

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

Linear equations $A\mathbf{x} = \mathbf{b}$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $d\mathbf{u}/dt = A\mathbf{u}$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra, based on $A\mathbf{x} = \lambda\mathbf{x}$. All matrices in this chapter are square.

A good model comes from the powers A, A^2, A^3, \dots of a matrix. Suppose you need the hundredth power A^{100} . The starting matrix A becomes unrecognizable after a few steps, and A^{100} is very close to $\begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$:

$$\begin{array}{ccccccc} \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} & \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} & \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} & \dots & \begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix} \\ A & A^2 & A^3 & & A^{100} \end{array}$$

A^{100} was found by using the *eigenvalues* of A , not by multiplying 100 matrices. Those eigenvalues (here they are 1 and $1/2$) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A . ***Certain exceptional vectors \mathbf{x} are in the same direction as $A\mathbf{x}$. Those are the “eigenvectors”.*** Multiply an eigenvector by A , and the vector $A\mathbf{x}$ is a number λ times the original \mathbf{x} .

The basic equation is $A\mathbf{x} = \lambda\mathbf{x}$. The number λ is an eigenvalue of A .

The eigenvalue λ tells whether the special vector \mathbf{x} is stretched or shrunk or reversed or left unchanged—when it is multiplied by A . We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1 . The eigenvalue λ could be zero! Then $A\mathbf{x} = 0\mathbf{x}$ means that this eigenvector \mathbf{x} is in the nullspace.

If A is the identity matrix, every vector has $A\mathbf{x} = \mathbf{x}$. All vectors are eigenvectors of I . All eigenvalues “lambda” are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.

This section will explain how to compute the \mathbf{x} 's and λ 's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use $\det(A - \lambda I) = 0$ to find the eigenvalues for this first example, and then derive it properly in equation (3).

Example 1 The matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1) \left(\lambda - \frac{1}{2} \right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\lambda = 1$ and $\lambda = \frac{1}{2}$. For those numbers, the matrix $A - \lambda I$ becomes *singular* (zero determinant). The eigenvectors \mathbf{x}_1 and \mathbf{x}_2 are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)\mathbf{x}_1 = 0$ is $A\mathbf{x}_1 = \mathbf{x}_1$ and the first eigenvector is $(.6, .4)$.

$(A - \frac{1}{2}I)\mathbf{x}_2 = 0$ is $A\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$ and the second eigenvector is $(1, -1)$:

$$\mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \mathbf{x}_1 \quad (A\mathbf{x} = \mathbf{x} \text{ means that } \lambda_1 = 1)$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \frac{1}{2}\mathbf{x}_2 \text{ so } \lambda_2 = \frac{1}{2}).$$

If \mathbf{x}_1 is multiplied again by A , we still get \mathbf{x}_1 . Every power of A will give $A^n\mathbf{x}_1 = \mathbf{x}_1$. Multiplying \mathbf{x}_2 by A gave $\frac{1}{2}\mathbf{x}_2$, and if we multiply again we get $(\frac{1}{2})^2$ times \mathbf{x}_2 .

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same \mathbf{x}_1 and \mathbf{x}_2 . The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} = \text{very small number}$.

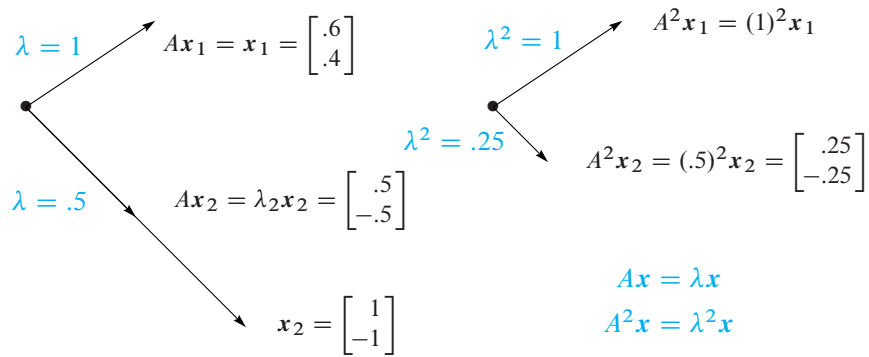


Figure 6.1: The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $\mathbf{x}_1 + (.2)\mathbf{x}_2$:

