Ditopological texture spaces and fuzzy topology, II. Topological considerations

Lawrence M. Brown\textsuperscript{a,∗}, Riza Ertürk\textsuperscript{a}, Şenol Dost\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Faculty of Science, Hacettepe University, Beytepe, Ankara 06532, Turkey
\textsuperscript{b}Department of Secondary Science and Mathematics Education, Hacettepe University, Beytepe, Ankara, Turkey

Available online 12 March 2004

Abstract

This is the second of three papers which develop various fundamental aspects of the theory of ditopological texture spaces in a categorical setting and present important links with the theory of \textit{L}-topological spaces. In the first paper in this series, subtitled Basic Concepts, the authors presented the notions of direlation and difunction between textures and introduced the category \textit{dfTex}, the construct \textit{fTex} and several related categories. In this paper the category \textit{dfDitop} of ditopological texture spaces and bicontinuous difunctions is defined and the forgetful functor \(\mathcal{U} : \textit{dfDitop} \rightarrow \textit{dfTex}\) shown to be topological. Several properties are discussed, including the existence of products and coproducts. An equivalence with the category of classical Hutton spaces is presented and the paper ends with a consideration of \textit{L}-valued sets and topologies from the viewpoint of ditopological texture spaces.

© 2004 Elsevier B.V. All rights reserved.

\textbf{MSC:} 54A40; 54B30; 03E72; 06A99

\textbf{Keywords:} Texture; Difunction; Ditopology; Base; Bicontinuity; Category; Initial source; Topological functor; Product; Coproduct; Hutton Space; \textit{L}-topological space; Chang–Goguen fuzzy topology; Topological \textit{L}-topology; Lowen functors; Hypergraph functor

1. Introduction

This is the second of three papers which develop various fundamental aspects of the theory of ditopological texture spaces in a categorical setting and present important links with the theory of \textit{L}-topological spaces.

∗ Corresponding author.

E-mail addresses: brown@hacettepe.edu.tr (L.M. Brown), rerturk@hacettepe.edu.tr (R. Ertürk), dost@hacettepe.edu.tr (Şenol Dost).

0165-0114/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.fss.2004.02.010
In the previous paper in this series, subtitled Basic Concepts, the authors presented the notions of direlation and difunction between textures, and introduced the category \( \text{dfTex} \) of textures and difunctions between textures. It was shown that when the texture \((S, \mathcal{S})\) is plain, or the texture \((T, \mathcal{S})\) is simple, a difunction \((f, F):(S, \mathcal{S}) \to (T, \mathcal{S})\) may be represented by a function \(\phi: S \to T\) satisfying the conditions

(a) \(s, s' \in S, \ P_s \not\in Q_{s'} \Rightarrow P_{\phi(s)} \not\in Q_{\phi(s')}\).

(b) \(P_{\phi(s)} \not\in B, B \in \mathcal{S} \Rightarrow \exists s' \in S \text{ with } P_s \not\in Q_{s'} \text{ for which } P'_{s'} \subseteq B\).

This lead to the definition of the construct \(\text{dfTex}\) of textures and functions between the base sets satisfying (a) and (b). Restricting to complemented textures and difunctions produced the category \(\text{cdfTex}\) and the construct \(\text{cfTex}\). Further restrictions to plain or simple textures also lead to important subcategories.

In this paper the category \(\text{dfDitop}\) of ditopological texture spaces and bicontinuous difunctions is defined and the forgetful functor \(U: \text{dfDitop} \to \text{dfTex}\) shown to be topological. Several related categories and constructs are introduced and studied. Various properties are also discussed, including the existence of products and coproducts. Finally an equivalence involving the category of classical Hutton spaces is presented and the paper ends with a consideration of \(L\)-valued sets and topologies from the viewpoint of ditopological texture spaces.

Frequent reference will be made to the first paper of this series [9]. Otherwise the series is largely self-contained, although the reader may wish to refer to [4–8,12–16,20,28] for additional background and motivation. The reader is referred to [17] for terms from lattice theory not defined here. Generally we follow the terminology of [1] for general concepts relating to category theory. If \(A\) is a category, \(\text{Ob} A\) will denote the class of objects and \(\text{Mor} A\) the class of morphisms of \(A\). We will sometimes use the notation \(A(A_1, A_2)\) for the set of \(A\) morphisms from \(A_1\) to \(A_2\). The categorical results presented in this paper form part of the third author’s continuing Ph.D. studies in the Department of Mathematics of Hacettepe University.

### 2. Dichotomous topologies and the category dfDitop

A **dichotomous topology**, or **ditopology** for short, on a texture \((S, \mathcal{S})\) is a pair \((\tau, \kappa)\) of subsets of \(\mathcal{S}\), where the set of **open sets** \(\tau\) satisfies

1. \(S, \emptyset \in \tau\),
2. \(G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau\) and
3. \(G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau\),

and the set of **closed sets** \(\kappa\) satisfies

1. \(S, \emptyset \in \kappa\),
2. \(K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa\) and
3. \(K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa\).

Hence a ditopology is essentially a “topology” for which there is no a priori relation between the open and closed sets. For \(A \in \mathcal{S}\) we define the **closure** \([A]\) and the **interior** \(]A[\) of \(A\) under \((\tau, \kappa)\) by the equalities

\[
[A] = \bigcap\{K \in \kappa \mid A \subseteq K\} \quad \text{and} \quad ]A[ = \bigvee\{G \in \tau \mid G \subseteq A\}.
\]
On the other hand, suppose that \((S, \mathcal{S})\) has a complementation \(\sigma\). Then if \(\tau\) and \(\kappa\) are related by \(\kappa = \sigma[\tau]\) we say that \((\tau, \kappa)\) is a complemented ditopology on \((S, \mathcal{S}, \sigma)\) \([5, 6, 12, 13, 20]\). In this case we have \(\sigma([A]) = [\sigma(A)]\) and \(\sigma([A]) = (\sigma(A))\).

**Example 2.1.** (1) For any texture \((S, \mathcal{S})\) a ditopology \((\tau, \kappa)\) with \(\tau = \mathcal{S}\) is called discrete, and one with \(\kappa = \mathcal{S}\) is called codiscrete.

(2) For any texture \((S, \mathcal{S})\) a ditopology \((\tau, \kappa)\) with \(\tau = \emptyset, S\) is called indiscrete, and one with \(\kappa = \emptyset, S\) is called coindiscrete.

(3) For any topology \(\mathcal{T}\) on \((X, (\mathcal{T}, \mathcal{T}^c)), \mathcal{T}^c = \{X \setminus G \mid G \in \mathcal{T}\}\), is a complemented ditopology on \((X, \mathcal{P}(X), \pi_X)\).

(4) For any bitopology \((u, v)\) on \((X, (u, v))\) is a ditopology on \((X, \mathcal{P}(X))\).

(5) \(\tau_1 = \{(0, r) \mid 0 \leq r \leq 1\} \cup \{\emptyset\}\), \(\kappa_1 = \{(0, r) \mid 0 \leq r \leq 1\} \cup \{\emptyset\}\) defines a complemented ditopology, called the natural ditopology on \((1, \mathcal{I}, 1)\).

(6) Consider the Hutton texture \((W_a, \mathcal{U}_a, \mathcal{O}_a)\) of \([A]\) for \(A = \{a, b\}\). Hence \(W_a = \{a, b\} \times L\), where \(L = (0, 1]\), \(\mathcal{U}_a\) is the product of the texturings \(\mathcal{P}(L)\) and \(\mathcal{L}\), and \(\mathcal{O}_a\) is the product of the complementations \(\pi_a\) and \(\lambda\) (see [9, Examples 1.1(2,3,4)]). Then \(\tau = \emptyset, \{a\} \times (0, 1/2], W_a\), \(\kappa = \mathcal{O}_a[\tau] = \{W_a, (\{b\} \times L) \cup (A \times (0, 1/2)], \emptyset\}\) is a complemented ditopology on \((W_a, \mathcal{O}_a, \mathcal{O}_a)\).

Another important example of a ditopological texture space may be found in Example 2.12, and further examples will be given in the last paper of this series.

Let us now define appropriate notions of continuity for difunctions between ditopological texture spaces.

**Definition 2.2.** Let \((S_k, \mathcal{I}_k, \tau_k, \kappa_k)\), \(k = 1, 2\), be ditopological texture spaces and \((f, F)\) a difunction from \((S_1, \mathcal{I}_1)\) to \((S_2, \mathcal{I}_2)\). Then

1. \((f, F)\) is continuous if \(G \in \tau_2 \Rightarrow F^{-} G \in \tau_1\).
2. \((f, F)\) is cocontinuous if \(K \in \kappa_2 \Rightarrow f^{-} K \in \kappa_1\).
3. \((f, F)\) is bicontinuous if it is continuous and cocontinuous.

**Lemma 2.3.** Let \(\mathcal{P}\) denote any of the properties continuous, cocontinuous, bicontinuous. Then

1. For any ditopological texture space \((S, \mathcal{I}, \tau, \kappa)\) the identity difunction \((i, I)\) has property \(\mathcal{P}\).
2. If \((S_k, \mathcal{I}_k, \tau_k, \kappa_k)\), \(k = 1, 2, 3\) are ditopological texture spaces, \((f, F) : (S_1, \mathcal{I}_1, \tau_1, \kappa_1) \rightarrow (S_2, \mathcal{I}_2, \tau_2, \kappa_2)\) and \((g, G) : (S_2, \mathcal{I}_2, \tau_2, \kappa_2) \rightarrow (S_3, \mathcal{I}_3, \tau_3, \kappa_3)\) are difunctions satisfying property \(\mathcal{P}\) then \((g, G) \circ (f, F)\) also has property \(\mathcal{P}\).
3. If \((S_k, \mathcal{I}_k, \sigma_k, \tau_k, \kappa_k)\), \(k = 1, 2\) are complemented ditopological texture spaces, \((f, F) : (S_1, \mathcal{I}_1, \sigma_1) \rightarrow (S_2, \mathcal{I}_2, \sigma_2)\) a complemented difunction, then \((f, F)\) is continuous if and only if it is cocontinuous. Hence all the properties \(\mathcal{P}\) are equivalent in this case.

**Proof.** (1) Immediate from [9, Lemma 2.9(3)].

(2) Clear since \((g \circ f)^{-} B = f^{-} (g^{-} B), (F \circ G)^{-} B = F^{-} (G^{-} B), \forall B \in \mathcal{I}_3\) by [9, Lemma 2.16(2)].

(3) For \(B \in \mathcal{I}_2\) we have \(F^{-} B = (f \circ F)^{-} \sigma_1 (f^{-} \sigma_2 B)\) by [9, Lemma 2.20], since \((f, F)\) is a complemented difunction. Hence for \(G \in \tau_2, F^{-} G \in \tau_1 \iff f^{-} \sigma_1 (G) \in \sigma_1 (\tau_1) = \kappa_1\) since \((\tau_1, \kappa_1)\) is
complemented. The result now follows on noting that \( \kappa_2 = \{ \sigma_2(G) \mid G \in \tau_2 \} \) since \((\tau_2, \kappa_2)\) is complemented. \(\square\)

In particular

**Theorem 2.4.** Ditopological texture spaces and bicontinuous difunctions form a category.

**Definition 2.5.** The category whose objects are ditopological texture spaces and morphisms are bicontinuous difunctions is denoted by \( \text{dfDitop} \).

The same notational conventions are used when restricting to plain, simple or complemented ditopological texture spaces as were used for \( \text{dfTex} \). Hence we have the categories \( \text{dfSDitop} \), \( \text{dfPDitop} \) and \( \text{dfPSDitop} \). Where no confusion can arise the same names will be used as previously for the functors between the corresponding categories.

For \( \varphi \in \text{fTex}((S_1, \mathcal{S}_1), (S_2, \mathcal{S}_2)) \) we recall from [9, Lemma 3.9] that for \( B \in \mathcal{S}_2 \) we have \( \varphi^{-1}[B] = \varphi^* B \in \mathcal{S}_1 \). Hence we may say that \( \varphi \) is \((\tau_1 - \tau_2)\)-continuous (respectively, \((\kappa_1 - \kappa_2)\)-cocontinuous) if \( G \in \tau_2 \Rightarrow \varphi^{-1}[G] \in \tau_1 \) \( (F \in \kappa_2 \Rightarrow \varphi^{-1}[F] \in \kappa_1) \). Again, \( \varphi \) will be called bicontinuous if it is continuous and cocontinuous.

**Definition 2.6.** The construct whose objects are ditopological texture spaces and whose morphisms are bicontinuous \( \text{fDitop} \)-morphisms is denoted by \( \text{fDitop} \). Restricting to ditopological simple or plain textures gives the subconstructs \( \text{fSDitop} \), \( \text{fPDitop} \) and \( \text{fPSDitop} \). Likewise, restricting to complemented ditopological texture spaces and complemented bicontinuous morphisms gives the construct \( \text{cfDitop} \) and its subconstructs \( \text{cfSDitop} \), \( \text{cfPDitop} \) and \( \text{cfPSDitop} \).

On analogy with the functor \( \mathcal{D} : \text{fTex} \to \text{dfTex} \) defined in [9] we now let \( \mathcal{D} \) map \((S, \mathcal{S}, \tau, \kappa)\) to itself and \( \varphi \in \text{fDitop}((S_1, \mathcal{S}_1, \tau_1, \kappa_1), (S_2, \mathcal{S}_2, \tau_2, \kappa_2)) \) to \((f_\varphi, F_\varphi) \in \text{dfDitop}((S_1, \mathcal{S}_1, \tau_1, \kappa_1), (S_2, \mathcal{S}_2, \tau_2, \kappa_2)) \). Then

**Theorem 2.7.** \( \mathcal{D} : \text{fDitop} \to \text{dfDitop} \) defined above is a functor. The restriction \( \mathcal{D}_\mathfrak{s} : \text{fSDitop} \to \text{dfSDitop} \) is an isomorphism with inverse \( \mathfrak{V}_\mathfrak{s} : \text{dfSDitop} \to \text{fSDitop} \) defined by \( \mathfrak{V}_\mathfrak{s}(f, F)(S, \mathcal{S}, \tau, \kappa) = (S, \mathcal{S}, \tau, \kappa) \) and \( \mathfrak{V}_\mathfrak{s}(f, F) = \varphi\). Likewise we have isomorphisms between \( \text{fPDitop} \) and \( \text{dfPDitop} \), \( \text{cfDitop} \) and \( \text{cfDitop} \), \( \text{cfDitop} \) and \( \text{cfDitop} \), \( \text{cfDitop} \) and \( \text{cfDitop} \), and between \( \text{cfDitop} \) and \( \text{cfDitop} \).

**Proof.** In view of [9, Theorem 3.10] we need only note that the bicontinuity of \( \varphi \) implies that of \((f_\varphi, F_\varphi)\) since \( f_\varphi^* B = F_\varphi^* B = \varphi^* B = \varphi^{-1}[B] \) for all \( B \in \mathcal{S}_2 \) by [9, Lemmas 3.4 and 3.9]. \(\square\)

A morphism in \( \text{fTex} \) which is a constant point function is clearly bicontinuous with respect to arbitrary ditopologies and hence is a morphism in \( \text{fDitop} \). Hence the connectedness of \( \text{fPTex} \) and \( \text{dfPTex} \) proved in [9] imply that of \( \text{fPDitop} \) and \( \text{dfPDitop} \), respectively.
Example 2.8. The construct □-Top (where □ is the Hutton algebra [0,1] with order reversing involution $r \mapsto 1-r$) has as objects □-topological spaces and as morphisms □-continuous maps. Here an □-topological space is a pair $(X, T)$ where $X$ is a set and $T$ a subset of the Hutton algebra □ closed under arbitrary joins and finite meets and (therefore) containing the top $\top$ and bottom $\bot$ of □.

For □-topological spaces $(X, T), (Y, V)$ a map $\varphi : X \to Y$ is called □-continuous if $\{ g \circ \varphi \mid g \in V \} \subseteq T$ (cf. [22], see also [28]).

As in [9, Example 3.11(4)], see also [7], the Hutton texture $(W_X, \mathcal{W}_X, \omega_X)$ of $(\bot, r)$ is the complemented product of the discrete texture $(X, \mathcal{P}(X), \pi_X)$ and the Hutton texture $(L, \mathcal{L}, \lambda)$ of $(\bot, l)$ [9, Example 1.1(3)]. Given an □-topological space $(X, T)$ we may define a ditopology $(\tau_T, \kappa_T)$ on $(W_X, \mathcal{W}_X, \omega_X)$ by setting $\tau_T = \{ \tilde{g} \mid g \in T \}$ and $\kappa_T = \{ \tilde{k} \mid k \in T' \}$ where $T' = \{ f' \mid f \in T \}$. It is trivial to verify that $(\tau_T, \kappa_T)$ is a complemented ditopology. Given $\varphi : X \to Y$ the function $(\varphi, 1_L)$ from $W_X$ to $W_Y$ defined by $(x, r) \mapsto (\varphi(x), r)$ is a cfSTex-morphism because it trivially satisfies the conditions (i)–(iv) of [9, Example 3.11(4)]. By Lemma 2.3(3) it is also a cfDitop-morphism if and only if $\varphi$ is □-continuous. Denoting by □-Ditop the construct of Hutton textures of $(\bot, r)$, equipped with a complemented ditopology, and bicontinuous maps of the form $(\varphi, 1_L)$, we obtain at once:

(i) The constructs □-Top and □-Ditop are isomorphic with respect to the functor $\mathcal{T}$ taking $(X, T)$ to $(W_X, \mathcal{W}_X, \omega_X, \tau_T, \kappa_T)$ and $\varphi$ to $(\varphi, 1_L)$.

