we will require the following result.

**Lemma 3.4.** Suppose that the point function \(\varphi\) on \(S\) to \(T\) satisfies the condition \(P_s \not\subseteq Q_v \Rightarrow P_{\varphi(s)} \not\subseteq Q_{\varphi(v)}\) for all \(s,s' \in S\). Then the equalities

\[
\begin{align*}
f &= \bigvee \{P_{(s,t)} \mid \exists u \in S \text{ satisfying } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\}, \\
F &= \bigcap \{Q_{(s,t)} \mid \exists u \in S \text{ satisfying } P_u \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(u)}\},
\end{align*}
\]

define a difunction \((f,F)\) from \((S,\mathcal{S})\) to \((T,\mathcal{T})\). Moreover, for \(B \in \mathcal{T}, F^\leftarrow B = \varphi^\leftarrow B = f^\leftarrow B\), where \(\varphi^\leftarrow B = \bigvee \{P_s \mid P_{\varphi(u)} \subseteq B \forall u \in S \text{ with } P_s \not\subseteq Q_u\}\).

**Proof.** It is trivial to verify that \((f,F)\) is a direlation, and we omit the details.

Let us first prove DF1. Take \(s,s' \in S\) with \(P_s \not\subseteq Q_{s'}\). There exists \(w \in S\) with \(P_s \not\subseteq Q_w\), \(P_w \not\subseteq Q_{s'}\), and we let \(t = \varphi(w) \in T\). To establish \(f \not\subseteq Q_{(s,t)}\), take \(u,v \in S\) satisfying \(P_s \not\subseteq Q_u\), \(P_v \not\subseteq Q_{s}\), and \(P_t \not\subseteq Q_{\varphi(u)}\). Then \(P_{\varphi(t)} \not\subseteq Q_{\varphi(u)}\), by the given condition on \(\varphi\), so \(P_{(s,t)} \not\subseteq f\). Also \(P_{\varphi(u)} \not\subseteq Q_{\varphi(w)} = Q_t\), which gives \(f \not\subseteq Q_{(s,t)}\). The proof of \(P_{(s',t)} \not\subseteq F\) is similar.

By Lemma 2.23(1 iv) we have \(f^\leftarrow B \subseteq F^\leftarrow B\) for all \(B \in \mathcal{T}\). It remains, therefore, to prove

\[
F^\leftarrow B \subseteq \varphi^\leftarrow B \subseteq f^\leftarrow B
\]

for all \(B \in \mathcal{T}\), for then \((f,F)\) will be a difunction by Theorem 2.24(3) and the desired equality for \(\varphi^\leftarrow B\) will also follow.

If \(F^\leftarrow B \not\subseteq \varphi^\leftarrow B\) then for some \(s \in S\) there exists, on the one hand, \(t \in T\) with \(P_{(s,t)} \not\subseteq F\), \(B \not\subseteq Q_t\), and on the other some \(u \in S\) with \(P_s \not\subseteq Q_u\) for which \(P_{\varphi(u)} \not\subseteq B\). Now we have \(t' \in T\) with \(P_{(s,t')} \not\subseteq Q_{(s,t')}\) for which \(P_{t'} \not\subseteq Q_{\varphi(u')}\) for some \(u' \in S\) satisfying \(P_{u'} \not\subseteq Q_t\). We deduce \(P_{u'} \not\subseteq Q_u\), and hence \(P_{\varphi(u')} \not\subseteq Q_{\varphi(u)}\) by the given condition on \(\varphi\). Therefore, \(B \subseteq Q_{\varphi(u)} \subseteq Q_{\varphi(u')}\), so \(P_{t'} \not\subseteq B\) which contradicts \(B \not\subseteq Q_t\) since \(P_t \not\subseteq Q_{t'}\).

Finally, suppose \(\varphi^\leftarrow B \not\subseteq f^\leftarrow B\). Now we have \(s \in S\) with \(P_s \not\subseteq f^\leftarrow B\), for which \(P_{\varphi(u)} \subseteq B \forall u\) with \(P_s \not\subseteq Q_u\). Take \(s' \in S\) with \(P_s \not\subseteq Q_{s'}\) and \(P_{s'} \not\subseteq f^\leftarrow B\). Now we have \(t \in T\) with \(f \not\subseteq Q_{(s',t)}\) and \(P_t \not\subseteq B\), and then \(t' \in T\) with \(P_{(s',t')} \not\subseteq Q_{(s',t')}\) and \(u \in S\) with \(P_{s'} \not\subseteq Q_u\) and \(P_{\varphi(u)} \not\subseteq Q_{t'}\). Since \(P_s \not\subseteq Q_u\) we deduce \(P_{\varphi(u)} \not\subseteq B\), and hence obtain the contradiction \(P_t \subseteq P_{t'} \subseteq P_{\varphi(u)} \subseteq B\). □

It may be shown that for \(B \in \mathcal{T}\), \(\varphi^\leftarrow B = \varphi^{-1}[B] \iff \varphi^{-1}[B] \in \mathcal{S}\). The proof is left to the interested reader.

We begin by considering the converse situation for simple textures.

**Lemma 3.5.** Let \((f,F)\) be a difunction from \((S,\mathcal{S})\) to \((T,\mathcal{T})\). Then if \(M\) is a molecule in \(\mathcal{S}\), \(f^\rightarrow M\) is a molecule in \(\mathcal{T}\).
Proof. Let $M \in \mathcal{F}$ be a molecule. Then $M \neq \emptyset$ and so we have $f \rightarrow M \neq \emptyset$ by Proposition 2.28(1 a).