Separate into eigenvectors
$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \mathbf{x}_1 + (.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \quad (1)$$

Multiplying by A gives $(.7, .3)$, the first column of A^2 . Do it separately for \mathbf{x}_1 and $(.2)\mathbf{x}_2$. Of course $A\mathbf{x}_1 = \mathbf{x}_1$. And A multiplies \mathbf{x}_2 by its eigenvalue $\frac{1}{2}$:

$$\text{Multiply each } \mathbf{x}_i \text{ by } \lambda_i \quad A \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad \mathbf{x}_1 + \frac{1}{2}(.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A . We didn't need these eigenvectors to find A^2 . But it is the good way to do 99 multiplications. At every step \mathbf{x}_1 is unchanged and \mathbf{x}_2 is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99} \begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \mathbf{x}_1 + (.2)\left(\frac{1}{2}\right)^{99}\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100} . The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector \mathbf{x}_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector \mathbf{x}_2 is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A , the closer its columns approach the steady state.

We mention that this particular A is a **Markov matrix**. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ (as we found). Its eigenvector $\mathbf{x}_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 8.3 shows how Markov matrices appear in applications like Google.

For projections we can spot the steady state ($\lambda = 1$) and the nullspace ($\lambda = 0$).

Example 2 The projection matrix $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

Its eigenvectors are $\mathbf{x}_1 = (1, 1)$ and $\mathbf{x}_2 = (1, -1)$. For those vectors, $P\mathbf{x}_1 = \mathbf{x}_1$ (steady state) and $P\mathbf{x}_2 = \mathbf{0}$ (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special λ 's and \mathbf{x} 's:

1. Each column of $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is **singular**, so $\lambda = 0$ is an eigenvalue.
3. P is **symmetric**, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\mathbf{x} = \mathbf{0}\mathbf{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\mathbf{x} = \mathbf{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

$$\text{Project each part} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{projects onto} \quad P\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Special properties of a matrix lead to special eigenvalues and eigenvectors. That is a major theme of this chapter (it is captured in a table at the very end).

Projections have $\lambda = 0$ and 1 . Permutations have all $|\lambda| = 1$. The next matrix R (a reflection and at the same time a permutation) is also special.

Example 3 The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1 .

The eigenvector $(1, 1)$ is unchanged by R . The second eigenvector is $(1, -1)$ —its signs are reversed by R . A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P , because *reflection* = $2(\text{projection}) - I$:

$$R = 2P - I \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2)$$

Here is the point. If $Px = \lambda x$ then $2Px = 2\lambda x$. The eigenvalues are doubled when the matrix is doubled. Now subtract $Ix = x$. The result is $(2P - I)x = (2\lambda - 1)x$. **When a matrix is shifted by I , each λ is shifted by 1 .** No change in eigenvectors.

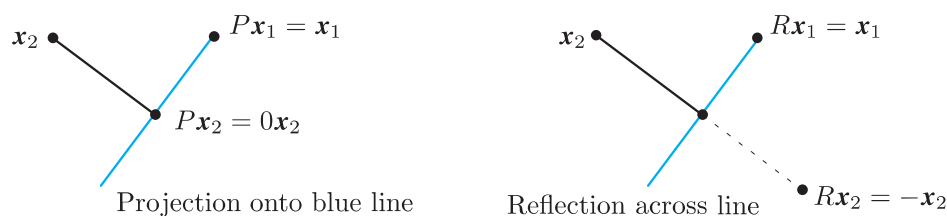


Figure 6.2: Projections P have eigenvalues 1 and 0 . Reflections R have $\lambda = 1$ and -1 . A typical x changes direction, but not the eigenvectors x_1 and x_2 .

Key idea: The eigenvalues of R and P are related exactly as the matrices are related:

The eigenvalues of $R = 2P - I$ are $2(1) - 1 = 1$ and $2(0) - 1 = -1$.

The eigenvalues of R^2 are λ^2 . In this case $R^2 = I$. Check $(1)^2 = 1$ and $(-1)^2 = 1$.