(ii) □-Ditop is a subconstruct of cfSDitop, and hence of cfDitop.

By [9, Example 3.11(4)] it is clear that □-Ditop is not embedded as a full subconstruct of cfDitop. We will return to a discussion of □-Top in the final section of this paper.

Let $(X, \mathcal{T})$ be a topological space. The functor $\mathcal{T}_c : \text{Set} \to \text{cdfPSDitop}$ defined in [9] becomes a functor $\mathcal{T}_C : \text{Top} \to \text{cdfPSDitop}$ if we set $\mathcal{T}_C((S, \mathcal{T}), (X, \mathcal{P}(X), \pi_X, \mathcal{T}, \mathcal{T}')) = (\mathcal{T}_c(\mathcal{T}), (\varphi, \varphi'))$, since clearly $(\mathcal{T}, \mathcal{T}')$ is a complemented ditopology on $(X, \mathcal{P}(X), \pi_X)$ and $(\varphi, \varphi')$ is a bicontinuous difunction whenever $\varphi$ is a morphism in Top. Also the forgetful functor $\mathcal{A}_c : \text{cfPSDitop} \to \text{Top}$ becomes a functor $\mathcal{A}_c : \text{cfPSDitop} \to \text{Top}$ if we set $\mathcal{A}_c((S, \mathcal{T}, \sigma, \tau, \sigma[\tau]) = (S, \tau)$ and $\mathcal{A}_c(\varphi) = \varphi$. This makes essential use of the fact that $(S, \mathcal{S})$ is plain, since this ensures that $\tau$ is closed under arbitrary unions and is therefore a topology on $S$ in the usual sense. Note also that the bicontinuity of $\varphi$ in cfPSDitop clearly implies its continuity in Top.

\[
\begin{align*}
\text{Top} & \xrightarrow{\mathcal{T}_C} \text{cdfPSDitop} \\
\text{Ditop} & \xrightarrow{\mathcal{A}_c} \text{cfPSDitop} \\
\text{Bitop} & \xrightarrow{\mathcal{A}} \text{Top}
\end{align*}
\]
\( \mathfrak{A} : \text{fPSDitop} \to \text{Bitop} \) on setting \( \mathfrak{A}(S, \mathcal{S}, \tau, \kappa) = (S, \tau, \kappa^c) \) and \( \mathfrak{A}(\varphi) = \varphi \). The definition of \( \mathfrak{A} \) again uses the fact that \((S, \mathcal{S})\) is plain.

**Theorem 2.9.** In the right-hand diagram above, \( \mathfrak{A} \) is an adjoint of \( \mathfrak{Y}_{ps} \circ \mathfrak{I} \) and \( \mathfrak{I} \) a co-adjoint of \( \mathfrak{A} \circ \mathfrak{Y}_{ps} \).

**Proof.** Take \((X, u, v) \in \text{Ob Bitop}\). We show that \((t_X, (X, \mathcal{P}(X), u, v^c))\) is an \( \mathfrak{A} \)-universal arrow. It is clearly an \( \mathfrak{A} \)-structured arrow, so take an object \((S, \mathcal{S}, \tau, \kappa)\) in \( \text{fPSDitop} \) and \( \varphi \in \text{Bitop}((X, u, v), (S, \mathcal{S}, \tau, \kappa^c)) \). As in [9, Theorem 3.12] we know that \( \varphi \in \text{Mor fPSTex} \), and that it is the unique such morphism satisfying \( \mathfrak{A}(\varphi) \circ t_X = \varphi \), so it remains to verify that \( \varphi : (X, \mathcal{P}(X), u, v^c) \to (S, \mathcal{S}, \tau, \kappa) \) is bicontinuous. However, for \( G \in \tau \) we have \( \varphi^{-1} G = \varphi^{-1}[G] \) by [9, Lemma 3.9], and \( \varphi^{-1}[G] \in u \) since \( \varphi \) is \( u \)-continuous. Likewise, for \( K \in \kappa \) we have \( \varphi^{-1} K = X \backslash \varphi^{-1}[S \setminus K] \in v^c \) since \( \varphi \) is \( v \)-continuous.

**Note 2.10.** Neither the definition of the functor \( \mathfrak{A} \) nor the proof of Theorem 2.9 depend explicitly on the simplicity of the textures involved. Hence we may define a corresponding functor \( \mathfrak{A} : \text{fSDitop} \to \text{Bitop} \), and now \( \mathfrak{A} \) is an adjoint of \( \mathfrak{Y}_{ps} \circ \mathfrak{G}_p \circ \mathfrak{I} \) and \( \mathfrak{G}_p \circ \mathfrak{I} \) a co-adjoint of \( \mathfrak{A} \circ \mathfrak{Y}_{ps} \).

If the ditopological space \((S, \mathcal{S}, \tau, \kappa)\) is not plain then \( \tau \) need not be a topology on \( S \) in the usual sense, as we have noted above. The simplest way to obtain a topology on \( S \) is to note that \( \tau \) is closed under finite intersections and is therefore a base for a topology \( \langle \tau \rangle \) on \( S \). Moreover, if \( \varphi : S_1 \to S_2 \) is \( \tau_1 \)-continuous and is clearly \( \langle \tau_1 \rangle \)-continuous. We have already used the fact that \( \kappa^c = \{ S \subseteq K | K \in \kappa \} \) is a topology because \( \kappa \) is closed under arbitrary intersections and finite unions. Hence if \( \langle \tau, \kappa \rangle \) is a complemented ditopology, \( \kappa^c = (\sigma[\tau])^c \) is also a topology on \( S \) associated with \( \tau \). We use both of these techniques in the notes below.

**Notes 2.11.** (i) In place of the functor \( \mathfrak{A} : \text{fSDitop} \to \text{Bitop} \) defined above consider the functor \( \mathfrak{A} : \text{fSDitop} \to \text{Bitop} \) given by \( \mathfrak{A}(S, \mathcal{S}, \tau, \kappa) = (S, \langle \tau \rangle, \kappa^c) \) and \( \mathfrak{A}(\varphi) = \varphi \).

By [9, Notes 3.13(1)], the functor \( \mathfrak{A} \) cannot be an adjoint of \( \mathfrak{Y}_s \circ \mathfrak{Y}_p \circ \mathfrak{I} \). Note that using the isomorphism \( \mathfrak{I} \) between \( \mathbb{T}-\text{Top} \) and the subconstruct \( \mathbb{I}-\text{Ditop} \) of \( \text{fSDitop} \), and then applying \( \mathfrak{A} \), gives the mapping \( \mathfrak{A} \circ \mathfrak{I} \) from \( \mathbb{T}-\text{Top} \) to \( \text{Bitop} \) considered by the second author in [14,15].

(ii) Functors \( \mathfrak{A}_c \) and \( \mathfrak{A}^c \) from \( \text{cfSDitop} \) to \( \text{Top} \) may be defined by setting

\[
\mathfrak{A}_c(S, \mathcal{S}, \sigma, \tau, \sigma(\tau)) = (S, \langle \tau \rangle), \quad \mathfrak{A}_c(\varphi) = \varphi,
\]
\[
\mathfrak{A}^c(S, \mathcal{S}, \sigma, \tau, \sigma(\tau)) = (S, \sigma(\tau)^c), \quad \mathfrak{A}^c(\varphi) = \varphi.
\]
Again, $\mathfrak{A}_c$, $\mathfrak{A}_c^*$ cannot be adjoints of $\mathfrak{V}_{cs} \circ \mathfrak{P}_{cs} \circ \mathfrak{T}_c$. If we apply $\mathcal{I}$ and $\mathfrak{A}_c$, $\mathfrak{A}_c^*$ as in (i) we obtain mappings from $\mathcal{I}$-$\textbf{Top}$ to $\textbf{Top}$. These mappings will be considered in greater detail in the final section.

Finally we look at some special morphisms in $\textbf{dfDitop}$ and $\textbf{fDitop}$.

**Proposition 2.12.** In the category $\textbf{dfDitop}$:
1. Every section [1,21] is injective.
2. Every injective morphism is a monomorphism [1,21].
3. Every retraction [1,21] is surjective.
4. Every surjective morphism is an epimorphism [1,21].
5. A morphism is an isomorphism [1,21] if and only if it is bijective and it, and its inverse, are bicontinuous.

**Proof.** Clear from [9, Proposition 3.14]. □

The first four results also hold trivially in $\textbf{fDitop}$, since it is a construct. We will say that a textural isomorphism between ditopological texture spaces is a textural homeomorphism if it, and its inverse, are bicontinuous. The following result is now a trivial consequence of [9, Proposition 3.15].

**Proposition 2.13.** A point function is an isomorphism in $\textbf{fDitop}$ if and only if it is a textural homeomorphism.

The following example shows very strikingly the potential difference between isomorphisms in $\textbf{dfDitop}$ and in $\textbf{fDitop}$.

**Example 2.14.** Recall from [9, Example 1.1(5)], that $\mathcal{J} = \{[0,r] | r \in [0,1]\} \cup \{(0,r) | r \in [0,1]\}$ and that the mapping $\iota: \mathcal{J} \rightarrow \mathcal{J}$ is defined by $\iota([0,r]) = [0, 1-r]$, $\iota((0,r)) = [0, 1-r]$. Then $(\mathcal{J}, \iota)$ is a Hutton algebra and we may consider the Hutton texture $(M_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}}, \mu_{\mathcal{J}})$ of $(\mathcal{J}, \iota)$ as in [9, Example 1.1(2)]. Since every non-empty element of $\mathcal{J}$ is a molecule in $\mathcal{J}$ we may represent $M_{\mathcal{J}}$ in the form $\{(0,1) \times \{0\}\} \cup \{(0,1) \times \{1\}\}$, and then write $M_{\mathcal{J}} = \{A_r | r \in [0,1]\} \cup \{B_r | r \in [0,1]\}$, where

\[
A_r = ([0,r] \times \{0\}) \cup ((0,r] \times \{1\}),
\]

\[
B_r = ([0,r) \times \{0\}) \cup ((0,r] \times \{1\}).
\]

Moreover,

\[
\mu_{\mathcal{J}}(A_r) = B_{1-r} \quad \text{and} \quad \mu_{\mathcal{J}}(B_r) = A_{1-r}
\]

gives the complementation on $(M_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}})$. It is easy to verify that

\[
P_{(r,0)} = A_r, \quad Q_{(r,0)} = B_r \quad \text{and} \quad P_{(r,1)} = B_r, \quad Q_{(r,1)} = B_r.
\]
Hence \((M_J, \mathcal{M}_J, \mu_J)\) is simple but not plain. Finally if we set
\[
\tau_J = \{B_r \mid r \in [0, 1]\} \cup \{M_J\} \quad \text{and} \quad \kappa_J = \{A_r \mid r \in [0, 1]\} \cup \{\emptyset\}
\]
we obtain a complemented ditopological space.

Consider the plain ditopological texture space \((\mathcal{I}, J, I_0, J_0, \kappa_0)\) of Example 2.1(5) and the mapping \(\varphi : \mathcal{I} \to M_J\) defined by \(\varphi(r) = (r, 0) \forall r \in \mathcal{I}\). Clearly \(\varphi\) satisfies condition (a) and (trivially) (b), so it is a morphism in the category \(\mathcal{C}\). Also \(\varphi^{-1}[B_r] = Q_r \in \tau_1\) and \(\varphi^{-1}[A_r] = P_r \in \kappa_1\) so \(\varphi\) is bicontinuous and hence a morphism in \(\mathcal{D}\).

Certainly \(\varphi\) is injective but it is not surjective. Let us examine the properties of the corresponding difunction \((f_\varphi, F_\varphi)\). Using the formulae given in [9, Lemma 3.4] we easily obtain
\[
\begin{align*}
&f_\varphi = \{(s, (r, k)) \mid 0 \leq r < s \leq 1, \ k = 0 \text{ or } 0 < r < s \leq 1, \ k = 1\}, \\
&F_\varphi = \{(s, (r, k)) \mid 0 \leq r < s \leq 1, \ k = 0 \text{ or } 0 < r < s \leq 1, \ k = 1\}.
\end{align*}
\]

As expected, the difunction \((f_\varphi, F_\varphi)\) is injective in the sense of [9, Definition 2.30]. But it is also surjective. To see this latter result take \(P_\varphi(r, k) \not\in Q_\varphi(r', k')\) in \((M_J, \mathcal{M}_J)\). Then \(r' \leq r\) if \(k = 0\) and \(r' < r\) if \(k = 1\). In the first case let \(s = r'\) and in the second choose \(s\) satisfying \(r' < s < r\). It is now straightforward to verify that \(f_\varphi \not\in Q_\varphi(s, (r', k'))\) and \(P_\varphi(s, (r, k)) \not\in F_\varphi\), which establishes that \((f_\varphi, F_\varphi)\) is indeed surjective. By [9, Proposition 3.14(5)] the difunction \((f_\varphi, F_\varphi)\) is an isomorphism in the category \(\mathcal{D}\). Since it is bicontinuous, to show that it is an isomorphism in \(\mathcal{D}\) it remains to show that its inverse is bicontinuous. However it is trivial to verify that \((f_\varphi^{-1})^{-1}Q_s = f_\varphi^{-1}Q_s = B_s\), \((F_\varphi^{-1})^{-1}P_s = F_\varphi^{-1}P_s = A_s\), whence the inverse is bicontinuous as required. This verifies that the ditopological texture spaces \((\mathcal{I}, J, I_0, J_0, \kappa_0)\) and \((M_J, \mathcal{M}_J, \tau_J, \kappa_J)\) are isomorphic objects of \(\mathcal{D}\). In particular this shows that the properties of being simple or plain are not invariant under \(\mathcal{D}\)-isomorphisms. We also note in passing that for \(A \in \mathcal{I} \setminus \{\emptyset\}\) we have \(\varphi[A] \notin \mathcal{M}_J\) and so certainly \(f_\varphi^\rightarrow A \neq \varphi[A]\) and \(F_\varphi^\rightarrow A \neq \varphi[A]\) in this case.

The inverse \((F_\varphi^{-1}, f_\varphi^{-1})\) of \((f_\varphi, F_\varphi)\) is an example of a difunction which does not arise from a \(\mathcal{C}\)-morphism, for such a morphism would have to be the inverse of \(\varphi\), which does not exist. The interested reader may easily obtain the following explicit formulae for the inverse difunction.
\[
\begin{align*}
&f_\varphi^{-1} = \{((r, k), s) \mid 0 \leq s < r \leq 1, \ k = 0 \text{ or } 0 < s \leq r \leq 1, \ k = 1\}, \\
&F_\varphi^{-1} = \{((r, k), s) \mid 0 \leq s \leq r \leq 1, \ k = 0 \text{ or } 0 < s < r \leq 1, \ k = 1\}.
\end{align*}
\]

By Proposition 2.13 it is clear that the spaces \((\mathcal{I}, J, I_0, J_0, \kappa_0)\) and \((M_J, \mathcal{M}_J, \tau_J, \kappa_J)\) are not isomorphic objects in \(\mathcal{D}\) because textural homeomorphisms clearly preserve p-sets and q-sets by the proof of [9, Proposition 3.15], and hence the properties of being simple or plain are invariant under such mappings.

Finally we note that the difunction \((f_\varphi, F_\varphi)\) is easily shown to be complemented, whence we may replace \(\mathcal{D}\), \(\mathcal{D}\) by \(\mathcal{D}\), \(\mathcal{D}\), respectively, in the discussion above.

3. Initial sources, products and coproducts

We begin by defining bases and subbases for ditopologies.
Definition 3.1. Let \((\tau, \kappa)\) be a ditopology on \((S, \mathcal{S})\). Then a subset \(\beta\) of \(\tau\) is called a base of \(\tau\) if every set in \(\tau\) can be written as a join of sets in \(\beta\), while a subset \(\beta\) of \(\kappa\) is a base of \(\kappa\) if every set in \(\kappa\) can be written as an intersection of sets in \(\beta\).

As usual, a subbase of \(\tau\) is a subset of \(\tau\), the set of finite intersections of which is a base of \(\tau\), while a subbase of \(\kappa\) is a subset of \(\kappa\), the set of finite unions of which is a base of \(\kappa\). In the case of a complemented ditopology, the complementation will clearly carry a base (subbase) of \(\tau\) into a base (subbase) of \(\kappa\), and conversely.

The p-sets and q-sets enable us to give a point-based characterization of bases as detailed below.

Theorem 3.2. Let \((\tau, \kappa)\) be a ditopology on the texture space \((S, \mathcal{S})\).

1. Let \(\beta \subseteq \tau\). Then the following are equivalent:
   (i) \(\beta\) is a base of \(\tau\).
   (ii) \(G \in \tau, G \notin Q_s \Rightarrow \exists B \in \beta \text{ with } B \notin Q_s \text{ and } B \subseteq G\).
   (iii) \(G \in \tau, G \notin Q_s \Rightarrow \exists B \in \beta \text{ with } P_s \subseteq B \subseteq G\).

2. Let \(\beta \subseteq \kappa\). Then the following are equivalent:
   (i) \(\beta\) is a base of \(\kappa\).
   (ii) \(K \in \kappa, P_s \notin K \Rightarrow \exists B \in \beta \text{ with } K \subseteq B \text{ and } P_s \notin B\).
   (iii) \(K \in \kappa, P_s \notin K \Rightarrow \exists B \in \beta \text{ with } K \subseteq B \subseteq Q_s\).

3. If \(\beta\) is a base of \(\tau\) then \(G \in \tau \iff (G \notin Q_s \Rightarrow \exists B \in \beta \text{ with } B \notin Q_s \text{ and } B \subseteq G)\).