Now take $A, B \in \mathcal{F}$ with $f \rightarrow M \subseteq A \cup B$. Then

$$M \subseteq f \leftarrow (f \rightarrow M) \subseteq f \leftarrow (A \cup B) = f \leftarrow (A) \cup f \leftarrow (B)$$

by Corollary 2.26. Since $M$ is a molecule $M \subseteq f \leftarrow A$ or $M \subseteq f \leftarrow B$, so by Lemma 2.9(1), $f \rightarrow M \subseteq f \rightarrow (f \leftarrow A) \subseteq A$ or $f \rightarrow M \subseteq f \rightarrow (f \leftarrow B) \subseteq B$. This shows that $f \rightarrow M$ is a molecule. □

Now let $(S, \mathcal{F})$ and $(T, \mathcal{F})$ be textures with $(T, \mathcal{F})$ simple, and let $(f, F)$ be a difunction from $(S, \mathcal{F})$ to $(T, \mathcal{F})$. For each $s \in S$, $P_s$ is a molecule in $\mathcal{F}$ and so by Lemma 3.5, $f \rightarrow P_s$ is a molecule in $\mathcal{F}$. It follows that there exists $t \in T$, necessarily unique since $\mathcal{F}$ separates the points of $T$, such that $f \rightarrow P_s = P_t$. In this way we obtain a function $\varphi = \varphi_{(f, F)} : S \rightarrow T$ characterized by the equality $P_{\varphi(s)} = f \rightarrow P_s$ for all $s \in S$.

**Proposition 3.6.** The function $\varphi : S \rightarrow T$ corresponding as above to the difunction $(f, F) : (S, \mathcal{F}) \rightarrow (T, \mathcal{F})$, with $(T, \mathcal{F})$ simple, has the properties

(a) $s, s' \in S$, $P_s \nsubseteq Q_{s'} \Rightarrow P_{\varphi(s)} \nsubseteq Q_{\varphi(s')}$.

(b) $P_{\varphi(s)} \nsubseteq B$, $B \in \mathcal{F} \Rightarrow \exists s' \in S$ with $P_s \nsubseteq Q_{s'}$ for which $P_{\varphi(s')} \nsubseteq B$.

Conversely, if $\varphi : S \rightarrow T$ is any function satisfying (a) and (b) then there exists a unique difunction $(f, F) : (S, \mathcal{F}) \rightarrow (T, \mathcal{F})$ satisfying $\varphi = \varphi_{(f, F)}$.

Proof. (a) Let $\varphi = \varphi_{(f, F)}$ and take $P_s \nsubseteq Q_{s'}$. By DF1 we have $t \in T$ satisfying $f \nsubseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \subseteq F$. Let us prove that $f \rightarrow P_s \subseteq P_t$. Assume the contrary and take $t' \in T$ satisfying $f \rightarrow P_s \nsubseteq Q_{t'}$ and $P_{t'} \nsubseteq P_t$. This gives $f \nsubseteq \overline{Q}_{(s',t')}$ and applying DF2 gives $P_t \nsubseteq Q_{t'}$ and hence the contradiction $P_{t'} \subseteq P_t$.

Hence $P_{\varphi(s')} = f \rightarrow P_s \subseteq P_t$, and so $Q_{\varphi(s')} \subseteq Q_t$. However, from $f \nsubseteq \overline{Q}_{(s,t)}$ we have $f \rightarrow P_s \nsubseteq Q_t$ by Lemma 2.6(1), whence $P_{\varphi(s)} \nsubseteq Q_t$ and we obtain $P_{\varphi(s)} \nsubseteq Q_{\varphi(s')}$, so proving (a).

For (b) take $B \in \mathcal{F}$ and $s \in S$ satisfying $P_{\varphi(s)} \nsubseteq B$. Now we have $f \rightarrow P_s \nsubseteq B$ and so $f \rightarrow P_s \nsubseteq Q_t$, $P_t \nsubseteq B$ for some $t \in T$. This gives us $f \nsubseteq \overline{Q}_{(s,t)}$ by Lemma 2.6 (1), and so we have $s' \in S$ satisfying $P_s \nsubseteq Q_{s'}$ and $f \nsubseteq \overline{Q}_{(s',t)}$ by R2. Now $P_{\varphi(s')} = f \rightarrow P_s \nsubseteq Q_t$, so $P_t \subseteq P_{\varphi(s')}$ which gives $P_{\varphi(s')} \nsubseteq B$, as required.

Conversely, let $\varphi : S \rightarrow T$ satisfy conditions (a) and (b). Since $\varphi$ satisfies (a), by Lemma 3.4

$$f = f_\varphi = \bigvee \{ P_{(s,t)} | \exists v \in S \text{ satisfying } P_s \nsubseteq Q_v \text{ and } P_{\varphi(v)} \nsubseteq Q_t \}.$$  

$$F = F_\varphi = \bigcap \{ \overline{Q}_{(s,t)} | \exists v \in S \text{ satisfying } P_s \nsubseteq Q_s \text{ and } P_t \nsubseteq Q_{\varphi(v)} \},$$

defines a difunction $(f, F) : (S, \mathcal{F}) \rightarrow (T, \mathcal{F})$. In order to show $\varphi = \varphi_{(f, F)}$ we must verify that $f \rightarrow P_s = P_{\varphi(s)}$ for all $s \in S$.