The Equation for the Eigenvalues

For projections and reflections we found λ 's and x 's by geometry: $Px = x$, $Px = \mathbf{0}$, $Rx = -x$. Now we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving $Ax = \lambda x$.

First move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. **The eigenvectors make up the nullspace of $A - \lambda I$.** When we know an eigenvalue λ , we find an eigenvector by solving $(A - \lambda I)x = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)x = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. **The determinant of $A - \lambda I$ must be zero.** This is how to recognize an eigenvalue λ :

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \quad (3)$$

This “characteristic equation” $\det(A - \lambda I) = 0$ involves only λ , not \mathbf{x} . When A is n by n , the equation has degree n . Then A has n eigenvalues and each λ leads to \mathbf{x} :

For each λ solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \lambda\mathbf{x}$ to find an eigenvector \mathbf{x} .

Example 4 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is already singular (zero determinant). Find its λ 's and \mathbf{x} 's.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\mathbf{x} = 0\mathbf{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A - \lambda I) = 0$ is the way to find *all* λ 's and \mathbf{x} 's. Always subtract λI from A :

$$\text{Subtract } \lambda \text{ from the diagonal to find } A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}. \quad (4)$$

Take the determinant “ $ad - bc$ ” of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the “ ad ” part is $\lambda^2 - 5\lambda + 4$. The “ bc ” part, not containing λ , is 2 times 2.

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(2) = \lambda^2 - 5\lambda. \quad (5)$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues } \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5.$$

Now find the eigenvectors. Solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\mathbf{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is $A\mathbf{x} = \lambda\mathbf{x}$.

We need to emphasize: *There is nothing exceptional about $\lambda = 0$.* Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*.

In the example, the shifted matrix $A - 5I$ is singular and 5 is the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. **Compute the determinant of $A - \lambda I$.** With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n .
2. **Find the roots of this polynomial,** by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A . They make $A - \lambda I$ singular.
3. For each eigenvalue λ , **solve $(A - \lambda I)x = \mathbf{0}$ to find an eigenvector x .**

A note on the eigenvectors of 2 by 2 matrices. When $A - \lambda I$ is singular, both rows are multiples of a vector (a, b) . *The eigenvector is any multiple of $(b, -a)$.* The example had $\lambda = 0$ and $\lambda = 5$:

$\lambda = 0$: rows of $A - 0I$ in the direction $(1, 2)$; eigenvector in the direction $(2, -1)$

$\lambda = 5$: rows of $A - 5I$ in the direction $(-4, 2)$; eigenvector in the direction $(2, 4)$.

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of x is as good as x . MATLAB's `eig(A)` divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Similarly some n by n matrices don't have n independent eigenvectors. Without n eigenvectors, we don't have a basis. We can't write every v as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Good News, Bad News

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the λ 's.* The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A ! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product λ_1 times λ_2 and the sum $\lambda_1 + \lambda_2$ can be found quickly from the matrix.* For this A , the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is 0 + 7. That agrees with the sum down the main diagonal (the **trace** is 1 + 6). These quick checks always work:

The product of the n eigenvalues equals the determinant.

The sum of the n eigenvalues equals the sum of the n diagonal entries.

The sum of the entries on the main diagonal is called the *trace* of A :

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \quad (6)$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ 's. But when the computation is wrong, they generally tell us so. To compute the correct λ 's, go back to $\det(A - \lambda I) = 0$.

The determinant test makes the *product* of the λ 's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the λ 's is not the sum of the pivots—as the example showed. The individual λ 's have almost nothing to do with the pivots. In this new part of linear algebra, the key equation is really *nonlinear*: λ multiplies \mathbf{x} .

Why do the eigenvalues of a triangular matrix lie on its diagonal?

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 5 The 90° rotation $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no real eigenvectors. Its eigenvalues are $\lambda = i$ and $\lambda = -i$. Sum of λ 's = trace = 0. Product = determinant = 1.

After a rotation, *no vector* $Q\mathbf{x}$ stays in the same direction as \mathbf{x} (except $\mathbf{x} = \mathbf{0}$ which is useless). There cannot be an eigenvector, unless we go to *imaginary numbers*. Which we do.