4. If \(\beta\) is a base of \(\kappa\) then \(K \in \kappa \iff (P_s \notin K \Rightarrow \exists B \in \beta \text{ with } P_s \notin B \text{ and } K \subseteq B)\).

Proof. We prove (1), leaving the dual proof of (2) to the reader.

(i) \(\Rightarrow\) (ii). Take \(G \in \tau\). Then \(G = \bigvee_{j \in J} B_j\) for some \(B_j \in \beta, j \in J\). If \(G \notin Q_s\) then \(B_j \notin Q_s\) for some \(j \in J\), and clearly \(B_j \subseteq G\).

(ii) \(\Rightarrow\) (iii) Trivial by [9, Theorem 1.2(1)].

(iii) \(\Rightarrow\) (i). Take \(G \in \tau\) and for each \(s \in S\) with \(G \notin Q_s\) choose \(B_s \in \beta\) with \(s \in B_s \subseteq G\). Then by [9, Theorem 1.2(7)] we have \(G = \bigvee\{P_s \mid G \notin Q_s\} \subseteq \bigvee\{B_s \mid G \notin Q_s\} \subseteq G\). This shows that \(\beta\) is a base of \(\tau\).

(3), (4) Straightforward. \(\Box\)

This theorem shows clearly the nature of the duality between bases of \(\tau\) and \(\kappa\) with respect to the p-sets and q-sets. Generally we will denote a dual concept with the prefix co. In particular, we call a base of \(\tau\) a base for the ditopology \((\tau, \kappa)\), and a base of \(\kappa\) a cobase for \((\tau, \kappa)\). A similar terminology will be applied to subbases. The following theorem gives necessary and sufficient conditions for a subset of \(\mathcal{S}\) to be a base (cobase) for some ditopology on \((S, \mathcal{S})\).

Theorem 3.3. Let \(\beta \subseteq \mathcal{S}\). Then

1. \(\beta\) is a base for some ditopology on \((S, \mathcal{S})\) if and only if
   (a) \(\bigvee \beta = S\), and
   (b) For \(B_1, B_2 \in \beta\) and \(s \in S\) with \(B_1 \cap B_2 \notin Q_s\) we have \(B_3 \in \beta\) with \(B_3 \subseteq B_1 \cap B_2\) and \(B_3 \notin Q_s\).

2. \(\beta\) is a base for some ditopology on \((S, \mathcal{S})\) if and only if
   (a) \(\bigvee \beta = S\), and
   (b) For \(B_1, B_2 \in \beta\) and \(s \in S\) with \(B_1 \cap B_2 \notin Q_s\) we have \(B_3 \in \beta\) with \(P_s \subseteq B_3 \subseteq B_1 \cap B_2\).
(3) $\beta$ is a cobase for some ditopology on $(S, \mathcal{S})$ if and only if
   (a) $\bigcap \beta = \emptyset$, and
   (b) For $B_1, B_2 \in \beta$ and $s \in S$ with $P_s \notin B_1 \cup B_2$ we have $B_3 \in \beta$ with $B_1 \cup B_2 \subseteq B_3$ and $P_s \notin B_3$.

(4) $\beta$ is a cobase for some ditopology on $(S, \mathcal{S})$ if and only if
   (a) $\bigcap \beta = \emptyset$, and
   (b) For $B_1, B_2 \in \beta$ and $s \in S$ with $P_s \notin B_1 \cup B_2$ we have $B_3 \in \beta$ with $B_1 \cup B_2 \subseteq B_3 \subseteq Q_3$.

Proof. Left to the reader. $\square$

Lemma 3.4. Let $(f, F): (S, \mathcal{S}, \tau, \kappa) \to (S_j, \mathcal{S}_j, \tau_j, \kappa_j)$ be a difunction, $\beta$ a subbase and $\gamma$ a cosubbase of $(\tau, \kappa)$. Then $(f, F)$ is continuous (cocontinuous) if $G \in \beta \Rightarrow F \leftarrow G \in \tau$ ($F \in \gamma \Rightarrow f^\rightarrow F \in \kappa$).

Proof. $G \in \tau_j$ has the form $G = \bigvee_{a \in A}(\bigcap_{\gamma \in C_\gamma} G_\gamma^2), C_\gamma$ finite, $G_\gamma^2 \in \beta$. Hence,

$$F \leftarrow G = \bigvee_{a \in A} F \leftarrow \left( \bigcap_{\gamma \in C_\gamma} G_\gamma^2 \right) = \bigvee_{a \in A} f \leftarrow \left( \bigcap_{\gamma \in C_\gamma} G_\gamma^2 \right)$$

$$= \bigcup_{a \in A} \left( \bigcap_{\gamma \in C_\gamma} f \leftarrow G_\gamma^2 \right) = \bigcup_{a \in A} \left( \bigcap_{\gamma \in C_\gamma} F \leftarrow G_\gamma^2 \right)$$

by [9, Corollary 2.26 and Theorem 2.24]. The proof for cocontinuity is similar. $\square$

Now let $\mathcal{U}: \text{dfDitop} \to \text{dfTex}$ be the forgetful functor.

Theorem 3.5. The source $((S, \mathcal{S}, \tau, \kappa), ((S, \mathcal{S}, \tau, \kappa) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \tau_j, \kappa_j))_{j \in J})$ in $\text{dfDitop}$ is $\mathcal{U}$-initial iff

$$\beta = \{F_j^\rightarrow G \mid G \in \tau_j, j \in J\}$$

is a subbase and

$$\gamma = \{f_j^\rightarrow K \mid K \in \kappa_j, j \in J\}$$

a cosubbase for $(\tau, \kappa)$. That is, iff $(\tau, \kappa)$ is the coarsest ditopology on $(S, \mathcal{S})$ for which the difunctions $(f_j, F_j), j \in J$, are bicontinuous.

Proof. Suppose that the given source is $\mathcal{U}$-initial [1,10]. Since each $(f_j, F_j)$ is a morphism in $\text{dfDitop}$ it is bicontinuous, hence

$$\beta = \{F_j^\rightarrow G \mid G \in \tau_j, j \in J\} \subseteq \tau,$$

$$\gamma = \{f_j^\rightarrow F \mid F \in \kappa_j, j \in J\} \subseteq \kappa.$$

Now let $(\tau^*, \kappa^*)$ be the ditopology on $(S, \mathcal{S})$ with subbase $\beta$ and cosubbase $\gamma$. Since the given source is $\mathcal{U}$-initial the morphism $(i, I)$ in $\text{dfTex}$ occurring in the commutative diagram on the right lifts to a morphism in $\text{dfDitop}$ making the diagram on the left commute.
Hence $\tau^* \subseteq \tau$, $\kappa^* \subseteq \kappa$, which proves $(\tau, \kappa) = (\tau^*, \kappa^*)$, as required.

Conversely let $(\tau, \kappa) = (\tau^*, \kappa^*)$ and consider the following diagrams in $\text{dfDitop}$ and $\text{dfTex}$, respectively.

Given that the morphism $(k, K)$ in $\text{dfTex}$ makes the right hand diagram commutative, it will clearly be sufficient, in view of the fact that $\Xi$ is faithful, to show that $(k, K)$ is a morphism in $\text{dfDitop}$. Hence it will be sufficient to show that $K^+ B \in \tau' \ \forall B \in \beta$ and $k^+ C \in \kappa' \ \forall C \in \gamma$. But $B \in \beta$ has the form $F_j^+ G = F_j \circ K \in \tau' \ G \in \tau_j$ and $K^+ (F_j^+ G) = (F_j \circ K)^+ G = H_j^+ G \in \tau'$ since $(h_j, H_j)$ is continuous. Likewise $k^+ (f_j^+ F) = (f_j \circ k)^+ F = h_j^+ F \in \kappa'$ since $(h_j, H_j)$ is cocontinuous. Hence $(k, K)$ is bicontinuous, as required. $\square$

**Theorem 3.6.** The functor $\Xi : \text{dfDitop} \rightarrow \text{dfTex}$ is topological. In other words, $\text{dfDitop}$ is topological over $\text{dfTex}$ with respect to the functor $\Xi$.

**Proof.** Take $(S_j, \mathcal{S}_j, \tau_j, \kappa_j) \in \text{Ob dfDitop}$, $j \in J$, and $(S, \mathcal{S}) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j)$ in $\Xi(\text{dfDitop}) = \text{dfTex}$. Let $(\tau, \kappa)$ be the ditopology on $(S, \mathcal{S})$ with subbase $\beta = \{F_j^+ G \mid G \in \tau_j, j \in J\}$ and cosubbase $\gamma = \{f_j^+ F \mid F \in \kappa_j, j \in J\}$. Then by Theorem 3.5,

$$(S, \mathcal{S}, \tau, \kappa), ((S, \mathcal{S}, \tau, \kappa) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \tau_j, \kappa_j))_{j \in J}$$

is the unique $\Xi$-initial source, which maps to $((S, \mathcal{S}), ((S, \mathcal{S}) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \tau_j, \kappa_j))_{j \in J})$ under $\Xi$. $\square$

In order to characterize the products in $\text{dfDitop}$ we begin by recalling that a product of a family $(D_j)_{j \in J}$ of objects in a category $\mathbf{A}$ may be viewed as a limit $(L, (L \xrightarrow{L_j} D_j)_{j \in J})$ of the functor $D : \mathbf{J} \rightarrow \mathbf{A}$, where $\mathbf{J}$ is the class $J$ regarded as a discrete category and $D(j) = D_j \ \forall j \in J = \text{Ob J}$. Also, if $U : \mathbf{A} \rightarrow \mathbf{B}$ is topological, $(L, (L \xrightarrow{L_j} D_j)_{j \in J})$ is a limit of $D : \mathbf{J} \rightarrow \mathbf{A}$ iff $(L, (L \xrightarrow{L_j} D_j)_{j \in J})$ is $U$-initial and $(U L, (U L \xrightarrow{U L_j} U D_j)_{j \in J})$ is a limit of $U \circ D : \mathbf{J} \rightarrow \mathbf{B}$. Applying this to the topological functor $\Xi : \text{dfDitop} \rightarrow \text{dfTex}$ and using Theorem 3.5 we obtain at once.

**Proposition 3.7.** The source $((S, \mathcal{S}, \tau, \kappa), ((S, \mathcal{S}, \tau, \kappa) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \tau_j, \kappa_j))_{j \in J})$ is a product of the family $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)_{j \in J}$ in $\text{dfDitop}$ iff $(\tau, \kappa)$ has subbase $\{F_j^+ (G) \mid G \in \tau_j, j \in J\}$, cosubbase $\gamma =$...
We note the following.

Lemma 3.8. Let $(S, \mathcal{F})$, $(T, \mathcal{G})$ be textures and $\varphi : S \to T$ a point function satisfying conditions (a) and (b), and in addition

(c) For $A \in \mathcal{F}$ and $s \in S^\circ$ we have $A \not\subseteq Q_{\varphi(s)} \Rightarrow A \not\subseteq Q_{\varphi(u)}$ for some $P_u \not\subseteq Q_s$. Then the function $(f_\varphi, F_\varphi)$ corresponding to $\varphi$ satisfies the equalities

$$f_\varphi = \bigvee \{P_{(s,\varphi(s))} \mid s \in S\} \quad \text{and} \quad F_\varphi = \bigcap \{Q_{(s,\varphi(s))} \mid s \in S^\circ\}.$$ 

Proof. The first equality holds for $\varphi$ satisfying (a) and (b). We leave the proof to the interested reader and establish the second equality. Recall that $F_\varphi = \bigcap \{Q_{(s,\varphi(s))} \mid P_u \not\subseteq Q_s \Rightarrow P_t \not\subseteq Q_{\varphi(u)}\}$ and take $P_u \not\subseteq Q_s$, $P_t \not\subseteq Q_{\varphi(u)}$. Then $P_{\varphi(u)} \not\subseteq Q_{\varphi(s)}$ by condition (a), whence $Q_{(s,\varphi(s))} \subseteq Q_{(s,\varphi(u))}$ and we obtain $\bigcap \{Q_{(s,\varphi(s))} \mid s \in S^\circ\} \subseteq F_\varphi$ since $s \in S^\circ$.

Now suppose that $F_\varphi \not\subseteq \bigcap \{Q_{(s,\varphi(s))} \mid s \in S^\circ\}$. Then we have $F_\varphi \not\subseteq Q_{(s,\varphi(s))}$, $P_{(s,\varphi(s))} \not\subseteq \bigcap \{Q_{(s,\varphi(s))} \mid s \in S^\circ\}$. Hence $P_{(s,\varphi(s))} \not\subseteq Q_{(s,\varphi(s))}$, and so $P_t \not\subseteq Q_{\varphi(s)}$, whence by condition (c) there exists $u \in S$ satisfying $P_u \not\subseteq Q_s$ and $P_t \not\subseteq Q_{\varphi(u)}$. This now gives $F_\varphi \subseteq Q_{(s,\varphi(s))}$, which is a contradiction. □

Now we have:

Lemma 3.9. The projection functions $\rho_j : S = \prod_{j \in J} S_j \to S_j$ satisfy conditions (a)–(c) of Lemma 3.8, and hence the equalities

$$\pi_j = \bigvee \{P_{(s_j,j)} \mid s = (s_j) \in S\}, \quad \Pi_j = \bigcap \{Q_{(s_j,j)} \mid s = (s_j) \in S^\circ\},$$

define a function $(\pi_j, \Pi_j)$ from $(S, \mathcal{F})$ to $(S_j, \mathcal{F}_j)$ for which $\pi_j^{-1}(A) = \Pi_j^{-1}(A) = \rho_j^{-1}[A]$ for all $A \in \mathcal{F}_j$. We will refer to $(\pi_j, \Pi_j)$ as the $j$th-projection difunction.

Proof. By [9, Corollary 1.4] we have $P_s \not\subseteq Q_s' \Rightarrow P_{s_j} \not\subseteq Q_{s_j}$, which is just $P_{\rho_j(s)} \not\subseteq Q_{\rho_j(s')}$. Hence condition (a) holds. For (b) take $A \in \mathcal{F}_j$ and $s = (s_j) \in S$ satisfying $P_{\rho_j(s)} \not\subseteq A$, that is $P_{s_j} \not\subseteq A$. Now we have $s_j' \in S_j$ with $P_{s_j} \not\subseteq Q_{s_j'}$ and $P_{s_j} \not\subseteq A$. Also, for $k \in J, k \neq j$, we have $s_k' \in S_k$ with $P_{s_k} \not\subseteq Q_{s_k'}$ since $P_{s_k} \not\subseteq \emptyset$. This gives us a point $s' \in S$ satisfying $P_s \not\subseteq Q_s'$ by [9, Corollary 1.4] for which $P_{\rho_j(s')} \not\subseteq A$, as required. Finally, for (c) take $A \in \mathcal{F}_j$ and $s \in S^\circ$ with $A \not\subseteq Q_{\rho_j(s)}$, that is $A \not\subseteq Q_{s_j}$. Now we have $s_j' \in S_j$ satisfying $A \not\subseteq Q_{s_j'}$, $P_{s_j'} \not\subseteq Q_{s_j'}$, and for $k \in J, k \neq j$, $s_k \in S_k$ since $S^\circ = \prod_{j \in J} S^\circ_j$, and so $Q_{s_k} \not\subseteq S_k$ and we have $s_k' \in S_k$ satisfying $P_{s_k} \not\subseteq Q_{s_k}$. This gives us a point $s' \in S$ with $P_s \not\subseteq Q_s$ and $A \not\subseteq Q_{\rho_j(s')}$, so verifying (c). The result now follows from Lemma 3.8. □
Theorem 3.10. Let \((S, \mathcal{S})\) be the product of the textures \((S_j, \mathcal{S}_j)\), \(j \in J\), where \(J\) is a set. Then \(((S, \mathcal{S}), ((S, \mathcal{S}) \xrightarrow{(\pi_j, \Pi_j)} (S_j, \mathcal{S}_j))_{j \in J}\) is a product of the family \((S_j, \mathcal{S}_j)_{j \in J}\) in dfTex.

Proof. Take \((Z, \mathcal{Z}) \in \text{Ob dfTex}\) and \((f_j, F_j) \in \text{dfTex}((Z, \mathcal{Z}), (S_j, \mathcal{S}_j))\), \(j \in J\). We must establish the existence of a unique difunction \((h, H)\), usually denoted by \(\langle (f_j, F_j) \rangle\), which makes the diagram below commutative for each \(j \in J\).

\[
\begin{array}{ccc}
(Z, \mathcal{Z}) & \xrightarrow{(h, H)} & (S, \mathcal{S}) \\
\downarrow (f_j, F_j) & & \downarrow (\pi_j, \Pi_j) \\
(S_j, \mathcal{S}_j) & & (S_j, \mathcal{S}_j)
\end{array}
\]

Define \((h, H)\) as follows:

\[
h = \bigvee \{P_{(z, t)} \mid \exists u \in Z \text{ with } P_u \neq Q_u \text{ and } f_k \notin \overline{Q}_{(u, k)} \forall k \in J\},
\]

\[
H = \bigcap \{\overline{Q}_{(z, t)} \mid \exists u \in Z \text{ with } P_u \neq Q_u \text{ and } F_k \notin \overline{Q}_{k, u} \forall k \in J\}.
\]

We show that \((h, H)\) has the required properties. It is trivial to verify that this is a direlation, so let us show that it is a difunction.