Suppose that $f \rightarrow P_s \nsubseteq P_{\varphi(s)}$ and take $t \in T$ with $f \rightarrow P_s \nsubseteq Q_t$ and $P_t \nsubseteq P_{\varphi(s)}$. This gives $f \nsubseteq \overline{Q}_{(s,t)}$ and so we have $t' \in T$, $s' \in S$ satisfying $\overline{P}_{(s',t')} \subseteq \overline{Q}_{(s,t)}$, $P_s \nsubseteq Q_{s'}$ and $P_{\varphi(s')} \subseteq Q_t$. By (a), $P_{\varphi(s')} \nsubseteq Q_{s'}$, whence $P_{\varphi(s)} \nsubseteq Q_t$ and we obtain the contradiction $P_t \subseteq P_{\varphi(s)}$.

Now assume that $P_{\varphi(s)} \nsubseteq f \rightarrow P_s$ and take $t \in T$ with $P_{\varphi(s)} \nsubseteq Q_t$, $P_t \nsubseteq f \rightarrow P_s$. Applying (b) with $B = Q_t$ gives $s' \in S$ satisfying $P_s \nsubseteq Q_{s'}$ and $P_{\varphi(s')} \subseteq Q_t$, from which we obtain $\overline{P}_{(s,t)} \subseteq f$. If we take
t' \in T$ with $P_t \not\subseteq Q_{t'}$, $P_t \not\subseteq f^{-1}P_s$ we obtain $f \not\subseteq \overline{Q}_{(s,t')}$.

But now $f^{-1}P_s \not\subseteq Q_{t'}$, which gives the contradiction $P_t \not\subseteq f^{-1}P_s$.

This establishes $\varphi = \varphi_{(f,F)}$, so it remains to show uniqueness. Suppose we also have $\varphi = \varphi_{(g,G)}$ for some difunction $(g,G):(S,\mathcal{S}) \rightarrow (T,\mathcal{T})$. Now $f^{-1}P_s = g^{-1}P_s$ for all $s \in S$, whence $f^{-1}A = g^{-1}A$ for all $A \in \mathcal{S}$ by Corollary 2.12(2) since $A = \bigvee_{s \in S} P_s$. By Lemma 2.7 we obtain $f = g$ since $f$ and $g$ are relations. This establishes $(f,F) = (g,G)$ by Proposition 2.27. \hfill $\square$

The above proposition singles out point functions satisfying conditions (a) and (b) as being of particular interest. Let us now see that we obtain the same class of point functions if we consider plain textures. Again consider a difunction $(f,F):(S,\mathcal{S}) \rightarrow (T,\mathcal{T})$, and assume this time that $(S,\mathcal{S})$ is plain. For $s \in S$ we now have $P_s \not\subseteq Q_s$, so by DF1 there exists $t \in T$ satisfying $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq F$. Moreover, $t$ is uniquely determined by $s$, for if $t' \in T$ also satisfies $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t')} \not\subseteq F$, then $P_{t'} \not\subseteq Q_{t'}$, $P_t \not\subseteq Q_t$ by DF2, whence $P_t = P_{t'}$, that is $t = t'$. In this way we obtain a point function $\psi = \psi_{(f,F)}:S \rightarrow T$ characterized by $f \not\subseteq \overline{Q}_{(s,\psi(s))}$ and $\overline{P}_{(s,\psi(s))} \not\subseteq F$.

**Proposition 3.7.** The function $\psi:S \rightarrow T$ corresponding as above to the difunction $(f,F):(S,\mathcal{S}) \rightarrow (T,\mathcal{T})$, with $(S,\mathcal{S})$ plain, has the properties

(a) $s,s' \in S$, $P_s \not\subseteq Q_{s'}$ $\implies$ $P_{\psi(s)} \not\subseteq Q_{\psi(s')}$,

(b) $P_{\psi(s)} \not\subseteq B$, $B \in \mathcal{T}$ $\implies$ $\exists s' \in S$ with $P_s \not\subseteq Q_{s'}$ for which $P_{\psi(s')} \not\subseteq B$.

Conversely, if $\psi:S \rightarrow T$ is any function satisfying (a) and (b) then there exists a unique difunction $(f,F):(S,\mathcal{S}) \rightarrow (T,\mathcal{T})$ satisfying $\psi = \psi_{(f,F)}$.

**Proof.** We show that $f^{-1}P_s = P_{\psi(s)}$ for all $s \in S$, whence the proof follows from the proof of Proposition 3.6 since this does not use the simplicity of $(T,\mathcal{T})$ directly.

Suppose first that $f^{-1}P_s \not\subseteq P_{\psi(s)}$, so that we have $t \in T$ satisfying $f^{-1}P_s \not\subseteq Q_t$ and $P_t \not\subseteq P_{\psi(s)}$. By Lemma 2.6(1) we obtain $f \not\subseteq \overline{Q}_{(s,t)}$, and since $\overline{P}_{(s,\psi(s))} \not\subseteq F$ by hypothesis we have $P_{\psi(s)} \not\subseteq Q_t$ by DF2. This contradicts $P_t \not\subseteq P_{\psi(s)}$.