To see how i can help, look at Q^2 which is $-I$. If Q is rotation through 90° , then Q^2 is rotation through 180° . Its eigenvalues are -1 and -1 . (Certainly $-I\mathbf{x} = -1\mathbf{x}$.) Squaring Q will square each λ , so we must have $\lambda^2 = -1$. The eigenvalues of the 90° rotation matrix Q are $+i$ and $-i$, because $i^2 = -1$.

Those λ 's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are i and $-i$. We meet the imaginary number i also in the eigenvectors:

$$\text{Complex eigenvectors} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = -i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Somehow these complex vectors $\mathbf{x}_1 = (1, i)$ and $\mathbf{x}_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and $-i$ also illustrate two special properties of Q :

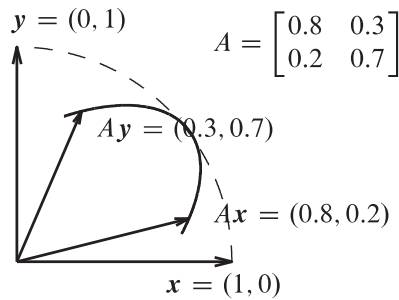
1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.
2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix ($A^T = A$) can be compared to a real number. A skew-symmetric matrix ($A^T = -A$) can be compared to an imaginary number. An orthogonal matrix ($A^T A = I$) can be compared to a complex number with $|\lambda| = 1$. For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4.

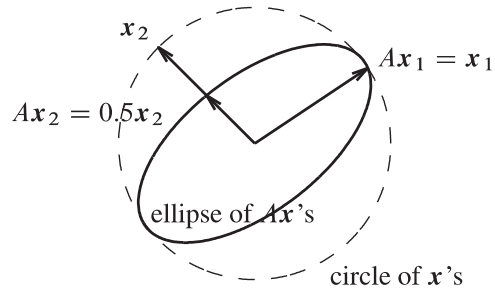
The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Chapter 10 explains the dot product of complex vectors).

Eigshow in MATLAB

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\mathbf{x} = (1, 0)$. *The mouse makes this vector move around the unit circle.* At the same time the screen shows $A\mathbf{x}$, in color and also moving. Possibly $A\mathbf{x}$ is ahead of \mathbf{x} . Possibly $A\mathbf{x}$ is behind \mathbf{x} . *Sometimes $A\mathbf{x}$ is parallel to \mathbf{x} .* At that parallel moment, $A\mathbf{x} = \lambda\mathbf{x}$ (at \mathbf{x}_1 and \mathbf{x}_2 in the second figure).



These are not eigenvectors



$A\mathbf{x}$ lines up with \mathbf{x} at eigenvectors

The eigenvalue λ is the length of $A\mathbf{x}$, when the unit eigenvector \mathbf{x} lines up. The built-in choices for A illustrate three possibilities: 0, 1, or 2 directions where $A\mathbf{x}$ crosses \mathbf{x} .

- 0. There are *no real eigenvectors*. $A\mathbf{x}$ stays behind or ahead of \mathbf{x} . This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q .
- 1. There is only *one* line of eigenvectors (unusual). The moving directions $A\mathbf{x}$ and \mathbf{x} touch but don't cross over. This happens for the last 2 by 2 matrix below.
- 2. There are eigenvectors in *two* independent directions. This is typical! $A\mathbf{x}$ crosses \mathbf{x} at the first eigenvector \mathbf{x}_1 , and it crosses back at the second eigenvector \mathbf{x}_2 . Then $A\mathbf{x}$ and \mathbf{x} cross again at $-\mathbf{x}_1$ and $-\mathbf{x}_2$.

You can mentally follow \mathbf{x} and $A\mathbf{x}$ for these five matrices. Under the matrices I will count their real eigenvectors. Can you see where $A\mathbf{x}$ lines up with \mathbf{x} ?

$$A = \begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{2} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{matrix}$$

When A is singular (rank one), its column space is a line. The vector $A\mathbf{x}$ goes up and down that line while \mathbf{x} circles around. One eigenvector \mathbf{x} is along the line. Another eigenvector appears when $A\mathbf{x}_2 = \mathbf{0}$. Zero is an eigenvalue of a singular matrix.