DF1. Take \(z, z' \in Z\) with \(P_z \neq Q_{z'}\), then \(u \in Z\) with \(P_u \neq Q_u\), \(P_u \neq Q_{z'}\), and finally \(u_1, u_2 \in Z\) satisfying \(P_z \neq Q_{u_1}\), \(P_{u_1} \neq Q_u\) and \(P_{u_1} \neq Q_{z'}\). For each \(j \in J\), \((f_j, F_j)\) is a difunction so by DF1 we have \(s_j, s'_j \in S_j\) satisfying \(f_j \neq \overline{Q}_{(u_1, s_j)}\), \(P_{(u_1, s_j)} \neq F_j\) and \(f_j \neq \overline{Q}_{(u_2, s'_j)}\), \(P_{(u_2, s'_j)} \neq F_j\). By DF2 for \((f_j, F_j)\) we have \(P_j \neq Q_s\) and so for each \(j \in J\) we have \(v_j \in S_j\) satisfying \(P_j \neq Q_j\) and \(P_j \neq Q_{s'_j}\). If we let \(v = (v_j) \in S\) it is not difficult to verify that \(h \notin \overline{Q}_{(z, v)}\) and \(\overline{P}_{(z', v')} \notin H\).

DF2. Take \(z \in Z\), \(t, t' \in S\) with \(h \notin \overline{Q}_{(z, t)}\) and \(\overline{P}_{(z', t')} \notin H\). Now we have \(v \in S\) and \(u \in Z\) so that \(P_{(z, v)} \notin \overline{Q}_{(z, u)}\), \(P_z \neq Q_u\), \(f_j \neq \overline{Q}_{(u, v)} \forall j \in J\) and \(v' \in S\) and \(u' \in Z\) so that \(P_{(z', v')} \notin \overline{Q}_{(z', v')}\), \(P_{v'} \neq Q_z\) and \(P_{(u', v')} \notin F_j \forall j \in J\). Clearly \(f_j \notin \overline{Q}_{(z, v)}\), \(\overline{P}_{(z', v')} \notin F_j\) so by DF2 for \((f_j, F_j)\) we have \(P_j \neq Q_j \forall j \in J\), whence \(P_{v'} \neq Q_{v'}\). Since \(P_{v'} \neq Q_{v'}\) and \(P_t \notin Q_u\), we obtain \(P_{v'} \neq Q_{v'}\) as required.

We now fix \(j \in J\) and establish the equality \(f_j = \pi_j \circ h\).

First suppose that \(f_j \neq \pi_j \circ h\) and take \(z \in Z\), \(s_j \in S_j\) so that \(f_j \neq \overline{Q}_{(z, s_j)}\) and \(\overline{P}_{(z, s_j)} \neq \pi_j \circ h\). Since \(f_j\) is a relation, by R2 there exists \(z' \in Z\) satisfying \(P_z \neq Q_{z'}\) and \(f_j \neq \overline{Q}_{(z', s_j)}\). Take \(z'' \in Z\) satisfying \(P_z \neq Q_{z''}\), \(P_{z''} \neq Q_{z'}\) and for \(k \in J\) apply DF1 for \((f_k, F_k)\) to obtain \(s_k \in S_k\) with \(f_k \neq \overline{Q}_{(z'', s_k)}\). Note that also \(f_j \neq \overline{Q}_{(z'', s_j)}\) so for all \(k \in J\) we may take \(s'_k \in S_k\) satisfying \(f_k \neq \overline{Q}_{(z'', s'_k)}\) and \(\overline{P}_{(z'', s'_k)} \neq \overline{Q}_{(z', s'_k)}\). Finally take \(u_k \in S_k\) with \(P_{(u_k, s'_k)} \neq P_{(u_k, s_k)}\) and consider \(s = (s_k), s' = (s'_k), u = (u_k) \in S\). Now \(P_z \neq Q_{z''}\), \(f_k \neq \overline{Q}_{(z'', s'_k)} \forall k \in J\) gives \(\overline{P}_{(z', v')} \subseteq h\), and since clearly \(P_{v'} \neq Q_u\) we obtain \(h \notin \overline{Q}_{(z, u)}\). Also, \(\overline{P}_{(u, u')} = \pi_j\) so \(\pi_j \notin \overline{Q}_{(u, u')}\) and we obtain the contradiction \(\overline{P}_{(z, s_j)} \subseteq \pi_j \circ h\).

Now suppose that \(\pi_j \circ h \neq f_j\) and take \(z \in Z\), \(s_j \in S_j\) with \(\pi_j \circ h \neq \overline{Q}_{(z, s_j)}\) and \(\overline{P}_{(z, s_j)} \neq f_j\). Hence for some \(s'_j \in S_j\) and \(t = (t_k) \in S\) we have \(\overline{P}_{(z, s_j)} \neq \overline{Q}_{(z, t)}\), \(h \notin \overline{Q}_{(z, t)}\) and \(\pi_j \notin \overline{Q}_{(t, s_j)}\). From the definition of \(\pi_j\) we have \(P_j \neq Q_j\), so \(P_j \neq Q_{s_j}\). Also, from \(h \notin \overline{Q}_{(z, t)}\) we have \(t' \in S\) with \(\overline{P}_{(z', t')} \neq \overline{Q}_{(z, t')}\) and
Lemma 3.14. Let \( \tau_j \) be the reverse inclusion to the interested reader. Suppose that for some \( s \in S_j \), \( t \in S_j \) with \( \Pi''_j \subseteq \pi_j \) and \( \Pi'(u_t) \notin \Omega(u'_t) \) and \( \Pi'(u_t) \notin \pi_j \). Now we have \( t' \in S_j \) with \( \Pi'(u_t') \notin \Omega(u'_t) \) and \( \Pi'(u_t') \notin \pi_j \). Using the formula \( \sigma(P_i) = \bigcup_{i \in J} E(i, \sigma(P_i)) \)
given in [7, Corollary to Lemma 2.7] it may be verified that \( P_j \not\subseteq Q_t \), whence \( P_j \not\subseteq Q_t \) and we obtain the contradiction \( \overline{P}_{(s,t)} \subseteq \overline{P}_{(s,s_j)} \subseteq \pi_j \). \( \square \)

For complemented textures we obtain the following diagram, where \((f_j, F_j)\) is complemented and \((h,H)\) defined as in Theorem 3.10.

\[
\begin{array}{ccc}
(Z, \mathcal{Z}, \zeta) & \xrightarrow{(h,H)} & (S, \mathcal{S}, \sigma) \\
& \downarrow{(f_j, F_j)} & \downarrow{(\pi_j, \Pi_j)} \\
(S_j, \mathcal{S}_j, \sigma_j)
\end{array}
\]

We wish to show that \((h,H)\) is complemented, and therefore a morphism in \textit{cdfTex}. However, from [9, Proposition 2.21(3)] and Lemma 3.14 we have

\[
(f_j, F_j) = (f_j, F_j)' = ((\pi_j, \Pi_j) \circ (h,H))' = (\pi_j, \Pi_j)' \circ (h,H)' = (\pi_j, \Pi_j) \circ (h,H)',
\]

whence \((\pi_j, \Pi_j) \circ (h,H) = (\pi_j, \Pi_j) \circ (h,H)'\) for all \( j \in J \), and we obtain \((h,H) = (h,H)'\) from the uniqueness property of \((h,H)\). This verifies that set indexed products of non-empty complemented textures are products in \textit{cdfTex}, and clearly the same is true for complemented ditopological texture spaces. Hence, we have

\section*{Theorem 3.15.}

The categories \textit{cdfTex} and \textit{cdfDitop} have products.

It may be shown that the categories \textit{dfSTex}, \textit{dfSDitop}, \textit{dfPTex} and \textit{dfPDitop} also have products. The details are left to the interested reader.

Before leaving the topic of products we note some further results for future reference. We recall that in a ditopological texture space, \([A]\) denotes the closure of a set \( A \) and \( ]A[ \) denotes the interior.

\section*{Proposition 3.16.}

For \( A_j \in \mathcal{F}_j, j \in J \), we have

(i) \( [\bigcap_{j \in J} E(j, A_j)] = \bigcap_{j \in J} E(j, [A_j]) \). That is, \( \prod_{j \in J} A_j = [\prod_{j \in J} A_j] \in \kappa \).

(ii) \( \bigcup_{j \in J} E(j, A_j) = \bigcup_{j \in J} [E(j, A_j)] \).

\section*{Proof.}

(i) is straightforward, so let us establish (ii). Clearly

\[
\bigcup_{j \in J} E(j, [A_j]) \subseteq \bigcup_{j \in J} E(j, A_j).
\]

Let \( A = \bigcup_{j \in J} E(j, A_j) \) and suppose that \( [A] \not\subseteq \bigcup_{j \in J} E(j, [A_j]) \). Then we have \( t = (t_j) \in S \) satisfying \( [A] \not\subseteq Q_t \) and \( P_t \not\subseteq \bigcup_{j \in J} E(j, [A_j]) \). Also note that we must have \( A_j \neq S_j \) for all \( j \in J \) so we may choose \( w_j \in S_j \) with \( P_{w_j} \not\subseteq A_j \).

Since \( [A] \in \tau \) we have \( J_0 \subseteq J \) finite and \( G_j \in \tau_j \) for \( j \in J_0 \) with \( \bigcap_{j \in J_0} E(j, G_j) \subseteq [A] \) and \( \bigcap_{j \in J_0} E(j, G_j) \not\subseteq Q_t \). In particular \( E(j, G_j) \not\subseteq E(j, Q_{t_j}) \) so \( G_j \not\subseteq Q_{t_j} \) for all \( j \in J_0 \). Also \( P_t = \bigcap_{j \in J} E(j, P_{t_j}) \not\subseteq \bigcup_{j \in J} E(j, [A_j]) \) gives \( P_t \not\subseteq [A_j] \), whence \( G_j \not\subseteq A_j \) for all \( j \in J_0 \). Choose \( u_j \in G_j \) with
Choose ai

1) First let us note that, from the definition of Proof.

complete distributivity of fors some A

\(P_{j} \notin A_{j}\) for \(j \in J_{0}\), and for \(j \in J \setminus J_{0}\) let \(u_{j} = w_{j}\). Then clearly \(u = (u_{j}) \in S\) satisfies \(u \in \bigcap_{j \in J_{0}} E(j, G_{j})\) and \(u \notin \bigcup_{j \in J} E(j, A_{j})\), which contradicts \(\bigcap_{j \in J_{0}} E(j, G_{j}) \subseteq A\) and completes the proof. □

Corollary 3.17. For \(s = (s_{j}) \in S\) we have

1) \([P_{s}] = \bigcap_{j \in J} E(j, [P_{j}])\).

2) \([Q_{s}] = \bigcup_{j \in J} E(j, [Q_{j}])\).

It will also be instructive to look at images and co-images under the projection difunctions.

Proposition 3.18. Let \((\pi_{j}, \Pi_{j})\) be the \(j\)th-projection difunction of the product \((S, \mathcal{S})\) of the textures \((S_{j}, \mathcal{S}_{j})\), \(j \in J\). Then

1) If \(A_{i} \in \mathcal{S}_{i}, i \in J\) and \(A_{i} \neq \emptyset, i \neq j\), then \(\pi_{j}^{-1} \bigcap_{i \in J} E(i, A_{i}) = A_{j}\).

2) If \(A_{i} \in \mathcal{S}_{i}, i \in J\) and \(A_{i} \neq S_{i}, i \neq j\), then \(\Pi_{j}^{-1} \bigcup_{i \in J} E(i, A_{i}) = A_{j}\).

3) For each \(A \in \mathcal{S}\) and \(j \in J\),

\[\Pi_{j}^{-1} A \subseteq \rho_{j}[A] \subseteq \pi_{j}^{-1} A.\]

In particular, \(\pi_{j}^{-1} A\) is the smallest element of \(\mathcal{S}_{j}\) containing \(\rho_{j}[A]\).

Proof. (1) First let us note that, from the definition of \(\pi_{j}\),

\[\pi_{j}^{-1} A = \bigcap_{t \in S_{j}} \forall s \in S, \pi_{j} \notin \overline{O}_{(s, t)} \Rightarrow A \subseteq O_{s} = \bigcap_{t \in S_{j}} \forall s \in S, P_{s} \notin Q_{t} \Rightarrow A \subseteq O_{s} = \bigcup_{t \in S_{j}} \forall s \in S, P_{s} \notin Q_{t} \Rightarrow A \subseteq O_{s}.\]

Suppose that \(\pi_{j}^{-1} \bigcap_{i \in J} E(i, A_{i}) \notin A_{j}\). Then \(\pi_{j}^{-1} \bigcap_{i \in J} E(i, A_{i}) \notin Q_{i}\) and \(P_{i} \notin A_{j}\) for some \(t \in S_{j}\). Hence, for some \(s \in S\) we have \(P_{s} \notin Q_{t}\) and \(\bigcap_{i \in J} E(i, A_{i}) \notin O_{s}\). But then \(P_{s} \subseteq E(j, A_{j})\), whence \(P_{i} \subseteq P_{s} \subseteq A_{j}\), which is a contradiction.

Conversely, suppose \(A_{j} \notin \pi_{j}^{-1} \bigcap_{i \in J} E(i, A_{i})\). Then there exists \(t \in S_{j}\) with \(A_{j} \notin Q_{t}\) so that \(\forall s \in S, P_{s} \notin Q_{t} \Rightarrow \bigcap_{i \in J} E(i, A_{i}) \subseteq O_{s}\). Define \(s \in S\) by choosing \(s_{j} \in S_{j}\) satisfying \(A_{j} \notin Q_{s_{j}}, P_{s_{j}} \notin Q_{t}\); and \(A_{j} \notin Q_{s_{j}}\) for \(i \in J \setminus \{j\}\), this latter being possible since \(A_{i} \neq \emptyset\). Now define \(s' \in S\) by choosing \(s' \in S_{j}\), \(i \in J\), satisfying \(A_{i} \notin Q_{s'}\) and \(P_{s'} \notin Q_{s}\). Clearly, \(P_{s} \notin Q_{s}\) by [9, Corollary 1.4], while \(\bigcap_{i \in J} E(i, A_{i}) \subseteq O_{s}\) by the above implication. Hence, \(P_{s} \notin E(i, A_{i})\) for some \(i \in J\). But then \(P_{s'} \notin A_{i}\), which leads to the contradiction \(A_{j} \subseteq Q_{s'}\).

(2) Dual to (1) on noting that, by the definition of \(\Pi_{j}\),

\[\Pi_{j}^{-1} A = \bigcup_{t \in S_{j}} \forall s \in S, P_{t} \notin Q_{s} \Rightarrow P_{s} \subseteq A\]

\[= \bigcup_{t \in S_{j}} \forall s \in S, P_{t} \notin Q_{s} \Rightarrow P_{s} \subseteq A.\]

(3) If \(\Pi_{j}^{-1} A = \emptyset\) the first inclusion is trivial, so take \(t \in S_{j}\) satisfying \(\forall s \in S, P_{t} \notin Q_{s} \Rightarrow P_{s} \subseteq A\). Choose \(a_{i} \in S_{i}, i \in J \setminus \{j\}\), and for \(z \in S_{j}\) let \(s' = a_{i}, i \neq j\) and \(s_{j} = z\). By hypothesis, if \(P_{t} \notin Q_{s}\) then \(P_{s} \subseteq A\). But, by [7, Lemma 2.3; 9, Theorem 1.2(7)] and the complete distributivity of \(\mathcal{S}\) we have \(P_{s'} = \bigcup\{P_{s'} | P_{t} \notin Q_{s}\} \subseteq A\). It follows that the join of the sets \(P_{s'}\) for all relevant \(t \in S_{j}\) is again contained in \(A\), and again using [7, Lemma 2.3] and complete distributivity we deduce that \(\Pi_{j}^{-1} A \subseteq \rho_{j}[A]\).
Take \( t \in \rho_j[A] \). Then we have \( s \in A \) with \( s_j = t \). Suppose that \( P_i \not\in \pi_j^{-1}A \). Then we have \( t' \in S_j \) with \( P_i \notin Q_{t'} \) so that \( P_{s'_j} \notin Q_{t'} \Rightarrow A \subseteq Q_{t'} \) for all \( s'_j \in S \). Define \( u \in S \) by taking \( u_j \in S_j \) satisfying \( P_{s'_j} \notin Q_{u_j} \), \( P_{u_j} \notin Q_{t'} \), and \( u_i \in S_i \), \( i \neq j \) satisfying \( P_i \notin Q_{u_i} \). Then \( P_i \notin Q_u \) by \cite[Corollary 1.4]{9}, so \( A \not\subseteq Q_u \) since \( P_s \subseteq A \). On the other hand taking \( s' = u \) in the above implication gives the contradiction \( A \subseteq Q_u \), and we have established that \( \rho_j[A] \subseteq \pi_j^{-1}A \).

Finally, let us note that \( \rho_j[A] \subseteq Q_t \Rightarrow \pi_j^{-1}A \subseteq Q_t \). By \cite[Theorem 1.2(6)]{9} we deduce that \( \pi_j^{-1}A \) is the smallest element of \( \mathcal{S}_j \) containing \( \rho_j[A] \). \( \square \)

It would be tempting to conjecture that \( \Pi_j^{-1}A \) is the largest element of \( \mathcal{S}_j \) contained in \( \rho_j[A] \). That this need not be the case is shown by the following example. This example also shows that we may have \( \rho_j[A] \not\in \mathcal{S}_j \), and so \( \pi_j^{-1}A \neq \rho_j[A] \).