Secondly, assume that $P_{\psi(s)} \not\subseteq f^{-1}P_s$. By Definition 2.5(1) we have $t \in T$ with $P_{\psi(s)} \not\subseteq Q_t$ for which $f \not\subseteq \overline{Q}_{(s,t)}$ $\implies$ $P_s \subseteq Q_z$ for all $z \in S$. By hypothesis $f \not\subseteq \overline{Q}_{(s,\psi(s))}$, whence $f \not\subseteq \overline{Q}_{(s,t)}$ and we obtain $P_s \subseteq Q_t$ by taking $z = s$ in the above implication. Since $(S,\mathcal{S})$ is plain this is a contradiction. \hfill $\square$

**Corollary 3.8.** If $(S,\mathcal{S})$ is plain and $(T,\mathcal{T})$ is simple then the point functions $\varphi$, $\psi:S \rightarrow T$ associated with a difunction $(f,F)$ from $(S,\mathcal{S})$ to $(T,\mathcal{T})$ are equal.

According to Corollary 3.8, no confusion will be caused by using $\varphi = \varphi_{(f,F)}$ to denote this function in both cases. We note in passing that when $(S,\mathcal{S})$ is plain, condition (b) is actually redundant because it may be satisfied by taking $s' = s$. We note in passing the following important result, which will be needed later:

**Lemma 3.9.** If $\varphi:S \rightarrow T$ satisfies (a) and (b) with respect to the textures $(S,\mathcal{S})$, $(T,\mathcal{T})$ then $\varphi^{-1}B = \varphi^{-1}[B]$ for all $B \in \mathcal{T}$. 


Proof. First take \( s \in \varphi^{-1}[B] \). Then if \( P_s \nsubseteq Q_u \) we have \( P_{\varphi(s)} \nsubseteq Q_{\varphi(u)} \) by condition (a), and so \( P_{\varphi(u)} \subseteq P_{\varphi(s)} \subseteq B \), which proves that \( s \in \varphi^+B \). Hence \( \varphi^{-1}[B] \subseteq \varphi^+B \).

Now suppose that \( \varphi^{-1}B \nsubseteq \varphi^{-1}[B] \). Then we have \( s \in \varphi^+B \) for which \( s \notin \varphi^{-1}[B] \). Hence \( \varphi(s) \notin B \), whence \( P_{\varphi(s)} \nsubseteq B \) and we have \( s' \in S \) satisfying \( P_s \nsubseteq Q_{s'} \) and \( P_{\varphi(s')} \subseteq B \) by condition (b). On the other hand \( P_s \subseteq \varphi^+B \) and \( P_s \nsubseteq Q_{s'} \) gives \( \varphi^+B \nsubseteq Q_{s'} \), so there exists \( s'' \in S \) satisfying \( P_{s''} \nsubseteq Q_{s'} \) and \( P_{\varphi(u)} \subseteq B \) for all \( P_{s''} \nsubseteq Q_u \). Taking \( u = s' \) gives the contradiction \( P_{\varphi(s')} \subseteq B \), so \( \varphi^{-1}B \subseteq \varphi^{-1}[B] \). \( \square \)

Although Propositions 3.6 and 3.7 only characterize difunctions in terms of point functions when the domain is plain or the range is simple, the conditions (a) and (b) satisfied by such functions \( \varphi \) are meaningful for general textures, and Lemma 3.4 enables us to associate a difunction \((f_\varphi,F_\varphi)\) with \( \varphi \), as in the proof of Proposition 3.6. If we take our objects to be textures and our morphisms to be point functions between the base sets satisfying (a) and (b), we obtain a construct that we denote by \( \text{fTex} \). Indeed the identity function on \( S \) (which we denote by \( I_S \) or \( I_S \)) clearly satisfies (a) and (b) for any texture \((S,\mathcal{S})\), so it remains to show that functional composition preserves these properties. This is trivial for (a), so consider textures \((S,\mathcal{S}),(T,\mathcal{F}),(U,\mathcal{U});\varphi:S \to T,\psi:T \to U\) satisfying (a) and (b), \( A \in \mathcal{U} \) and \( s \in S \) with \( P_{\psi \circ \varphi(s)} \nsubseteq A \). Then \( P_{\varphi(s)} \nsubseteq A \), so by condition (b) for \( \psi \) there exists \( t \in T \) with \( P_{\psi(s)} \nsubseteq Q_t, P_{\psi(t)} \nsubseteq A \). By condition (b) for \( \varphi \) applied to \( B = Q_t \) we have \( s' \in S \) satisfying \( P_{s'} \nsubseteq Q_{s'}, P_{\psi(s')} \nsubseteq B \), and \( P_{\varphi(s')} \nsubseteq B \), and hence \( P_{\varphi \circ \varphi(s')} \nsubseteq A \) as required.

In the case of complemented textures, we must require in addition that \( \varphi:(S,\mathcal{S},\sigma) \to (T,\mathcal{F},\tau) \) has the property \((f_\varphi,F_\varphi)'=(f_\varphi,F_\varphi)^\tau\). We refer to a point function with this property as \( \text{complemented} \), and denote the construct of complemented textures and complemented point functions satisfying conditions (a) and (b) by \( \text{c} \text{fTex} \).