■ REVIEW OF THE KEY IDEAS ■

1. $A\mathbf{x} = \lambda\mathbf{x}$ says that eigenvectors \mathbf{x} keep the same direction when multiplied by A .
2. $A\mathbf{x} = \lambda\mathbf{x}$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1} , with the same eigenvectors.
4. The sum of the λ 's equals the sum down the main diagonal of A (*the trace*). The product of the λ 's equals the determinant.
5. Projections P , reflections R , 90° rotations Q have special eigenvalues $1, 0, -1, i, -i$. Singular matrices have $\lambda = 0$. Triangular matrices have λ 's on their diagonal.

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2 .

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2 + 2$ agrees with $1 + 3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ which is $A\mathbf{x} = \lambda\mathbf{x}$:

$$\lambda = 1: \quad (A - I)\mathbf{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)\mathbf{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors* as A . Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A + 4I \text{ has } \frac{1+4=5}{3+4=7}$$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^T$ and the λ 's are positive.

6.1 B Find the eigenvalues and eigenvectors of this 3 by 3 matrix A :

Symmetric matrix

Singular matrix

Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of A add to zero, the vector $\mathbf{x} = (1, 1, 1)$ gives $A\mathbf{x} = \mathbf{0}$. This is an eigenvector for the eigenvalue $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda)(1 - \lambda) - 2(1 - \lambda) \\ = (1 - \lambda)[(2 - \lambda)(1 - \lambda) - 2] \\ = (1 - \lambda)(-\lambda)(3 - \lambda).$$

That factor $-\lambda$ confirms that $\lambda = 0$ is a root, and an eigenvalue of A . The other factors $(1 - \lambda)$ and $(3 - \lambda)$ give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue 0, 1, 3 corresponds to an eigenvector:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad A\mathbf{x}_1 = \mathbf{0}\mathbf{x}_1 \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad A\mathbf{x}_2 = \mathbf{1}\mathbf{x}_2 \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad A\mathbf{x}_3 = \mathbf{3}\mathbf{x}_3.$$

I notice again that eigenvectors are perpendicular when A is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$. We were lucky to find simple roots $\lambda = 0, 1, 3$. Normally we would use a command like **eig**(A), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command $[S, D] = \mathbf{eig}(A)$ will produce unit eigenvectors in the columns of the **eigenvector matrix** S . The first one happens to have three minus signs, reversed from $(1, 1, 1)$ and divided by $\sqrt{3}$. The eigenvalues of A will be on the diagonal of the **eigenvalue matrix** (typed as D but soon called Λ).

Problem Set 6.1

- 1 The example at the start of the chapter has powers of this matrix A :

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

- (a) Show from A how a row exchange can produce different eigenvalues.
 (b) Why is a zero eigenvalue *not* changed by the steps of elimination?

- 2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A . Its eigenvalues are _____ by 1.

- 3 Compute the eigenvalues and eigenvectors of A and A^{-1} . Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A . When A has eigenvalues λ_1 and λ_2 , its inverse has eigenvalues _____.

- 4 Compute the eigenvalues and eigenvectors of A and A^2 :

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A . When A has eigenvalues λ_1 and λ_2 , A^2 has eigenvalues _____. In this example, why is $\lambda_1^2 + \lambda_2^2 = 13$?

- 5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (are equal to)(are not equal to) eigenvalues of A plus eigenvalues of B .

- 6 Find the eigenvalues of A and B and AB and BA :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

- (a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B ?
 (b) Are the eigenvalues of AB equal to the eigenvalues of BA ?

- 7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.
- 8 (a) If you know that \mathbf{x} is an eigenvector, the way to find λ is to _____.
 (b) If you know that λ is an eigenvalue, the way to find \mathbf{x} is to _____.
- 9 What do you do to the equation $A\mathbf{x} = \lambda\mathbf{x}$, in order to prove (a), (b), and (c)?
- (a) λ^2 is an eigenvalue of A^2 , as in Problem 4.
 (b) λ^{-1} is an eigenvalue of A^{-1} , as in Problem 3.
 (c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.
- 10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞ . Explain from those answers why A^{100} is close to A^∞ :

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

- 11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector \mathbf{x}_2 . Any idea why this should be?
- 12 Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\text{Projection matrix} \quad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ , so do all their linear combinations. Find an eigenvector of P with no zero components.