**Example 3.19.** Consider the texture space \((L, \mathcal{L})\) of \cite[Example 1.1(3)]{9}. For each \( n \in \mathbb{N} \) let \( L_n = L \times \{n\} \) and \( \mathcal{L}_n = \{A \times \{n\} | A \in \mathcal{L}\} \), so that the \((L_n, \mathcal{L}_n)\) are disjoint isomorphic copies of \((L, \mathcal{L})\). Denote by \((M, \mathcal{M})\) the disjoint sum of the textures \((L_n, \mathcal{L}_n), n \in \mathbb{N}\), and consider the product \((S, \mathcal{S})\) of the texture spaces \((L, \mathcal{L})\) and \((M, \mathcal{M})\). Consider the element

\[
A = \bigcap_{n=2}^{\infty} \bigcap_{m=1}^{\infty} Q(1-\frac{1}{n^2} \cup \frac{1}{m^2} \cup \frac{1}{n^2} \cup \frac{1}{m^2})
\]

of \( \mathcal{S} \). By \cite[Corollary 1.4 and Proposition 1.5]{9} we note that for \( u \in L \) and \((v,k) \in M\),

\[
Q_{(u,v,k)} = (Q_u \times M) \cup \left(L \times \left(Q_v \times \{k\}\right) \cup \bigcup_{i \in \mathbb{N} \setminus \{k\}} L_i\right).
\]

It now follows easily that \( \rho_1[A] = (0,1) \notin \mathcal{L} \), whence \( \pi_1^{-1}A = (0,1) \) by Proposition 3.18. On the other hand it may be verified that \( \Pi_1^{-1}A = \emptyset \), which is certainly not the largest set in \( \mathcal{L} \) contained in \( \rho_1[A] \). This verifies the properties mentioned above.

The following definition generalizes openness and closedness to difunctions.

**Definition 3.20.** Let \((S_k, \mathcal{S}_k, \tau_k, \kappa_k), k = 1, 2, \) be ditopological texture spaces and \((f, F)\) a difunction from \((S_1, \mathcal{S}_1, \tau_1)\) to \((S_2, \mathcal{S}_2, \tau_2)\). Then

1. \((f, F)\) is open (co-open) if \( G \in \tau_1 \Rightarrow f^{-1}G \in \tau_2 \) \((F^{-1}G \in \tau_2)\).
2. \((f, F)\) is closed (coclosed) if \( K \in \kappa_1 \Rightarrow f^{-1}K \in \kappa_2 \) \((F^{-1}K \in \kappa_2)\).

It is clear that these properties are preserved by composition of difunctions. We now have:

**Proposition 3.21.** The projection difunctions on a product ditopological texture are open and coclosed.

**Proof.** We give the proof of openness, leaving the dual proof of coclosedness to the interested reader. Take \( G \in \tau \) and \( t \in \tau_j \) satisfying \( \pi_j^{-1}G \not\in \mathcal{Q}_t \). Now we have \( s \in S \) satisfying \( P_{s_j} \not\in \mathcal{Q}_t \) and \( G \not\in \mathcal{Q}_s \), so by Theorem 3.2 we have a set \( B \) in the standard base for the product ditopology so that \( B \subseteq G \) and \( B \not\in \mathcal{Q}_t \). Without loss of generality we may write \( B = \bigcap_{i \in J} E(i, G_i) \) where \( G_i \in \tau_i \) for
each \(i \in J\) and \(G_i = S_i\) for all but a finite number of \(i\). From Proposition 3.18(1) we have \(\pi_j^{-}B = G_j\), so \(G_j \subseteq \pi_j^{-}G\). On the other hand \(B \not\subseteq Q_j \Rightarrow P_{S_j} \subseteq G_j \Rightarrow G_j \not\subseteq Q_i\), and we conclude that \(\pi_j^{-}G \in \tau_j\), as required. \(\Box\)

We turn now to a discussion of coproducts \([1,21]\) in the categories \(\text{dfTex}\) and \(\text{df Ditop}\). Since the notion of topological functor is self-dual \([1,3,10]\) we may state the following dual of Proposition 3.7.

**Proposition 3.22.** The sink \(((S_j, \mathcal{S}_j, \tau_j, k_j) \xrightarrow{(f_j, F_j)} (S, \mathcal{S}, \tau, k))_{j \in J}\) in \(\text{df Ditop}\) is a coproduct of the family \(((S_j, \mathcal{S}_j, \tau_j, k_j))_{j \in J}\) in \(\text{dfTex}\) iff \((\tau, k)\) is the finest ditopology on \((S, \mathcal{S})\) for which the \((f_j, F_j)\) are bicontinuous, and \(((S_j, \mathcal{S}_j) \xrightarrow{(f_j, F_j)} (S, \mathcal{S}))_{j \in J}\) is a coproduct of the family \(((S_j, \mathcal{S}_j))_{j \in J}\) in \(\text{dfTex}\). Hence the requirement on \((\tau, k)\) is that \(\tau = \{G \in \mathcal{S} | F_j^{-}G \in \tau_j \ \forall j \in J\}\) and \(k = \{F \in \mathcal{S} | f_j^{-}F \in k_j \ \forall j \in J\}\).

We now show that we can identify set–indexed coproducts in \(\text{dfTex}\) with disjoint sums of textures. Let \(J\) be a set, suppose that \((S_j, \mathcal{S}_j)\), \(j \in J\) are textures, and that \((S, \mathcal{S})\) is the disjoint sum of these textures. Then, for \(j \in J\), we have the inclusion mapping \(e_j : S_j \rightarrow S\) defined by \(e_j(s_j) = (s_j, j)\), \(s_j \in S_j\). If we recall from [9, Proposition 1.5] that for \(s = (s_j, j)\), \(s_j \in S_j\) we have

1. \(P_s = P_{s_j} \times \{j\}\), and
2. \(Q_s = (Q_{s_j} \times \{j\}) \cup \bigcup_{k \in J \setminus \{j\}} (S_k \times \{k\})\),

then it is easy to check that \(e_j\) satisfies conditions (a), (b) and the condition (c) of Lemma 3.8. Hence

\[
e_j = \bigvee \{P_{(z, (z, j))} | z \in S_j\}, \quad E_j = \bigcap \{Q_{(z, (z, j))} | z \in S_j\},\]

define a difunction \((e_j, E_j)\) from \((S_j, \mathcal{S}_j)\) to \((S, \mathcal{S})\) satisfying \(e_j^{-}A = E_j^{-}A = e_j^{-1}[A]\) for all \(A \in \mathcal{S}_j\) and \(j \in J\). We note for future reference that for \(A \in \mathcal{S}\) we have \(e_j^{-1}[A] = A_j\), where \(A \cap (S_j \times \{j\}) = A_j \times \{j\}\).

**Theorem 3.23.** Let \((S, \mathcal{S})\) be the disjoint sum of the textures \((S_j, \mathcal{S}_j)_{j \in J}\), where \(J\) is a set. Then \(((S_j, \mathcal{S}_j) \xrightarrow{(e_j, E_j)} (S, \mathcal{S}))_{j \in J}\) is a coproduct of the family \(((S_j, \mathcal{S}_j))_{j \in J}\) in \(\text{dfTex}\).

**Proof.** Take \((Z, \mathcal{Q}) \in \text{Ob dfTex}\) and \((f_j, F_j) \in \text{dfTex}((S_j, \mathcal{S}_j), (Z, \mathcal{Q}))\), \(j \in J\). We must establish the existence of a unique difunction \((h, H)\), usually denoted by \([(f_j, F_j)]\), which makes the diagram below commutative for each \(j \in J\).
Define \((h,H)\) as follows:

\[
\begin{align*}
h &= \bigvee \{ \overline{\mathcal{P}(s,z)} | s = (s_j,j), s_j \in S_j, f_j \notin \overline{\mathcal{Q}(s_j,z)} \}, \\
H &= \bigcap \{ \overline{\mathcal{Q}(s,z)} | s = (s_j,j), s_j \in S_j, \mathcal{P}(s_j,z) \notin F_j \}.
\end{align*}
\]

We verify that \((h,H)\) has the required properties. In view of [9, Proposition 1.5] it is easy to show that \((h,H)\) is a difunction and we omit the details. To show that \(f_j = h \circ e_j\) let us first assume that \(f_j \notin h \circ e_j\) and take \(s_j \in S_j, z \in Z\) with \(f_j \notin \overline{\mathcal{Q}(s_j,z)}\) and \(\mathcal{P}(s_j,z) \notin h \circ e_j\). Since \(f_j\) is a relation we have \(s'_j \in S_j\) with \(P_{s_j} \notin \overline{\mathcal{Q}_{s_j}}\) and \(f_j \notin \overline{\mathcal{Q}(s'_j,z)}\). Also we have \(z' \in Z\) with \(f_j \notin \overline{\mathcal{Q}(s'_j,z')}\) and \(\mathcal{P}(s'_j,z') \notin \overline{\mathcal{Q}(s'_j,z)}\). Now we have \(\overline{\mathcal{P}(s'_j,z')} \subseteq h\), whence \(h \notin \overline{\mathcal{P}(s'_j,z)}\) and \(\mathcal{P}(s_j,s'_j)) \subseteq e_j\), whence \(e_j \notin \overline{\mathcal{Q}(s_j,s'_j))}\), which gives the contradiction \(\overline{\mathcal{P}(s_j,z)} \leq h \circ e_j\).

On the other hand suppose that \(h \circ e_j \neq f_j\). Then we have \(s_j \in S_j, z \in Z\) with \(\mathcal{P}(s_j,z) \neq f_j\) and \(t \in S\) with \(e_j \neq \overline{\mathcal{Q}(s_j,t)}\) and \(h \neq \overline{\mathcal{Q}(t,z)}\). From the definition of \(e_j\) we have \(\overline{\mathcal{P}(s_j,s_j)} \neq \overline{\mathcal{Q}(s_j,t)}\) and so by [9, Proposition 1.5], \(t = (t_j,j)\) and \(P_{s_j} \neq \overline{\mathcal{Q}_{s_j}}\). From \(h \neq \overline{\mathcal{Q}(t,z)}\) we now have \(z' \in Z\) with \(\overline{\mathcal{Q}(t,z')} \neq \overline{\mathcal{Q}(t,z)}\) and \(f_j \notin \overline{\mathcal{Q}(t_j,j'),z'}\). From this we deduce \(f_j \neq \overline{\mathcal{Q}(s_j,j),z}\), which contradicts \(\overline{\mathcal{P}(s_j,z)} \neq f_j\).

This establishes \(f_j = h \circ e_j\), and the dual proof of \(F_j = H \circ E_j\) is omitted.

The proof of uniqueness is left to the interested reader. \(\square\)

If we work in \textsf{cdfTex} instead of \textsf{dfTex} it may be shown that the injections \((e_j,E_j)\) are complemented, and hence the complemented disjoint sum [7] is a coproduct in \textsf{cdfTex}. Hence we have shown:

**Theorem 3.24.** The categories \textsf{dfTex} and \textsf{cdfTex} have coproducts.

According to Proposition 3.22 we need to take the finest ditopology making the inclusion difunctions \((e_j,E_j)\) bicontinuous in order to obtain a coproduct in \textsf{dfDitop}. We describe a ditopology on the coproduct in \textsf{dfTex}, and show it has the required property.

Let \((S_j,\mathcal{S}_j,\tau_j,\kappa_j), j \in J\), be ditopological texture spaces and \((S,\mathcal{S})\) the disjoint sum of the \((S_j,\mathcal{S}_j)\). The family \(\beta = \{ G_j \times \{ i \} | G_j \in \tau_j, j \in J \}\) clearly satisfies \(\bigvee \beta = S\), while if \(B_1, B_2 \in \beta\) and \(s \in S\) satisfy \(B_1 \cap B_2 \notin \overline{\mathcal{Q}}\), we have \(B_1 \cap B_2 \notin \beta\). It follows from Theorem 3.3 that \(\beta\) is a base for some ditopology on \((S,\mathcal{S})\). Next, let \(\gamma = \{ (F_j \times \{ i \}) \cup \bigcup_{i \in J \setminus \{ j \}} (S_i \times \{ i \}) | F_j \in \kappa_j, j \in J \}\). Clearly \(\bigcap \gamma = \emptyset\) and \(\gamma\) is closed under finite unions. Hence \(\gamma\) is a cobase for some ditopology on \((S,\mathcal{S})\), and we may make the following definition.

**Definition 3.25.** The ditopology \((\tau,\kappa)\) on the disjoint sum \((S,\mathcal{S})\) of the ditopological texture spaces \((S_j,\mathcal{S}_j,\tau_j,\kappa_j), j \in J, with base \(\beta = \{ G_j \times \{ i \} | G_j \in \tau_j, j \in J \}\) and cobase \(\gamma = \{ (F_j \times \{ i \}) \cup \bigcup_{i \in J \setminus \{ j \}} (S_i \times \{ i \}) | F_j \in \kappa_j, j \in J \}\) is called the **ditopological sum** on \((S,\mathcal{S})\). The ditopological sum may be denoted by \((\bigsqcup_{j \in J} S_j, \bigoplus_{j \in J} \mathcal{S}_j, \bigoplus_{j \in J} \tau_j, \bigoplus_{j \in J} \kappa_j)\).

**Proposition 3.26.** Let \((S,\mathcal{S},\tau,\kappa)\) be the disjoint sum of the ditopological texture spaces \((S_j,\mathcal{S}_j,\tau_j,\kappa_j), j \in J, where J is a set. Then

\[
((S,\mathcal{S},\tau,\kappa)) \xrightarrow{(\text{epimorphism})} (S,\mathcal{S},\tau,\kappa)) \quad \text{is a coproduct of the family } (S_j,\mathcal{S}_j,\tau_j,\kappa_j)_{j \in J} \text{ in } \textsf{dfDitop}.
\]
Proof. Straightforward. □

Corollary 3.27. The categories $\text{dfDitop}$ and $\text{cdfDitop}$ have coproducts.

Coproducts for the various subcategories can easily be developed—details are left to the interested reader.

Proposition 3.28. With the notation as in Definition 3.25 take $A \in \mathcal{S}$. Then

(i) $[A] = \bigcup_{j \in J} ([A]_j' \times \{j\})$ and

(ii) $[A] = \bigcup_{j \in J} ([A]_j' \times \{j\}),$

where $A \cap (S_j \times \{j\}) = A_j \times \{j\}$ and $[\cdot]'$, $[\cdot]$ denote closure and interior, respectively, in $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)$.

Proof. (i) Clearly,$$A \subseteq \bigcup_{j \in J} ([A]_j' \times \{j\}) = \bigcap_{j \in J} \left( [A]_j' \times \{j\} \cup \bigcup_{i \in \mathcal{J} \setminus \{j\}} (S_i \times \{i\}) \right) \in \kappa,$$so $[A] \subseteq \bigcup_{j \in J} ([A]_j' \times \{j\})$. If the reverse inclusion fails we have $k \in J$ and $B \in \gamma$ with $A \subseteq B$ for which $[A_k]_{\gamma} \times \{k\} \not\subseteq B$. Clearly $B$ must have the form $(F_k \times \{k\}) \cup \bigcup_{i \in \mathcal{J} \setminus \{k\}} (S_i \times \{i\})$ with $F_k \in \kappa_k$ satisfying $[A_k]_{\gamma} \not\subseteq F_k$. On the other hand $A \subseteq B$ gives $A_k \subseteq F_k$ and hence the contradiction $[A_k]_{\gamma} \subseteq F_k$.

This completes the proof of (i).

(ii) Straightforward. □

With regard to images and co-images we have:

Proposition 3.29. For $A \in \mathcal{S}_j$, $j \in J$, we have

$$e_j^{-1} A = A \times \{j\} \quad \text{and} \quad E_j^{-1} A = (A \times \{j\}) \cup \bigcup_{k \in \mathcal{J} \setminus \{j\}} (S_k \times \{k\}).$$

Hence, $e_j^{-1} A = e_j[A] \subseteq E_j^{-1} A$.

Proof. Straightforward application of [9, Proposition 1.5]. □

Proposition 3.30. The inclusion difunctions into a disjoint sum of ditopological texture spaces are open, co-open, closed and coclosed.

Proof. Immediate from Proposition 3.29. □

4. A point-free characterization of difunctions

In [9, Section 3] we gave a characterization of difunctions between simple textures and between plain textures in terms of point functions between the base sets. In this section we give a general
characterization of difunctions which involves only the texturings, and may therefore be regarded as “point-free”.

First let us observe that a difunction \((f, F): (S, \mathcal{I}) \to (T, \mathcal{F})\) gives rise to a mapping \(\theta = \theta_{(f, F)}: \mathcal{F} \to \mathcal{I}\) defined by \(\theta(B) = f^{-1}B = F^{-1}B \ \forall B \in \mathcal{F}\). In view of [9, Corollary 2.26], we see that \(\theta\) preserves arbitrary intersections and joins. Now let us show that, conversely, any mapping \(\theta: \mathcal{F} \to \mathcal{I}\) which preserves arbitrary intersections and joins gives rise to a difunction from \((S, \mathcal{I})\) to \((T, \mathcal{F})\).

**Proposition 4.1.** Let \(\theta: \mathcal{F} \to \mathcal{I}\) be a mapping which preserves arbitrary intersections and joins. Then there exists a unique difunction \((f, F) = (f^0, F^0): (S, \mathcal{I}) \to (T, \mathcal{F})\) which satisfies the equality \(\theta_{(f, F)} = \theta\).