The above discussion indicates how we might set up a functor \( \mathcal{D} \) from \( \text{fTex} \) to \( \text{dfTex} \). Namely, we let \( \mathcal{D} \) be the identity on objects and map \( \varphi:S \to T \) satisfying (a) and (b) to the difunction \((f_\varphi,F_\varphi):(S,\mathcal{S}) \to (T,\mathcal{F})\). Then

**Theorem 3.10.** \( \mathcal{D}:\text{fTex} \to \text{dfTex} \) defined above is a functor. The restriction \( \mathcal{D}_S:\text{fSTex} \to \text{dfSTex} \) is an isomorphism with inverse \( \mathcal{V}_S: \text{dfSTex} \to \text{fSTex} \) defined by \( \mathcal{V}_S(S,\mathcal{S})=(S,\mathcal{S}) \) and \( \mathcal{V}_S(f,F)=\varphi(f,F) \). Likewise we have isomorphisms between \( \text{fPTex} \) and \( \text{dfPTex} \), \( \text{c} \text{fSTex} \) and \( \text{c} \text{dfSTex} \), \( \text{c} \text{fPSTex} \) and \( \text{c} \text{dfPSTex} \), and between \( \text{c} \text{f} \text{PSTex} \) and \( \text{c} \text{df} \text{PSTex} \).

**Proof.** It is trivial to verify that \( \mathcal{D}(i_S) = (i_S,i_S) \).

Now let \((S,\mathcal{S}),(T,\mathcal{F}),(U,\mathcal{U})\) be textures, \( \varphi:S \to T,\psi:T \to U \) point functions satisfying (a) and (b). We must verify that \((f_\psi \circ f_\varphi,F_\psi \circ F_\varphi) = (f_\psi,F_\psi) \circ (f_\varphi,F_\varphi) \). However, in view of Theorem 2.24(3), Lemmas 2.16, 3.4 and 3.9, this follows at once from \((\psi \circ \varphi)^{-1}[C] = \varphi^{-1}[\psi^{-1}[C]] \) for all \( C \in \mathcal{U} \).

Hence \((f_\psi \circ f_\varphi,F_\psi \circ F_\varphi) = (f_\psi \circ f_\varphi,F_\psi \circ F_\varphi) \), so \( \mathcal{D}:\text{fTex} \to \text{dfTex} \) is a functor.

Now let us restrict to \( \mathcal{D}_S: fSTex \to dSTex \) and define \( \mathcal{V}_S: dSTex \to fSTex \) by \( \mathcal{V}_S(S,\mathcal{S})=(S,\mathcal{S}) \) and \( \mathcal{V}_S(f,F)=\varphi(f,F) \). The equalities \( \mathcal{V}_S \circ \mathcal{D}_S = 1_{\text{fSTex}}, \mathcal{D}_S \circ \mathcal{V}_S = 1_{\text{dSTex}} \) are an immediate consequence of Proposition 3.6, so \( \mathcal{V}_S \) is a functor which is the inverse of \( \mathcal{D}_S \). Hence, \( \mathcal{D}_S \) is an isomorphism. The remaining equivalences may be proved in a similar way. \( \square \)
Example 3.11. (1) For $L = (0, 1]$ consider the discrete texture $(L, \mathcal{P}(L))$, which is plain and simple, and the simple texture $(L, \mathcal{D})$ of Example 1.1(3). By Proposition 3.6, any difunction from $(L, \mathcal{P}(L))$ to $(L, \mathcal{D})$ corresponds to a point function $\varphi : L \to L$ satisfying (a) and (b). However for $s \in L, \{s\} \not\subseteq L \setminus \{s\}$ in $(L, \mathcal{P}(L))$ but $P_{\varphi(s)} \subseteq Q_{\varphi(s)}$ in $(L, \mathcal{D})$, so no point function $\varphi$ can satisfy (a) and there is no difunction from $(L, \mathcal{P}(L))$ to $(L, \mathcal{D})$. In the opposite direction, it is straightforward to show that the difunctions from $(L, \mathcal{D})$ to $(L, \mathcal{P}(L))$ correspond precisely to the constant point functions $r : L \to L$, $r(s) = r \quad \forall s \in L$.

(2) Let $(S, \mathcal{D})$, $(T, \mathcal{F})$ be plain textures. Then for $t \in T$, the constant function $t : S \to T$ clearly satisfies (a) and (b), and hence gives rise to a difunction from $(S, \mathcal{D})$ to $(T, \mathcal{F})$.

(3) For $B = \{a, b, c\}$ consider the discrete complemented texture $(B, \mathcal{P}(B), \pi)$ and the complemented texture $(B, \mathcal{B}, \beta)$ of Example 1.1(6). Both these textures are plain and simple. We consider the existence of complemented difunctions from $(B, \mathcal{P}(B), \pi)$ to $(B, \mathcal{B}, \beta)$. Clearly all point functions $\varphi : B \to B$ satisfy (a) and (b), so there are 27 difunctions between these textures. Since non-complemented difunctions occur in pairs, at least one of these difunctions must be complemented. An examination of all 27 difunctions with the help of the MAPLE computer algebra system shows that only the one corresponding to the constant function $b : B \to B$ is complemented.