- 13 From the unit vector $\mathbf{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \mathbf{u}\mathbf{u}^T$. This matrix has $P^2 = P$ because $\mathbf{u}^T\mathbf{u} = 1$.
- (a) $P\mathbf{u} = \mathbf{u}$ comes from $(\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(\text{_____})$. Then \mathbf{u} is an eigenvector with $\lambda = 1$.
 (b) If \mathbf{v} is perpendicular to \mathbf{u} show that $P\mathbf{v} = \mathbf{0}$. Then $\lambda = 0$.
 (c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.
- 14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos \theta \pm i \sin \theta$:

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\mathbf{x} = \mathbf{0}$. Use $i^2 = -1$.

- 15 Every permutation matrix leaves $\mathbf{x} = (1, 1, \dots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ 's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 16 **The determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$.** Start with the polynomial $\det(A - \lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\hspace{2cm}}.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

- 17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{\hspace{1cm}})/2$ and $\lambda = \underline{\hspace{1cm}}$. Their sum is $\underline{\hspace{1cm}}$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \underline{\hspace{1cm}}$.

- 18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

- 19 A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find three of these (give the answers where possible):

- the rank of B
- the determinant of $B^T B$
- the eigenvalues of $B^T B$
- the eigenvalues of $(B^2 + I)^{-1}$.

- 20 Choose the last rows of A and C to give eigenvalues 4, 7 and 1, 2, 3:

$$\text{Companion matrices} \quad A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

- 21 **The eigenvalues of A equal the eigenvalues of A^T .** This is because $\det(A - \lambda I)$ equals $\det(A^T - \lambda I)$. That is true because $\underline{\hspace{1cm}}$. Show by an example that the eigenvectors of A and A^T are *not* the same.

- 22 Construct any 3 by 3 Markov matrix M : positive entries down each column add to 1. Show that $M^T(1, 1, 1) = (1, 1, 1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M . Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ 's?

- 23 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that $A^2 = 0$.
- 24 This matrix is singular with rank one. Find three λ 's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [2 \ 1 \ 2] = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

- 25 Suppose A and B have the same eigenvalues $\lambda_1, \dots, \lambda_n$ with the same independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Then $A = B$. Reason: Any vector \mathbf{x} is a combination $c_1\mathbf{x}_1 + \dots + c_n\mathbf{x}_n$. What is $A\mathbf{x}$? What is $B\mathbf{x}$?
- 26 The block B has eigenvalues 1, 2 and C has eigenvalues 3, 4 and D has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix A :

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

- 27 Find the rank and the four eigenvalues of A and C :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

- 28 Subtract I from the previous A . Find the λ 's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

- 29 (Review) Find the eigenvalues of A , B , and C :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

- 30 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- 31** If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda = 11$. Rank one gives $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^T = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

- 32** Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors \mathbf{u} , \mathbf{v} , \mathbf{w} .
- Give a basis for the nullspace and a basis for the column space.
 - Find a particular solution to $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Find all solutions.
 - $A\mathbf{x} = \mathbf{u}$ has no solution. If it did then _____ would be in the column space.
- 33** Suppose \mathbf{u} , \mathbf{v} are orthonormal vectors in \mathbf{R}^2 , and $A = \mathbf{u}\mathbf{v}^T$. Compute $A^2 = \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T$ to discover the eigenvalues of A . Check that the trace of A agrees with $\lambda_1 + \lambda_2$.
- 34** Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Challenge Problems

- 35** There are six 3 by 3 permutation matrices P . What numbers can be the *determinants* of P ? What numbers can be *pivots*? What numbers can be the *trace* of P ? What *four numbers* can be eigenvalues of P , as in Problem 15?
- 36** Is there a real 2 by 2 matrix (other than I) with $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct a rotation matrix as A (which angle of rotation?).
- 37** (a) Find the eigenvalues and eigenvectors of A . They depend on c :

$$A = \begin{bmatrix} .4 & 1 - c \\ .6 & c \end{bmatrix}.$$

- Show that A has just one line of eigenvectors when $c = 1.6$.
- This is a Markov matrix when $c = .8$. Then A^n will approach what matrix A^∞ ?