**Proof.** We define

\[
    f = \bigvee \{ \overline{P}_{(s,t)} | \exists P_s \notin Q_u, P_v \notin Q_t \text{ with } P_u \subseteq \theta(C) \Rightarrow P_v \subseteq C, \ \forall C \in \mathcal{I} \},
\]

\[
    F = \bigwedge \{ \overline{Q}_{(s,t)} | \exists P_s \notin Q_v, P_t \notin Q_u \text{ with } \theta(C) \subseteq Q_u \Rightarrow C \subseteq Q_v, \ \forall C \in \mathcal{I} \}.
\]

It is immediate that \((f, F)\) is a direlation from \((S, \mathcal{I})\) to \((T, \mathcal{F})\). We verify that \(f^{-1}B = \theta(B)\) for all \(B \in \mathcal{F}\).

Suppose first that \(\theta(B) \notin f^{-1}B\) and take \(s \in S\) with \(\theta(B) \notin Q_s\), \(P_s \notin f^{-1}B\). Now we have \(t \in T\) with \(f \notin \overline{Q}_{(s,t)}\), \(P_t \notin B\), and hence \(t' \in T, u \in S, v \in T\) with \(\overline{P}_{(s',t')} \notin \overline{Q}_{(s,t)}\), \(P_s \notin Q_u\) and \(P_v \notin Q_{t'}\), for which \(P_u \subseteq \theta(C) \Rightarrow P_v \subseteq C \ \forall C \in \mathcal{I}\). From \(\theta(B) \notin Q_s\) and \(P_s \notin Q_u\) we have \(P_u \subseteq \theta(B)\), whence \(P_s \subseteq B\) by the above implication. However, this leads to \(P_t \subseteq B\), which is a contradiction.

Now suppose that \(f^{-1}B \subseteq \theta(B)\) and take \(s \in S\) satisfying \(f^{-1}B \subseteq Q_s\), \(P_s \subseteq \theta(B)\). Now we have \(s' \in S\) with \(P_{s'} \notin Q_s\) for which \(f \notin \overline{Q}_{(s',t)}\Rightarrow P_{t'} \subseteq B \ \forall t' \in T\). Let \(C_0 = \bigwedge \{ C \in \mathcal{I} | P_s \subseteq \theta(C) \}\). Then \(\theta(C_0) = \bigwedge \{ \theta(C) | C \in \mathcal{I}, P_s \subseteq \theta(C) \}\) since \(\theta\) preserves arbitrary intersections, so \(P_s \subseteq \theta(C_0)\). Since \(P_s \notin \theta(B)\) we have \(\theta(C_0) \notin \theta(B)\), whence \(C_0 \notin B\) since \(\theta\) clearly preserves inclusion. Now we have \(t \in T\) satisfying \(C_0 \notin Q_t\), \(P_t \notin B\), and hence \(v \in T\) with \(C_0 \notin Q_v\), \(P_v \notin Q_t\) and \(t' \in T\) satisfying \(P_v \notin Q_{t'}\) and \(P_{t'} \notin Q_{t'}\). Now

\[
    P_{s'} \notin Q_s, P_v \notin Q_t \quad \text{and} \quad P_s \subseteq \theta(C) \Rightarrow C_0 \subseteq C \Rightarrow C \notin Q_v \Rightarrow P_v \subseteq C
\]

so \(\overline{P}_{(s',t')} \subseteq f\). Since \(P_{t'} \notin Q_{t'}\) this gives \(f \notin \overline{Q}_{(s',t')}\), so by the above implication we obtain \(P_t \subseteq B\), which is a contradiction.

This establishes \(f^{-1}B = \theta(B)\) for all \(B \in \mathcal{F}\). A dual proof may be used to show that \(F^{-1}B = \theta(B)\) for all \(B \in \mathcal{F}\), and we omit the details. In particular we deduce that \(f^{-1}B = F^{-1}B \ \forall B \in \mathcal{F}\), whence \((f, F)\) is a difunction by [9, Theorem 2.24]. This proves the existence of a difunction \((f, F): (S, \mathcal{I}) \to (T, \mathcal{F})\) satisfying \(\theta_{(f, F)} = \theta\), and uniqueness may be established by an already familiar argument. \(\Box\)

Denote by \textbf{tmTex} the category of textures and textural morphisms. That is, the objects of \textbf{tmTex} are textures and a morphism \(\theta: (S, \mathcal{I}) \to (T, \mathcal{F})\) is a mapping \(\theta: \mathcal{F} \to \mathcal{I}\) which preserves arbitrary intersections and joins. Proposition 4.1 now enables us to set up an isomorphism between \textbf{dfTex} and \textbf{tmTex}°, as detailed below.
Theorem 4.2. Define \( \mathcal{M}: \text{dfTex} \to \text{tmTex}^{\text{op}} \) by \( \mathcal{M}(S, \mathcal{F}) = (S, \mathcal{F}) \), \( \mathcal{M}(f, F) = \theta_{(f,F)} \) and \( \mathcal{N}: \text{tmTex}^{\text{op}} \to \text{dfTex} \) by \( \mathcal{N}(S, \mathcal{F}) = (S, \mathcal{F}) \), \( \mathcal{N}(\theta) = (\mathcal{F}^{0}, \mathcal{F}^{0}) \). Then \( \mathcal{M} \) and \( \mathcal{N} \) are functors satisfying \( \mathcal{M} \circ \mathcal{N} = 1_{\text{tmTex}^{\text{op}}} \) and \( \mathcal{N} \circ \mathcal{M} = 1_{\text{dfTex}} \).

Proof. If \((i,I)\) is the identity on \((S, \mathcal{F})\) in \text{dfTex} then by [9, Proposition 2.9(3)] we have \( \theta_{(i,I)}(B) = i^{\leftarrow}B = B \) \( \forall B \in \mathcal{F} \) so \( \theta_{(i,I)} \) is also an identity in \text{tmTex}^{\text{op}}. Also, if \((S, \mathcal{F}) \xrightarrow{(g,G)} (T, \mathcal{F}) \xrightarrow{(f,F)} (U, \mathcal{F})\) and we consider the composition of mappings \( \mathcal{F} \xleftarrow{\theta_{(g,G)}} \mathcal{F} \xleftarrow{\theta_{(f,F)}} \mathcal{F} \) we obtain

\[
\theta_{(g,G)} \circ \theta_{(f,F)}(C) = \theta_{(g,G)}(f^{\leftarrow}C) = g^{\leftarrow}(f^{\leftarrow}C) = (f \circ g)^{\leftarrow}C = \theta_{(f,F) \circ (g,G)}(C)
\]

for all \( C \in \mathcal{F} \) by [9, Proposition 2.16(2)]. Since the order of composition of morphisms in \text{tmTex}^{\text{op}} is opposite to that for mappings, this shows that \( \mathcal{M}((f, F) \circ (g, G)) = \mathcal{M}(f, F) \circ \mathcal{M}(g, G) \), and so \( \mathcal{M} \) is a functor. Moreover, \( \mathcal{M} \) is faithful since if \((f, F)\) and \((g, G)\) are morphisms from \((S, \mathcal{F})\) to \((T, \mathcal{F})\) with \( \mathcal{M}(f, F) = \mathcal{M}(g, G) \) then \( \theta_{(f,F)} = \theta_{(g,G)} \) and so \((f, F)\) is \((g, G)\) by the uniqueness property in Proposition 4.1. Since we clearly have \( \mathcal{M} \circ \mathcal{N} = 1_{\text{tmTex}^{\text{op}}} \) and \( \mathcal{N} \circ \mathcal{M} = 1_{\text{dfTex}} \), again by Proposition 4.1, it is clear that \( \mathcal{M} \) is also a functor and so \( \mathcal{M} \) is an isomorphism with inverse \( \mathcal{N} \). \( \square \)

The above isomorphism between \text{dfTex} and \text{tmTex}^{\text{op}} restricts in the obvious way to an isomorphism between \text{dfSTex} and \text{tmSTex}^{\text{op}}, and between \text{dfPTex} and \text{tmPTex}^{\text{op}}. In order to be able to deal with complemented textures we need the following:

Proposition 4.3. Let \((S_{1}, \mathcal{F}_{1}, \sigma_{1})\), \((S_{2}, \mathcal{F}_{2}, \sigma_{2})\) be complemented textures and \( \theta: \mathcal{F}_{2} \to \mathcal{F}_{1} \) a mapping which preserves arbitrary intersections and joins. If we define \( \theta' = \sigma_{1} \circ \theta \circ \sigma_{2} \) then \((f^{0'}, F^{0'}) = (f^{0}, F^{0})'\).

Proof. For \( B \in \mathcal{F}_{2} \) we have \( (f^{0'})^{\leftarrow}B = \sigma_{2}(\sigma_{1}(B)) = \sigma_{1}(f^{0})^{\leftarrow}(\sigma_{2}(B)) = \sigma_{1}(\mathcal{F}^{0})^{\leftarrow}\sigma_{2}(B) = ((f^{0})')^{\leftarrow}B \) by [9, Theorem 2.20(2)]. Since \( f^{0'} \) and \((F^{0})' \) are both relations we deduce \( f^{0'} = (F^{0})' \) by [9, Lemma 2.7], and hence \( F^{0'} = (f^{0})' \) by [9, Proposition 2.27]. \( \square \)

Corollary 4.4. Let \((S_{1}, \mathcal{F}_{1}, \sigma_{1})\), \((S_{2}, \mathcal{F}_{2}, \sigma_{2})\) be complemented textures and \((f, F): (S_{1}, \mathcal{F}_{1}, \sigma_{1}) \to (S_{2}, \mathcal{F}_{2}, \sigma_{2})\) a difunction. Then \((f, F)\) is complemented if and only if \( \theta = \theta_{(f,F)}: \mathcal{F}_{2} \to \mathcal{F}_{1} \) preserves complementation, i.e. \( \theta \circ \sigma_{2} = \sigma_{1} \circ \theta \).

Proof. \((f, F) = (f^{0}, F^{0})\) is complemented iff \((f^{0}, F^{0})' = (f^{0}, F^{0})'\) iff \((f^{0'}, F^{0'}) = (f^{0}, F^{0})\) by Proposition 4.3 iff \( \theta' = \theta \) by Proposition 4.1 iff \( \sigma_{1} \circ \theta \circ \sigma_{2} = \theta \) iff \( \theta \circ \sigma_{2} = \sigma_{1} \circ \theta \). \( \square \)

If now we denote by \text{ctmTex} the category of complemented textures and textusal morphisms which preserve complementation, the functors \( \mathcal{M} \) and \( \mathcal{N} \) clearly specialize to an isomorphism between \text{cdfTex} and \text{ctmTex}^{\text{op}}.

Finally, in \text{tmTex}^{\text{op}}, let us take ditopological texture spaces in place of textures. Since a morphism from \((S_{1}, \mathcal{T}_{1}, \tau_{1}, \kappa_{1})\) to \((S_{2}, \mathcal{T}_{2}, \tau_{2}, \kappa_{2})\) is a mapping \( \theta: \mathcal{T}_{2} \to \mathcal{T}_{1} \) it will be appropriate to require that \( \theta[\tau_{2}] \subseteq \tau_{1} \) and \( \theta[\kappa_{2}] \subseteq \kappa_{1} \). Denoting the resulting category by \text{tmDitop} it is clear that \text{tmDitop} is isomorphic to \text{dfDitop}, and with similar results for the other associated categories.
5. Applications

In this section we present applications of the above characterization, and also investigate in greater detail the relation between the categories considered here and the theory of \(\mathbb{I}\)-topological spaces.

We begin with the following:

**Proposition 5.1.** The functor \(\mathcal{G} : \text{dfSTex} \to \text{dfTex}\) is an equivalence. Hence the categories \(\text{dfSTex}\) and \(\text{dfTex}\) are equivalent.

**Proof.** It will be sufficient to show that the inclusion functor \(\mathcal{G} : \text{tmSTex}^{\text{op}} \to \text{tmTex}^{\text{op}}\) is an equivalence. Since \(\mathcal{G}\) is clearly full and faithful it remains to show it is isomorphism-dense [1].

Recall from [9, Example 1.1(2)] that we may associate a simple texture \((\mathbb{L}, \mathcal{A})\) by setting \(\mathbb{L} = \{\hat{a} \mid a \in \mathbb{L}\}\), where \(\hat{a} = \{m \in M_{\mathbb{L}} \mid m \leq a\}\). Now given \((S, \mathcal{S}) \in \text{Ob tmTex}^{\text{op}}\), \(\mathcal{S}\) is a complete, completely distributive lattice and so as above we obtain a simple texture \((M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\). Hence \(\mathcal{M}_{\mathcal{S}} = \{\hat{A} \mid A \in \mathcal{S}\}\), where \(\hat{A} = \{M \in M_{\mathcal{S}} \mid M \subseteq A\}\). It is easy to see that the mapping \(A \mapsto \hat{A}\) is an isomorphism between the complete lattices \(\mathcal{S}\) and \(\mathcal{M}_{\mathcal{S}}\). It follows that the same mapping is an isomorphism between \((S, \mathcal{S})\) and \(\mathcal{G}(M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}) = (M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\) in the category \(\text{tmTex}^{\text{op}}\), which establishes that \(\mathcal{G}\) is isomorphism-dense. \(\square\)

**Corollary 5.2.** The categories \(\text{dfSDitop}\) and \(\text{dfDitop}\) are equivalent.

**Proof.** If \((S, \mathcal{S}, \tau, \kappa)\) is regarded as an object in \(\text{tmDitop}\) then we may define a ditopology \((\hat{\tau}, \hat{\kappa})\) on \((M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\) by setting \(\hat{\tau} = \{\hat{G} \mid G \in \tau\}\), \(\hat{\kappa} = \{\hat{K} \mid K \in \kappa\}\). Clearly the isomorphism between \(\mathcal{G}(M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}) = (M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\) and \((S, \mathcal{S})\) mentioned above becomes an isomorphism between \(\mathcal{G}(M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}, \hat{\tau}, \hat{\kappa}) = (M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}, \hat{\tau}, \hat{\kappa})\) and \((S, \mathcal{S}, \tau, \kappa)\) in \(\text{tmDitop}\). This establishes that \(\text{tmSDitop}\) and \(\text{tmDitop}\) are equivalent, whence the result follows. \(\square\)

**Corollary 5.3.** The categories \(\text{cdfTex}\) and \(\text{cdfSTex}\) are equivalent, as are the categories \(\text{cdfDitop}\) and \(\text{cdfSDitop}\).

**Proof.** In case \(\sigma\) is a complementation on \((S, \mathcal{S})\), \(\mu_{\sigma}\) defined by \(\mu_{\sigma}(A) = \sigma(A), A \in \mathcal{S}\) is a complementation on \((M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\). Since the mapping \(A \mapsto \hat{A}\) preserves complementation in the sense of Corollary 4.4, the corresponding isomorphism between \(\mathcal{G}(M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}) = (M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}})\) and \((S, \mathcal{S})\) becomes an isomorphism \(\mathcal{G}_{\text{ct}}\) between \(\mathcal{G}(M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}, \mu_{\sigma}) = (M_{\mathcal{S}}, \mathcal{M}_{\mathcal{S}}, \mu_{\sigma})\) and \((S, \mathcal{S}, \sigma)\) in \(\text{tmTex}^{\text{op}}\). This establishes that \(\text{ctmSTex}^{\text{op}}\) and \(\text{ctmTex}^{\text{op}}\) are equivalent, and likewise \(\text{tmSDitop}\) may be shown to be equivalent to \(\text{tmDitop}\), which establishes the required results. \(\square\)

Note that Example 2.14 provides a specific instance of the isomorphism mentioned above for the case \((S, \mathcal{S}, \sigma) = (\mathbb{I}, \mathcal{I}, 1)\).

As a second application of our point-free representation we establish an equivalence between \(\text{cdfSDitop}\) and the category of classical (complemented) Hutton spaces. We begin by recalling that a Hutton algebra (sometimes known as a fuzzy lattice) is a complete, completely distributive lattice
\( \mathbb{L} \) with an order-reversing involution \( ' \). We will denote by \( \text{HutAlg} \) the category of Hutton algebras and mappings between Hutton algebras that preserve arbitrary meets, joins and the involution, and make use of the correspondence between Hutton algebras and Hutton textures detailed in [9, Example 1.1(2)] to define a functor \( \mathcal{H} \) from \( \text{HutAlg} \) to the category \( \text{ctmSTex} \). Specifically we let \( \mathcal{H}(\mathbb{L},') = (M_1, \mathcal{M}_1, \mu_1, \mu_1') \), \( \mathcal{H}(\phi) = \hat{\phi} \), where \((M_1, \mathcal{M}_1, \mu_1)\) is the Hutton texture of \((\mathbb{L},') \in \text{Ob} \text{HutAlg} \), and for \( \phi \in \text{HutAlg}((\mathbb{L}_1,),(\mathbb{L}_2,')) \) we have \( \hat{\phi}(a) = \phi(a) \) for all \( a \in \mathbb{L}_1 \). Since \( a \mapsto \hat{a} \) is an isomorphism it is clear that \( \hat{\phi} \) is well defined and belongs to \( \text{ctmSTex}(\mathcal{M}_1, \mathcal{M}_1, \mu_1, \mu_1') \)

It is easy to verify that \( \mathcal{H} : \text{HutAlg} \to \text{ctmSTex} \) is a functor, and it is clearly full and faithful. It is also isomorphism-dense since with each complemented (simple) texture \((S, \mathcal{F}, \sigma)\) we may associate the Hutton algebra \((\mathcal{F}, \sigma)\) as previously, and \( \mathcal{H}(\mathcal{F}, \sigma) \) is clearly isomorphic to \((S, \mathcal{F}, \sigma)\) in \( \text{ctmSTex} \). Hence \( \mathcal{H} \) is an equivalence, and since equivalence is a self-dual property of functors [1, Proposition 3.43], the functor \( \mathcal{H}^{\text{op}} : \text{HutAlg}^{\text{op}} \to \text{ctmSTex}^{\text{op}} \) is also an equivalence. Finally, \( \text{cdfSTex} \) is isomorphic to \( \text{ctmSTex}^{\text{op}} \) and we have proved.