(4) Let $(W_X, \mathcal{H}_X, \omega_X)$, $(W_Y, \mathcal{H}_Y, \omega_Y)$ be the Hutton textures of $\mathbb{I}^X$, $\mathbb{I}^Y$, where $X$, $Y$ are non-empty sets and $\mathbb{I}$ the lattice $[0, 1]$ with involution $r' = 1 - r$, $r \in \mathbb{I}$. By [5, Theorem 2.9] we know that $(W_X, \mathcal{H}_X, \omega_X)$ is the complemented product of the textures $(X, \mathcal{P}(X), \pi_X)$ and $(L, \mathcal{D}, \lambda)$ [5], with a similar result for $(W_Y, \mathcal{H}_Y, \omega_Y)$. We may characterize the complemented difunctions from $(W_X, \mathcal{H}_X, \omega_X)$ to $(W_Y, \mathcal{H}_Y, \omega_Y)$ in terms of point functions $\varphi : X \times L \to Y \times L$ as follows:

(i) $\varphi(x, r) = (\varphi_1(x), \varphi_2(x, r))$, where $\varphi_1 : X \to Y$ and $\varphi_2 : X \times L \to L$.

(ii) $r, s \in L, r < s \implies \varphi_2(x, r) < \varphi_2(x, s) \quad \forall x \in X$.

(iii) $x \in X, s, t \in L, t < \varphi_2(x, s) \implies \exists r \in L$ with $r < s$ and $t < \varphi_2(x, r)$.

(iv) (A) $\varphi_2(x, s) + \varphi_2(x, 1 - s) \leq 1 \quad \forall x \in X, \forall s \in L \setminus \{1\}$.

(B) $\varphi_2(x, v) + \varphi_2(x, 1 - s) > 1 \quad \forall x \in X, \forall v, v \in L, v > s$.

Properties (i) and (ii) are easy consequences of condition (a), and (iii) follows from (b). Conversely, (a) and (b) follow from (i)-(iii). Condition (iv) is equivalent to the requirement that the difunction $(f_\varphi, F_\varphi)$ should be complemented. To see this first note that

$$f_\varphi = \bigvee \{P_{((x,s),(\varphi_1(x),t))} \mid x \in X, s, t \in L, t < \varphi_2(x, s)\},$$

and

$$F_\varphi = \bigcap \{Q_{((x,s),(\varphi_1(x),t))} \mid x \in X, s, t \in L, \exists v \in L \text{ with } s < v, \varphi_2(x, v) < t\}.$$

Since $\omega_X$ is the product of $\pi_X$ and $\lambda$ it is easy to verify $\omega_X(P_{(x,s)}) = Q_{(x,1-s)}$, whence $\omega_X(Q_{(x,s)}) = P_{(x,1-s)}$. Here, for $s = 1$, $P_{(x,1-s)}$ should be interpreted as $\emptyset$ and $Q_{(x,1-s)}$ as $(X \setminus \{x\}) \times L$. This, and the corresponding results for $\omega_Y$, enables us to verify that

$$f'_\varphi = \bigcap \{Q_{((x,s),(\varphi_1(x),t))} \mid x \in X, s, t \in L, \exists x \in L, x < 1 - s, 1 - t < \varphi_2(x, x)\}.$$
complementation, but it is not necessary. Indeed the function
\[
\varphi_2(x, r) = \begin{cases} 
  r/2, & 0 < r \leq 1/3, \\
  r/2 + 1/4, & 1/3 < r \leq 2/3, \\
  r/2 + 1/2, & 2/3 < r \leq 1,
\end{cases}
\]
defined for \(x \in X\) satisfies (iv), but \(\varphi_2(x, 1/3) + \varphi_2(x, 1 - 1/3) = 3/4\).

Example 3.11(1) shows that dfSTex is not connected [15] in the sense that there exist objects \((S, \mathcal{F})\), \((T, \mathcal{F})\) with dfSTex\((S, \mathcal{F})(T, \mathcal{F})\) = \(\emptyset\). In the same way, fSTex, dfTex, fTex are not connected, and since the textures considered in Example 3.11(1) have complements, the same is true of the complemented versions of these categories. On the other hand, Example 3.11(2) shows that dfPTex and fPTex are connected. It is not known if dfPTex and fPTex are connected, but certainly Example 2.11(3) shows that constant point functions between plain textures need not be complemented.

Now let \(\mathcal{A}\) be the forgetful functor \(\mathcal{A} : fPTex \to Set\) and consider the following diagram:

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{\alpha} & fPTex \\
\downarrow \mathcal{A} & & \uparrow \mathcal{A}'_p, \\
dfPTex & \xrightarrow{\mathcal{A}} & fPTex
\end{array}
\]

Then

**Theorem 3.12.** \(\mathcal{A}\) is an adjoint of \(\mathcal{A}'_p \circ \mathcal{I}\) and \(\mathcal{I}\) a co-adjoint of \(\mathcal{A} \circ \mathcal{A}'_p\).