**Theorem 5.4.** The categories \( \text{cdfSTex} \) and \( \text{HutAlg}^{\text{op}} \) are equivalent.

Since \( \text{cdfSTex} \) is equivalent to \( \text{cdfTex} \), and equivalence is transitive, we also have:

**Corollary 5.5.** The categories \( \text{cdfTex} \) and \( \text{HutAlg}^{\text{op}} \) are equivalent.

If \( J \) is a set and \((\mathbb{L}_j,')\) Hutton algebras it is well known that the Cartesian product \( \mathbb{L} = \prod_{j \in J} \mathbb{L}_j \) with the pointwise order and order reversing involution \((I_j)' = (I_j')_j \) is a product in \( \text{HutAlg} \). It follows, therefore, that \( \prod_{j \in J} \mathbb{L}_j, ' \) is a coproduct in \( \text{HutAlg}^{\text{op}} \). In view of the equivalence with \( \text{cdfTex} \), this coproduct will be isomorphic to a disjoint sum in \( \text{cdfTex} \). This correspondence can be made explicit by noting that the molecules in \( \prod_{j \in J} \mathbb{L}_j \) have the form \( (m_j) \), where for some \( k \in J \), \( m_k \in M_{\mathbb{L}_k} \) and \( m_j = \bot_j \) for \( j \neq k \), \( \bot_j \) being the bottom (smallest element) of \( \mathbb{L}_j \). Clearly this may be represented uniquely by the element \( (m_j)_k \) of the disjoint sum of the sets \( M_{\mathbb{L}_j} \).

If \( T \) is a topology on \( \mathbb{L} \), that is \( T \subseteq \mathbb{L} \) is closed under arbitrary joins and finite meets and contains the top and bottom elements of \( \mathbb{L} \), and we set \( \tau_T = \{ \hat{a} \mid a \in T \} \), \( \kappa_T = \{ \hat{b} \mid b' \in T \} \), it is clear that \( (\tau_T, \kappa_T) \) is a complemented ditopology on \((M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_1)\). Recall that the category of classical (complemented) Hutton spaces is obtained from \( \text{HutAlg}^{\text{op}} \) by replacing Hutton algebras \( (\mathbb{L},') \) with Hutton spaces \((\mathbb{L},', T)\), and requiring that a morphism \( \phi : (\mathbb{L}_1, ', T_1) \to (\mathbb{L}_2, ', T_2) \) should satisfy the continuity condition \( \phi[T_1] \subseteq T_2 \). This category is denoted by \( \mathcal{H} \) in [2], and we use the same notation here. It is now trivial to verify that \( \mathcal{H}^{\text{op}} \) becomes a functor from \( \mathcal{H} \) to \( \text{ctmSDitop} \) if we set \( \mathcal{H}^{\text{op}}(\mathbb{L},', T) = (M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \tau_T, \kappa_T) \), and restrict to morphisms in \( \mathcal{H} \). Moreover, \( \mathcal{H}^{\text{op}} \) is again an equivalence. Hence, since the categories \( \text{ctmSDitop} \) and \( \text{cdfSDitop} \) are isomorphic, we have proved.

**Proposition 5.6.** The categories \( \text{cdfSDitop} \) and \( \mathcal{H} \) are equivalent.

Finally, since \( \text{cdfSDitop} \) and \( \text{cdfDitop} \) are equivalent,

**Corollary 5.7.** The categories \( \text{cdfDitop} \) and \( \mathcal{H} \) are equivalent.
The study of categories such as $H$ represents a “point-free” approach to the study of lattice-valued topology, which is favoured by many mathematicians. However, points are often a useful tool and the equivalence between $H$ and $\text{cdfSDtop}$ shows that not only do simple complemented textures give a point-based representation of the objects of $H$, the morphisms also have a concrete pointed realization in the form of bicontinuous complemented difunctions. It is no coincidence, therefore, that the study of ditopological texture spaces tends to produce concepts and results which are appropriate to a point-free setting. This will be clearly illustrated by our treatment of separation axioms in the third paper in this series, where we show that most of the pointed axioms we introduce correspond to a point-free setting. Naturally, other approaches, such as the theory of topological molecular lattices of Wang [31], can be expressed in the framework of ditopological texture spaces, but we will not follow this line of enquiry here.

For the remainder of this section we restrict our attention to Hutton algebras of the form $\mathbb{L}^X$, where $\mathbb{L}$ is a fixed Hutton algebra and $X$ an arbitrary set. The Hutton texture of $\mathbb{L}^X$ has the form $(W_X, \mathcal{W}_X, \omega_X) = (X, \mathcal{P}(X), \pi_X) \otimes (M_\mathbb{L}, \mathcal{M}_X, \mu_X)$ (cf. [9, Example 1.1(4), 7]). A $\mathbb{L}$-topology $T$ on $X$, that is a $\mathbb{L}$-fuzzy topology in the sense of Chang and Goguen [11,18,19], corresponds as before to a complemented ditopology $(\tau_T, \kappa_T)$, this time on $(W_X, \mathcal{W}_X, \omega_X)$. We denote a restriction to objects of the form $\mathbb{L}^X$ or $(W_X, \mathcal{W}_X, \omega_X)$, as the case may be, by appending $\mathbb{L}$ as a subscript to the category in question. Hence the correspondence between Hutton algebras of the form $(\mathbb{L}^X)'$ and textures $(W_X, \mathcal{W}_X, \omega_X)$ may be expressed by saying that the functor $\mathbb{H}_{\mathbb{L}}$ discussed above becomes an isomorphism from $\text{HutAlg}_{\mathbb{L}}^{\text{op}}$ to $\text{ctmSTex}_{\mathbb{L}}^{\text{op}}$. Likewise we obtain an isomorphism between $H_{\mathbb{L}}$ and $\text{ctmDitop}_{\mathbb{L}}$.

The special nature of the structures considered above enables us to define a functor from $\text{Set}$ to $\text{ctmSTex}$ which is essentially different from the embedding considered earlier. To see this consider the following mappings for sets $X$ and $Y$:

$$
egin{array}{ccc}
X & \xleftarrow{\varphi} & \mathbb{W}_X = \mathbb{L}^X & \xrightarrow{\varphi^{-1}} & \mathbb{W}_Y \\
\downarrow & & \varphi^{-1} & & \downarrow \\
Y & \xleftarrow{\varphi} & \mathbb{L}^Y & \xrightarrow{\varphi^{-1}} & \mathbb{W}_Y
\end{array}
$$

Here $\varphi^{-1}(\mu)(x) = \mu(\varphi(x))$ for all $\mu \in \mathbb{W}_Y$ and $x \in X$, while $\varphi^{-1}(\hat{\mu}) = \varphi^{-1}(\mu)$ for all $\mu \in \mathbb{W}_Y$. It is well known that $\varphi^{-1}$ preserves complementation and arbitrary meets and joins, so in view of the complementation preserving isomorphism $\mu \mapsto \hat{\mu}$ the mapping $\varphi^{-1}$ does the same. It is easy to verify that $\iota_x$ is the identity on $\mathbb{W}_X$, while for $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ we have $(\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1}$. Hence we have proved:

**Theorem 5.8.** The mapping $\mathfrak{F} = \mathfrak{F}_{(\mathbb{L}, \cdot)} : \text{Set} \rightarrow \text{ctmSTex}^{\text{op}}$ defined by setting $\mathfrak{F}(X) = (W_X, \mathcal{W}_X, \omega_X)$ and $\mathfrak{F}(\varphi) = \varphi^{-1}$ is a functor.

Now consider a fixed complemented ditopology $(\tau_0, \kappa_0)$ on $(M_\mathbb{L}, \mathcal{M}_\mathbb{L}, \mu_\mathbb{L})$. For any topology $\mathcal{T}$ on $X$, $(\mathcal{T}, \mathcal{T}^c)$ is a complemented ditopology on $(X, \mathcal{P}(X), \pi_X)$ and we may consider the product ditopology $(\tau^\mathcal{T}, \kappa^\mathcal{T}) = (\mathcal{T} \otimes \tau_0, \mathcal{T}^c \otimes \kappa_0)$ on $(W_X, \mathcal{W}_X, \omega_X)$. Clearly $(\tau^\mathcal{T}, \kappa^\mathcal{T})$ is complemented. If $\varphi : (X, \mathcal{T}) \rightarrow (Y, \mathcal{V})$ is continuous we claim that $\varphi^{-1}[\tau^\mathcal{T}] \subseteq \tau^\mathcal{V}$ and $\varphi^{-1}[\kappa^\mathcal{T}] \subseteq \kappa^\mathcal{V}$. The first result
follows by noting that a subbase for $\tau'$ consists of sets of the form $G \times M_\ell$, $G \in \mathcal{Y}$ and $Y \times \hat{\gamma}$, 
$\gamma \in \tau_0$, while $\phi^{-1}(G \times M_\ell) = \phi^{-1}[G] \times M_\ell \in \tau'$ and $\phi^{-1}(Y \times \hat{\gamma}) = X \times \hat{\gamma} \in \tau'$. The second result is proved in the same way, and we omit the details. We deduce that

**Theorem 5.9.** The functor $\mathfrak{F}$ defined above specializes to a functor

$$
\mathfrak{F} = \mathfrak{F}(\mathfrak{I}, \tau_0, \kappa_0) : \text{Top} \to \text{ctmSDitop}
$$

for every Hutton algebra $(\mathfrak{I}, \tau')$ and complemented ditopology $(\tau_0, \kappa_0)$ on $(M_\ell, M_\ell, \mu_\ell)$.

It is straightforward to prove that $\mathfrak{F}$ is faithful, but in general it will not be full. It will be interesting to obtain more information about this functor, and to simplify the discussion we will confine ourselves to the case $\mathfrak{I} = \mathfrak{I}$. Now we have $(M_1, M_\ell, \mu_\ell) = (L, \mathcal{L}, \iota)$, the complemented texture of [9, Example 1.1(3)]. We will first regard $\mathfrak{F} = \mathfrak{F}(\mathfrak{I}, \tau')$ as a functor from $\text{Set}$ to the category $\text{ctmTex}^{\text{op}}$. Hence

$$
\text{Set} \xrightarrow{\mathfrak{F}} \text{ctmTex}^{\text{op}} \xrightarrow{\mathfrak{M}} \text{cdfTex}_1 \xrightarrow{\mathfrak{W}} \text{cfTex}_1,
$$

where $\mathfrak{M}$ and $\mathfrak{W}$ are isomorphisms.

**Lemma 5.10.** For $\psi : X \to Y$ in $\text{Set}$,

1. If $\mathfrak{F}(\psi) = \theta$, so that $\theta = \mathfrak{F}^{-1}$, then $\mathfrak{M}(\theta) = (f^0, F^0)$, where

$$
f^0 = \bigvee \{\overline{P}_{(x,r),(\psi(x),r)}) | x \in X, r \in L\}, \quad F^0 = \bigcap \{\overline{Q}_{(x,r),(\psi(x),r)}) | x \in X, r \in L\}.
$$

2. If $\mathfrak{W}(f^0, F^0) = \varphi$, so that $\varphi = \mathfrak{W}(f^0, F^0)$, then $\varphi(x, r) = (\psi(x), r)$, i.e. $\varphi = (\psi_1)_L$.

**Proof.** Straightforward. $\square$

The general form of the morphisms in $\text{cfTex}_1$ is given in [9, Example 3.11(4)]. In particular, we note that with each such morphism $\varphi$ is associated a morphism $\varphi_1$ in $\text{Set}$. Moreover, if $\varphi$ is the identity mapping on $X \times L$ then $\varphi_1$ is the identity on $X$, while for $X \times L \xrightarrow{\varphi} Y \times L \xrightarrow{\psi} Z \times L$ we have $(\psi \circ \varphi)_1 = \psi_1 \circ \varphi_1$. It follows that we may define a functor $\mathfrak{G} : \text{cfTex}_1 \to \text{Set}$ by letting $\mathfrak{G}(W_X, \mathcal{W}_X, \mathcal{O}_X) = X$ and $\mathfrak{G}(\varphi) = \varphi_1$.

$$
\begin{array}{ccc}
\text{Set} & \xrightarrow{\mathfrak{F}} & \text{ctmTex}^{\text{op}} \\
\mathfrak{G} & \downarrow & \mathfrak{M} \\
\text{cfTex}_1 & \xleftarrow{\mathfrak{W}} & \text{cdfTex}_1
\end{array}
$$

Even though $\mathfrak{G} \circ \mathfrak{W} \circ \mathfrak{M} \circ \mathfrak{F} = 1_{\text{Set}}$ it is not difficult to show that $\mathfrak{G}$ is not an adjoint of $\mathfrak{W} \circ \mathfrak{M} \circ \mathfrak{F}$ precisely because there are morphisms in $\text{cfTex}_1$ which are not of the form $(\psi_1)_L$ for $\psi \in \text{Mor Set}$. For the same reason $\mathfrak{W} \circ \mathfrak{M} \circ \mathfrak{F}$ is not the identity on morphisms and we deduce easily that $\mathfrak{G}$ is not a co-adjoint of $\mathfrak{W} \circ \mathfrak{M} \circ \mathfrak{F}$ either.

Now let $(\tau, \kappa)$ be a ditopology on $(W_X, \mathcal{W}_X)$ and define

$$
\tau^1 = \{G \in \mathcal{P}(X) | G \times L \in \tau\} \quad \text{and} \quad \kappa^1 = \{K \in \mathcal{P}(X) | K \times L \in \kappa\}.
$$
Clearly, \((\tau^1, \kappa^1)\) is a ditopology on \((X, \mathcal{P}(X))\). Likewise, \((\tau^2, \kappa^2)\) defined by

\[
\tau^2 = \{G \in \mathcal{L} \mid X \times G \in \tau\} \quad \text{and} \quad \kappa^2 = \{K \in \mathcal{L} \mid X \times K \in \kappa\}
\]

is a ditopology on \((L, \mathcal{L})\). The product of \((\tau^1, \kappa^1)\) and \((\tau^2, \kappa^2)\) is a ditopology on \((W_X, \mathcal{F}_X)\) which is clearly coarser than \((\tau, \kappa)\). The following result gives necessary and sufficient conditions under which these ditopologies coincide.

**Lemma 5.11.** The following are equivalent:

1. The product of \((\tau^1, \kappa^1)\) and \((\tau^2, \kappa^2)\) coincides with \((\tau, \kappa)\).
2. The following conditions hold:
   a. Given \(H \in \tau, (x,r) \in W_X\) with \(H \not\subseteq \mathcal{O}(x,r)\), there exist \(G_1 \in \tau^1, G_2 \in \tau^2\) with \((G_1 \times L) \cap (X \times G_2) \subseteq H\) and \((G_1 \times L) \cap (X \times G_2) \not\subseteq \mathcal{O}(x,r)\).
   b. Given \(K \in \kappa, (x,r) \in W_X\) with \(P(x,r) \not\subseteq K\), there exist \(K_1 \in \kappa^1, K_2 \in \kappa^2\) with \(K \subseteq (K_1 \times L) \cup (X \times K_2)\) and \(P(x,r) \not\subseteq (K_1 \times L) \cup (X \times K_2)\).

**Proof.** Immediate by the definition of product ditopology. \(\square\)

If \((\tau, \kappa)\) is complemented then \((\tau^1, \kappa^1)\) and \((\tau^2, \kappa^2)\) are also complemented, so in particular \(\tau^1\) is a topology on \(X\) in the usual sense and \(\kappa^1\) is the set of closed sets of \(\tau^1\). This enables us to specialize \(\mathcal{G}\) to a functor \(\mathcal{G} : \text{cfditop}_1 \rightarrow \text{Top}\) by setting \(\mathcal{G}(W_X, \mathcal{F}_X, \omega_X, \tau, \kappa) = (X, \tau^1)\). Of course we must verify that if \(\varphi \in \text{cfditop}_1((W_X, \mathcal{F}_X, \omega_X, \tau_X, \kappa_X),(W_Y, \mathcal{F}_Y, \omega_Y, \tau_Y, \kappa_Y))\) then \(\mathcal{G}(\varphi) = \varphi_1 \in \text{Top}(\tau^1_X, \tau^1_Y)\).

However, if we note that for \(A \subseteq Y\) we have \(\varphi^{-1}(A \times L) = \varphi^{-1}_1[A] \times L\) then,

\[
G \in \tau^1_Y \Rightarrow G \times L \in \tau_Y \Rightarrow \varphi^{-1}(G \times L) = \varphi^{-1}_1[G] \times L \in \tau_X \Rightarrow \varphi^{-1}[G] \in \tau^1_X,
\]

which gives the required result. For any fixed ditopology \((\tau_0, \kappa_0)\) on \((L, \mathcal{L}, \lambda)\) we may also regard the functor \(\mathfrak{F} = \mathfrak{F}_{(\lambda^1, \tau_0, \kappa_0)}\) as going from \(\text{Top}\) to \(\text{ctmditop}_1\), and so we have the following diagram:

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\mathfrak{J}} & \text{ctmditop}_1 \\
\mathcal{G} \uparrow & & \downarrow \mathfrak{J} \\
\text{cfditop}_1 & \xleftarrow{\mathfrak{F}} & \text{cfditop}_2
\end{array}
\]

According to the discussion above there is no possibility of an adjoint situation involving \(\mathcal{G}\) and \(\mathfrak{F}\) unless we restrict our morphisms in \(\text{cfditop}_1\) to have the form \(\langle \psi, \lambda \rangle\), effectively replacing \(\text{cfditop}_1\) by the subconstruct \(\text{l-ditop}\) of Example 2.8. To simplify the notation we will regard \(\mathfrak{F}\) as a functor from \(\text{Top}\) to \(\text{l-ditop}\) by redefining \(\mathfrak{F}(\psi)\) for \(\psi \in \text{Mor}\, \text{Top}\) to have the value \(\langle \psi, \lambda \rangle \in \text{Mor}\, \text{l-ditop}\). In view of the isomorphism \(\mathfrak{J} : \text{l-ditop} \rightarrow \text{Top}\) introduced in Example 2.8, we also have the functors \(\mathfrak{J}^{-1} \circ \mathfrak{F} : \text{Top} \rightarrow \text{l-ditop}, \mathcal{G} \circ \mathfrak{J} : \text{l-ditop} \rightarrow \text{Top}\), which we will also abbreviate to \(\mathfrak{F}\), \(\mathcal{G}\), respectively, when there is no possibility of confusion.