**Proof.** Take \(X \in \text{Ob}\ Set\). Then \((\mathcal{I}_X, (X, \mathcal{P}(X)))\) is an \(\mathcal{A}\)-universal arrow with domain \(X\). Indeed, it is clearly an \(\mathcal{A}\)-structured arrow with domain \(X\) because \(\mathcal{A}(X, \mathcal{P}(X)) = X\) and \(\mathcal{I}_X : X \to X\) is a Set morphism, so it remains to prove the universal property. Take \((S, \mathcal{F}) \in \text{Ob} fPTex\) and let \(\varphi : X \to \mathcal{A}(S, \mathcal{F}) = S\) be a morphism in Set. Since \((X, \mathcal{P}(X))\) is plain, \(\varphi\) satisfies (a). It also satisfies (a) because for \(x, x' \in X\), \(P_x \subseteq Q_{x'} \implies x = x' \implies \varphi(x) = \varphi(x') \implies P_{\varphi(x)} \subseteq Q_{\varphi(x')}\) since \((S, \mathcal{F})\) is plain. Hence \(\varphi \in \text{Mor}\ fPTex\), and this is clearly the unique morphism satisfying \(\mathcal{A}(\varphi) \circ \mathcal{I}_X = \varphi\).

Since \((X, \mathcal{P}(X)) = \mathcal{A}'_p(\mathcal{I}_X(X))\) for each \(X\), it follows from [1, Theorem 19.1] that \(\mathcal{A}\) is an adjoint of \(\mathcal{A}'_p \circ \mathcal{I}\). Since \(\mathcal{A}'_p\) is an isomorphism, \(\mathcal{I}\) is also a co-adjoint of \(\mathcal{A} \circ \mathcal{A}'_p\). \(\square\)

**Notes 3.13.** (i) No use is made of simplicity in the proof of Theorem 3.12, so the forgetful functor \(\mathcal{A} : fPTex \to Set\) is also an adjoint of \(\mathcal{A}'_p \circ \mathcal{I}\), and \(\mathcal{I} \circ \mathcal{A}\) a co-adjoint of \(\mathcal{A} \circ \mathcal{A}'_p\) (see the left-hand diagram below). On the other hand the restriction to plain textures is essential. Indeed, if we let \(\mathcal{A} : fSTex \to Set\) be the forgetful functor, as in the right-hand diagram, this is not an adjoint of \(\mathcal{A}'_s \circ \mathcal{I}\).
(ii) If $A_c : \text{cfPSTex} \to \text{Set}$ is the forgetful functor, then $A_c$ is not an adjoint of $\mathcal{U}_{ps} \circ \Sigma_c$.

To see this assume the contrary. Then there is an $A_c$-universal arrow with domain $B = \{a, b, c\} \in \text{Ob Set}$ of the form $(\psi(B, \mathcal{P}(B), \pi_B))$. Consider the complemented plain, simple texture $(B; B; d_{FF})$ of Example 1.1(6), regarded as an object of $\text{cfPSTex}$, and the $\text{Set}$ morphism $i_B : B \to B$. By the universal property we have a unique $\varphi \in \text{cfPSTex}((B, \mathcal{P}(B), \pi_B), (B, \mathcal{B}, \beta))$ satisfying $\varphi \circ \psi = \mathcal{A}_c(\varphi) \circ \psi = i_B$. However, by Example 3.11(3), $\varphi = b$ and so $\varphi \circ \psi$ is a constant function, which is a contradiction.

We end this section by discussing some properties of the morphisms in $\text{dfTex}$. Various other properties of $\text{dfTex}$ and related categories will be given in subsequent sections.

**Proposition 3.14.** In the category $\text{dfTex}$:

1. Every section $[1,15]$ is injective.
2. Every injective morphism is a monomorphism $[1,15]$.
3. Every retraction $[1,15]$ is surjective.
4. Every surjective morphism is an epimorphism $[1,15]$.
5. A morphism is an isomorphism $[1,15]$ if and only if it is bijective.

**Proof.** Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction.

1. If $(f, F)$ is a section there exists a difunction $(g, G) : (T, \mathcal{T}) \to (S, \mathcal{S})$ satisfying

   $$(g, G) \circ (f, F) = (i_S, I_S).$$

   Since $(g, G)$ is a difunction it satisfies $\text{DF1}$, so $g^{-} \circ G \subseteq I_S$ by Lemma 2.23(1 ii). Taking the composition of both sides with $F$ on the right, and using $G \circ F = I_S$ gives $g^{-} \subseteq F$. Taking the inverse of both sides gives $F^{-} \subseteq g$ and so $F^{-} \circ f \subseteq g \circ f = I_S$. By Theorem 2.32(2), $(f, F)$ is injective.

2. Let $(f, F)$ be injective and take morphisms $(g, G), (h, H)$ in $\text{dfTex}$ satisfying

   $$(f, F) \circ (g, G) = (f, F) \circ (h, H).$$

   Since $F \circ G = F \circ H$, for $A$ in the domain texturing of $(g, G), (h, H)$, we have $F^{-}(G^{-}A) = F^{-}(H^{-}A)$ by Lemma 2.16(1). Now $G^{-}A = H^{-}A$ by Corollary 2.33(2), and so $G = H$ by Lemma 2.7(2). Finally, $(g, G) = (h, H)$ follows from Proposition 2.27. Hence $(f, F)$ is a monomorphism.

3. Dual to (1) and (2), respectively.

4. If $(f, F)$ is bijective then $(f, F)^{-}$ is a morphism in $\text{dfTex}$ by Theorem 2.31(3). It remains to show that

   $$(f, F) \circ (f, F)^{-} = (i_T, I_T) \quad \text{and} \quad (f, F)^{-} \circ (f, F) = (i_S, I_S).$$

   However $(f, F)$ is surjective and so $i_T \subseteq f \circ F^{-}$ by Theorem 2.32(1), while condition $\text{DF2}$ for $(f, F)$ gives $f \circ F^{-} \subseteq i_T$. Hence $f \circ F^{-} = i_T$. This is sufficient to establish the first equality since
taking the inverse of both sides gives \( F \circ f^{-1} = 1\). The proof of the second equality follows from injectivity and DF1 in the same way.