**Theorem 5.12.** If \((\tau_0, \kappa_0)\) denotes the indiscrete, co-indiscrete ditopology (see Example 2.1(2)) then \(\mathfrak{F} = \mathfrak{F}_{(\lambda^1, \tau_0, \kappa_0)}\) is an adjoint of \(\mathcal{G}\).
Proof. Consider \( \top \xrightarrow{\mathfrak{F}} \mathcal{D} \mathcal{T}_\mathcal{O}_\mathcal{P} \xrightarrow{\mathfrak{G}} \mathcal{T}_\mathcal{O} \) and take \( (W_X, \mathcal{U}_X, \omega_X, \tau, \kappa) \in \text{Ob} \mathcal{D} \mathcal{T}_\mathcal{O}_\mathcal{P} \). We show that \( ((\iota_X, \iota_L), (X, \tau^1)) \) is a \( \mathfrak{F} \)-universal arrow with domain \( (W_X, \mathcal{U}_X, \omega_X, \tau, \kappa) \). Certainly \( (\iota_X, \iota_L) : W_X \rightarrow W_X \) is \( \tau \tau^1 \setminus \{L, \emptyset\} \) continuous, whence it is bicontinuous by Lemma 2.3(3). Hence \( ((\iota_X, \iota_L), (X, \tau^1)) \) is a \( \mathfrak{F} \)-structured arrow with domain \( (W_X, \mathcal{U}_X, \omega_X, \tau, \kappa) \). If \( (\langle \psi, \iota_L \rangle, (Y, \tau')) \) is a \( \mathfrak{F} \)-structured arrow with the same domain, the diagram

\[
\begin{array}{ccc}
(W_X, W_X, \omega_X, \tau, \kappa) & \xrightarrow{(\iota_X, \iota_L)} & \mathfrak{F}(X, \tau^1) \\
\downarrow{\langle \psi, \iota_L \rangle} & & \downarrow{\mathfrak{F}(\psi)} \\
\mathfrak{F}(Y, \tau') & & \\
\end{array}
\]

is commutative since \( \psi : X \rightarrow Y \) is \( \tau^1 \tau' \) continuous \( (V \in \tau' \Rightarrow V \times L \in \tau' \setminus \{L, \emptyset\} \Rightarrow \psi^{-1}[V] \times L = \langle \psi, \iota_L \rangle^{-1}[V \times L] \in \tau \Rightarrow \psi^{-1}[\tau'] \in \tau^1) \). Moreover, \( \psi \) is clearly the unique \( \mathcal{T}_\mathcal{O} \) morphism with this property. Since \( (X, \tau^1) = \mathfrak{G}(W_X, \mathcal{U}_X, \omega_X, \tau, \kappa) \) this shows that \( \mathfrak{F} \) is an adjoint of \( \mathfrak{G} \). \( \square \)

The following results will be useful when working directly in terms of \( \mathcal{D} \mathcal{T}_\mathcal{O} \).

**Lemma 5.13.** Let \( (X, T) \) be an \( \mathcal{D} \) topology and \( (\tau_T, \kappa_T) \) the corresponding complemented ditopology on \( (W_X, \mathcal{U}_X, \omega_X) \). Then

1. \( \tau_T^1 = \{G \subseteq X \mid \chi_G \in T\}, \quad \tau_T^2 = \{\{0, r\} \mid r \in T\}. \)

2. The following are equivalent:
   
   (i) \( \tau_T = \tau_T^1 \otimes \tau_T^2 \).
   
   (ii) For any subbase \( B \) of \( T \), given \( h \in B, x \in X \) and \( r \in (0, 1] \) satisfying \( r < h(x) \), there exists \( s \in (0, 1] \) with \( r < s \), and \( Y \subseteq X \) with \( x \in Y \) so that \( \chi_Y \in T \), \( s \in T \) and \( \chi_Y \wedge s \leq h \).

(iii) \( (X, T) = \mathfrak{G}(W_X, \mathcal{U}_X, \omega_X, \tau, \kappa) \).

**Proof.** (1) The first equality follows from \( G \in \tau_T^1 \Leftrightarrow \chi_G = G \times L \in \tau_T \Leftrightarrow \chi_G \in T \), and the second from \( (0, r] \in \tau_T^2 \Leftrightarrow \hat{r} = X \times (0, r] \in \tau_T \Leftrightarrow r \in T \).

(2) (i) \( \Leftrightarrow \) (ii) Suppose that \( \tau_T = \tau_T^1 \otimes \tau_T^2 \) and take \( h, x \) and \( r \) satisfying the stated conditions. Then \( \hat{h} \in \tau_T \) clearly satisfies \( \hat{h} \notin O_{(x, r)} \) and so by Lemma 5.11 we have \( G_1 \in \tau_T^1, G_2 \in \tau_T^2 \) with \( (G_1 \times L) \cap (X \times G_2) \subseteq H \) and \( (G_1 \times L) \cap (X \times G_2) \notin O_{(x, r)} \). Setting \( Y = G_1 \) and noting that \( G_2 = (0, s] \) for some \( s \in L \) gives \( Y \) and \( s \) satisfying the required conditions. The proof of the converse is left to the interested reader.

(i) \( \Leftrightarrow \) (iii) Immediate from the definitions. \( \square \)

We will refer to an \( \mathcal{D} \) topology \( T \) satisfying the equivalent conditions of (2) as **productive**.

Let us recall that the mapping \( \omega \) of Lowen [26,27,30] takes a topology \( \mathcal{T} \) on \( X \) to the \( \mathcal{D} \) topology on \( X \) with subbase \( \{g \in \mathcal{D} \mid \{x \mid r < g(x)\} \in \mathcal{T} \quad \forall r \in \mathbb{L}\} \), while the mapping \( \iota \) takes an \( \mathcal{D} \) topology \( T \) on \( X \) to the topology on \( X \) with subbase \( \{\{x \mid r < g(x)\} \mid r \in \mathbb{L}, g \in T\} \). For simplicity of notation we will also denote by \( \omega \) and \( \iota \) the corresponding functors \( \omega : \mathcal{T}_\mathcal{O} \rightarrow \mathcal{D} \mathcal{T}_\mathcal{O} \) and \( \iota : \mathcal{D} \mathcal{T}_\mathcal{O} \rightarrow \mathcal{T}_\mathcal{O} \).

**Theorem 5.14.** (1) For any topology \( \mathcal{T} \) on \( X \) the \( \mathcal{D} \) topology \( \omega(\mathcal{T}) \) is productive and contains all the constant \( \mathbb{L} \)-sets.
(2) The ω functor of Lowen coincides with the functor $\mathcal{F}(\{\cdot\},\mathcal{F}) : \text{Top} \to \mathcal{I}\text{-Top}$ corresponding to the discrete, codiscrete complemented ditopology.

**Proof.** (1) It is clear from the definition that $\omega(\mathcal{T})$ contains all the constant $\mathcal{I}$-subsets of $X$. A direct proof of productivity may be obtained from Lemma 5.13(2ii) by letting $B$ be the subbase of $\omega(\mathcal{T})$ mentioned in the definition, taking $h \in B$, $x \in X$ and $r \in (0,1]$ satisfying $r < h(x)$. If we choose $r < s < h(x)$ and let $Y = \{ z \in X \mid s < h(z) \}$ it is immediate that $\chi_Y,s \in \omega(\mathcal{T})$ and $\chi_Y \wedge s \leq h$.

Since $\mathcal{I}$ is a chain, productivity is also a trivial consequence of [30, Theorem 3.3] which says that $\{ \chi_G \mid G \in \mathcal{F} \} \cup \{ r \mid r \in \mathcal{I} \}$ is also a subbase for $\omega(T)$.

(2) Immediate from (1). □

**Theorem 5.15.** The functor $\mathfrak{G} : \mathcal{I}\text{-Top} \to \text{Top}$ assigns to an $\mathcal{I}$-topology $(X,T)$ a (possibly strictly) weaker topology than does the Lowen functor $i$. In case $T$ is topological these topologies coincide.

**Proof.** For $H \in \tau^i_T$ we have $\chi_H \in T$ and so $H \in i(T)$ since $H = \{ x \mid r < \chi_H(x) \}$ for $r < 1$. Hence $\tau^i_T \subseteq i(T)$ which verifies the first statement. To show that we may have strict inclusion, consider the complemented ditopology $(\tau,\kappa)$ on $(W_A,\mathcal{W}_A,\omega_A)$, $A = \{ a,b \}$ defined in Example 2.1(6). Since $\tau = \{ \emptyset,\{ a \} \times (0,1/2],W_A \}$ the corresponding $\mathcal{I}$-topology on $A$ is clearly $T = \{ \emptyset,\chi_a \wedge 1/2,1 \}$. In this case $\tau^i_T = \{ \emptyset,A \} \subseteq \{ \emptyset,\{ a \},A \} = i(T)$.

Finally, suppose that $T$ is topological. Then by [30, Definition 3.9] we have $\omega(i(G_k(T))) = G_k(T)$, where $G_k$ assigns to $T$ the $\mathcal{I}$-topology generated by $T$ and the constant $\mathcal{I}$-valued sets. Now $H \in i(T)$ implies $G_k(T) \Rightarrow \chi_H \in\omega(i(G_k(T)))$ by [30, Theorem 3.3]. Hence $\chi_H \in G_k(T)$, which easily leads to $\chi_H \in T$ and so $H \in \tau^i_T$, as required. □

It is known [30] that $i$ is an adjoint of $\omega$, but we can use the above example to show that $\mathfrak{G}$ is not an adjoint of $\mathcal{F} = \mathcal{F}(\{\cdot\},\mathcal{F})$. Suppose that $(\psi,\mathcal{F}(\{\cdot\},\mathcal{F}))$ is a $\mathfrak{G}$-universal arrow with domain $(A,\mathcal{F})$, where $\mathcal{F}$ is as defined above. Clearly $\mathcal{F}(\mathfrak{G}(A,\mathcal{F})) = (A,\mathcal{F})$ and so $\psi : A \to A$, being a $\text{Top}$ morphism, must be $\mathcal{F} \setminus \mathcal{F}$ continuous. If $\phi : A \to A$ is defined by $\phi(a) = b$, $\phi(b) = a$ then $(\phi,(A,\mathcal{F}))$, with $T$ as above, is a $\mathfrak{G}$-structured arrow with domain $(A,\mathcal{F})$ since $\mathcal{F}(A,T)$ is indiscrete as noted above. However it is easy to see that there is no $\mathcal{I}\text{-Top}$ morphism $\eta : \mathcal{F}(A,\mathcal{F}) \to (A,T)$ satisfying $\mathfrak{G}(\eta) \circ \psi = \phi$, which contradicts the universal property.

**Theorem 5.16.** Let $(X,T)$ be an $\mathcal{I}$-topological space.

1. If $T$ is productive it is topological, but not conversely.
2. $T$ is topological if and only if $G_k(T)$ is productive.

**Proof.** (1) Let $T$ be productive. Since $G_k(T) \subseteq \omega(i(G_k(T)))$ always holds, take $h \in \omega(i(G_k(T)))$. By [30, Theorem 3.3] we may write $h = \bigvee_{j \in J} (\chi_{H_j} \wedge r_j)$, $H_j \subseteq i(G_k(T))$, $r_j \in \mathcal{I}$, so in order to prove that $h \in G_k(T)$ it will suffice to show that $H \in i(G_k(T)) \Rightarrow \chi_H \in T$. Hence it will be sufficient to show that for $x \in H$ we have $Y \subseteq X$ satisfying $x \in Y \subseteq H$ and $\chi_Y \in T$. But if $x \in H \in i(G_k(T)) = i(T)$ then we have $u_m \in T$, $r_m \in \mathcal{I}$, $m = 1,2,\ldots,n$, so that $x \in \bigcap_{m=1}^n \{ z \mid r_m < u_m(z) \} \subseteq H$. Without loss of generality we may assume $r_m > 0$, so by Lemma 5.13(2ii) applied to $r_m < u_m(x)$ we have $Y_m \subseteq X$ and $s_m \in \mathcal{I}$ satisfying $r_m < s_m$, $\chi_{Y_m} \in T$, $s_m \in T$ and $\chi_{Y_m} \wedge s_m \leq u_m$ for each $m$. Clearly $Y = \bigcap_{m=1}^n Y_m$ has the required property, and we have established $h \in G_k(T)$, as required.
To show that the converse does not hold it suffices to note that the $\mathbb{L}$-topology $T$ on $A = \{a, b\}$ defined by $T = \{\emptyset, X, 1/2, X\}$ is topological but not productive.

If $G_k(T)$ is productive then $G_k(T)$ is topological by (1), hence $T$ is topological also. Conversely, if $T$ is topological, from $\omega(i(G_k(T))) = G_k(T)$ we obtain \[ S\circ T \mathbb{L} \mathbb{L} \omega(i(G_k(T))) \mathbb{L} X, G_k(T) \] by Theorem 5.14(2) and Theorem 5.15 since $G_k(T)$ is topological. We deduce that \[ S\circ T \mathbb{L} \mathbb{L} \omega(i(G_k(T))) \mathbb{L} X, G_k(T) \text{ whence } G_k(T) \text{ is productive by Lemma 5.13(3)} \)[1].

The above results show very clearly the importance of the family of functors $\mathbb{F}_{\mathcal{L}, R, R'}$ defined by \[ X, G_k \rightarrow (\mathbb{F}(X, G_k(T))) = (X, G_k(T)) \text{ hence } G_k(T) \text{ is productive by Lemma 5.13(3)} \).

Theorem 5.17. The functor $\mathbb{F}_c$ is co-adjoint.

Proof. We follow the general outline of the proof of the main theorem in [22], making due allowance for the facts that [22] uses localic points instead of molecules and that $\mathbb{F}_c$ assigns its topology using complementation.

Take $(X, \mathcal{T}) \in \text{Ob } l\text{-Top}$ and set $M(X, \mathcal{T}) = \{\phi : L \rightarrow X | \phi^{-1}[\mathcal{T}] \subseteq L\}$. For $A \subseteq X$ define $\psi_L : M(X, \mathcal{T}) \rightarrow l$ by $\psi_L(\phi) = 1 - r \leftrightarrow \phi^{-1}[X \setminus A] = (0, r]$, whence $\phi^{-1}[\mathcal{T}] = (0, 1 - \psi_L(\phi))$ for all $\phi \in M(X, \mathcal{T})$. It is easy to verify that $T_\mathcal{T}^* = \{\psi_L | H \in \mathcal{T}\}$ is an $l$-topology on $M(X, \mathcal{T})$, and we omit the details. If we define $\eta : M(X, \mathcal{T}) \times L \rightarrow X$ by $\eta(\phi, r) = \phi(r)$, we claim that $((M(X, \mathcal{T}), T_\mathcal{T}^*), \eta)$ is an $\mathbb{F}_c$-co-universal arrow with codomain $(X, \mathcal{T})$.

Firstly, the continuity of $\eta : \mathbb{F}_c(M(X, \mathcal{T}), T_\mathcal{T}^*) \rightarrow (X, \mathcal{T})$ follows from the easily verified equality $\eta^{-1}[H] = (M(X, \mathcal{T}) \times L) \setminus 1 - \psi_L$. Hence $\eta \in \text{Mor } l\text{-Top}$. Secondly, to verify the universal property let $(Y, V)$ be an $l$-topology and $\zeta : \mathbb{F}_c(Y, V) \rightarrow (X, \mathcal{T})$ a $l\text{-Top}$ morphism. If we define $\mu : Y \rightarrow M(X, \mathcal{T})$ by $\mu(y)(r) = \zeta(y, r)$ then $\mu$ is an $l\text{-Top}$ morphism since for $H \in \mathcal{T}$ we have $(Y \times L) \setminus \zeta^{-1}[H] = \hat{k}$ for $k = 1 - \psi_L \circ \mu$. Finally, $\mu$ is clearly the unique $l\text{-Top}$ morphism making the diagram below commute.

\[ \mathbb{F}_c(M(X, \mathcal{T}), T_\mathcal{T}^*) \xrightarrow{\eta} (X, \mathcal{T}) \]

\[ \mathbb{F}_c(Y, V) \]

\[ \zeta \]

\[ \mu \]

$\square$

In the same way $\mathbb{F}_c$ is co-adjoint. The proof follows the same lines as that of Theorem 5.17, and is omitted.

Acknowledgements

The authors would like to express their sincere appreciation to Professor Rodabaugh and to the referees for their constructive comments which have helped enrich the content and improve the
exposition of this paper, and to thank Professor Brümmer for his infectious enthusiasm and unfailing encouragement.

References