Conversely, if \((f,F)\) is an isomorphism it is a section and a retraction, so the result follows from (1) and (3). □

The first four results in Proposition 3.14 hold trivially for \(\text{fTex}\) because it is a construct. With regard to isomorphisms in \(\text{fTex}\) we have the following:

**Proposition 3.15.** A point function is an isomorphism in \(\text{fTex}\) if and only if it is a textural isomorphism in the sense of [3].

**Proof.** We recall that \(\varphi\) is a textural isomorphism from \((S,\mathcal{S})\) to \((T,\mathcal{T})\) if it is a bijective point function from \(S\) to \(T\) satisfying \(A \in \mathcal{S} \implies \varphi[A] \in \mathcal{T}\) and for which \(A \mapsto \varphi[A]\) is a bijection from \(\mathcal{S}\) to \(\mathcal{T}\). Clearly this is equivalent to requiring that \(\varphi\) be bijective with inverse \(\psi\) and that \(A \in \mathcal{S} \implies \varphi[A] \in \mathcal{T}\) and \(B \in \mathcal{T} \implies \psi[B] \in \mathcal{S}\).

Firstly let \(\varphi\) be a \(\text{fTex}\)-isomorphism. Since \(\varphi\) is a section and a retraction it is certainly bijective. Denote by \(\psi\) the inverse of \(\varphi\) in \(\text{fTex}\). Then for \(A \in \mathcal{S}\) we have \(\varphi[A] = \psi^{-1}[A] = \psi^{-1}A \in \mathcal{T}\) by Lemma 3.9. Likewise \(\psi[B] \in \mathcal{S}\) for all \(B \in \mathcal{T}\), so \(\varphi\) is a textural isomorphism by the comment above.

Conversely, let \(\varphi:S \to T\) be a textural isomorphism from \((S,\mathcal{S})\) to \((T,\mathcal{T})\). We begin by showing that \(\varphi[P_s] = P_{\varphi(s)}\) for all \(s \in S\). Firstly \(\varphi(s) \in \varphi[P_s]\) and so \(P_{\varphi(s)} \subseteq \varphi[P_s]\) since \(\varphi[P_s] \in \mathcal{T}\) by hypothesis. Conversely suppose that \(\varphi[P_s] \not\subseteq P_{\varphi(s)}\) and take \(t \in T\) with \(\varphi[P_t] \not\subseteq Q_t\) and \(P_t \not\subseteq P_{\varphi(s)}\). If \(\psi\) denotes the inverse of \(\varphi\) then \(\psi[\varphi[P_t]] \not\subseteq \psi[Q_t]\), whence \(P_t \not\subseteq \psi[Q_t]\). Since \(P_{\varphi(s)} \subseteq Q_t\) we obtain \(P_t \not\subseteq \psi[P_{\varphi(s)}]\), and since \(\psi\) is also a textural isomorphism \(P_{\psi[\varphi(s)]} \subseteq \psi[P_{\varphi(s)}]\), which gives the contradiction \(P_t \not\subseteq P_s\).

This verifies \(\varphi[P_s] = P_{\varphi(s)}\), as required. Now \(\varphi[Q_t] = \varphi[\bigvee\{P_u | P_u \not\subseteq P_t\}] = \bigvee\{\varphi[P_u] | P_s \not\subseteq P_u\}\) by [3, Proposition 2.3]. Hence \(\varphi[Q_t] = \bigvee\{\varphi[P_u] | P_s \not\subseteq P_u\}\). However \(P_s \not\subseteq P_u\) is easily seen to be equivalent to \(P_{\varphi(s)} \not\subseteq P_{\varphi(u)}\), and since \(\varphi\) is surjective we deduce that \(\varphi[Q_t] = Q_{\varphi(s)}\) also. Hence \(P_s \not\subseteq Q_{\varphi(s)} \implies \varphi[P_s] \not\subseteq Q_{\varphi(s)}\), so \(\varphi\) satisfies (a). Finally, if \(B \in \mathcal{T}\) and \(P_{\varphi(s)} \not\subseteq B\) then \(P_s = P_{\varphi(s)} = \psi[\varphi[P_s]] \not\subseteq \psi[Q_{\varphi(s)}]\) by the above applied to the textural isomorphism \(\psi\), and now we have \(s' \in S\) satisfying \(P_s \not\subseteq Q_{\varphi(s')}\) and \(P_s \not\subseteq \psi[Q_{\varphi(s')}]\). It follows easily that \(P_{\varphi(s')} \not\subseteq B\), whence (b) is satisfied and so \(\varphi \in \text{Mor fTex}\). Likewise \(\psi \in \text{Mor fTex}\) and we have established that \(\varphi\) is a \(\text{fTex}\)-isomorphism. □

**Acknowledgements**

The authors would like to express their appreciation to the Area Editor Professor Rodabaugh for suggesting that the categorical underpinnings of the material presented here should be made explicit, to the referees for their constructive comments which have enriched the content and improved the exposition, and to Professor Brümmer for his infectious enthusiasm and unfailing encouragement.

**References**